

RECURSIVE FORMULAS GENERATING POWER MOMENTS OF KLOOSTERMAN SUMS: SYMPLECTIC CASE

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ABSTRACT. In this paper, we construct two infinite families of binary linear codes associated with double cosets with respect to certain maximal parabolic subgroup of the symplectic group $Sp(2n, q)$. Here q is a power of two. Then we obtain an infinite family of recursive formulas for the power moments of Kloosterman sums and those of 2-dimensional Kloosterman sums in terms of the frequencies of weights in the codes. This is done via Pless power moment identity and by utilizing the explicit expressions of exponential sums over those double cosets related to the evaluations of "Gauss sum" for the symplectic groups $Sp(2n, q)$.

Keywords: Kloosterman sum, symplectic group, double cosets, maximal parabolic subgroup, Pless power moment identity, weight distribution.

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1. Introduction

The purpose of this paper is to obtain an infinite family of recursive formulas for the power moments of Kloosterman sums and those of 2-dimensional Kloosterman sums in terms of the frequencies of weights in certain infinite families of binary linear codes. Indeed, we will construct two infinite families of binary linear codes associated with double cosets with respect to certain maximal parabolic subgroup of the symplectic group $Sp(2n, q)$. Here q is a power of two. Our construction is based on Pless power moment identity and on the explicit expressions of exponential sums over those double cosets related to the evaluations of "Gauss sums" for the symplectic groups $Sp(2n, q)$.

We emphasize here that there have been only a handful of recursive formulas generating power moments of Kloosterman and 2-dimensional Kloosterman sums (cf. [1], [6], [17], [19]). This paper is distinguished from the other studies in two respects: The first thing is that we succeeded in constructing not just a few but an infinite family of such recursive formulas. The second thing is that this is one of the first papers (see also [10], [11])

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in which some exponential sums associated with finite classical groups are used.

Let ψ be a nontrivial additive character of the finite field \mathbb{F}_q with $q = p^r$ elements (p a prime), and let m be a positive integer. Then the m -dimensional Kloosterman sum $K_m(\psi; a)$ ([14]) is defined by

$$K_m(\psi; a) = \sum_{\alpha_1, \dots, \alpha_m \in \mathbb{F}_q^*} \psi(\alpha_1 + \dots + \alpha_m + a\alpha_1^{-1} \dots \alpha_m^{-1}) \quad (a \in \mathbb{F}_q^*).$$

In particular, if $m = 1$, then $K_1(\psi; a)$ is simply denoted by $K(\psi; a)$, and is called the Kloosterman sum. The Kloosterman sum was introduced in 1926 to give an estimate for the Fourier coefficients of modular forms (cf. [12], [4]). It has also been studied to solve various problems in coding theory and cryptography over finite fields of characteristic two (cf. [3], [5]).

For each nonnegative integer h , by $MK_m(\psi)^h$ we will denote the h -th moment of the m -dimensional Kloosterman sum $K_m(\psi; a)$. Namely, it is given by

$$MK_m(\psi)^h = \sum_{a \in \mathbb{F}_q^*} K_m(\psi; a)^h.$$

If $\psi = \lambda$ is the canonical additive character of \mathbb{F}_q , then $MK_m(\lambda)^h$ will be simply denoted by MK_m^h . If further $m = 1$, for brevity MK_1^h will be indicated by MK^h .

Explicit computations on power moments of Kloosterman sums were begun with the paper [19] of Salié in 1931, where he showed, for any odd prime q ,

$$MK^h = q^2 M_{h-1} - (q-1)^{h-1} + 2(-1)^{h-1} (h \geq 1).$$

Here $M_0 = 0$, and for $h \in \mathbb{Z}_{>0}$,

$$M_h = |\{(\alpha_1, \dots, \alpha_h) \in (\mathbb{F}_q^*)^h \mid \sum_{j=1}^h \alpha_j = 1 = \sum_{j=1}^h \alpha_j^{-1}\}|.$$

For $q = p$ odd prime, Salié obtained MK^1 , MK^2 , MK^3 , MK^4 in [19] by determining M_1, M_2, M_3 . MK^5 can be expressed in terms of the p -th eigenvalue for a weight 3 newform on $\Gamma_0(15)$ (cf. [15], [18]). MK^6 can be expressed in terms of the p -th eigenvalue for a weight 4 newform on $\Gamma_0(6)$ (cf. [7]). Also, based on numerical evidence, in [6] Evans was led to propose a conjecture which expresses MK^7 in terms of Hecke eigenvalues for a weight 3 newform on $\Gamma_0(525)$ with quartic nebentypus of conductor 105. For more details about this brief history of explicit computations on power moments of Kloosterman sums, one is referred to Section 4 of [9].

From now on, let us assume that $q = 2^r$. Carlitz[1] evaluated MK^h for $h \leq 4$. Recently, Moisiso was able to find explicit expressions of MK^h , for the other values of h with $h \leq 10$ (cf. [17]). This was done, via Pless power moment identity, by connecting moments of Kloosterman sums and

the frequencies of weights in the binary Zetterberg code of length $q+1$ which were known by the work of Schoof and Vlught in [20].

In [9], the binary linear codes $C(SL(n, q))$ associated with finite special linear groups $SL(n, q)$ were constructed when n, q are both powers of two. Then obtained was a recursive formula for the power moments of multi-dimensional Kloosterman sums in terms of the frequencies of weights in $C(SL(n, q))$. In particular, when $n = 2$, this gives a recursive formula for the power moments of Kloosterman sums. Also, in order to get recursive formulas for the power moments of Kloosterman and 2-dimensional Kloosterman sums, we constructed in [10] three binary linear codes $C(SO^+(2, q))$, $C(O^+(2, q))$, $C(SO^+(4, q))$, respectively associated with $SO^+(2, q)$, $O^+(2, q)$, $SO^+(4, q)$, and in [11] three binary linear codes $C(SO^-(2, q))$, $C(O^-(2, q))$, $C(SO^-(4, q))$, respectively associated with $SO^-(2, q)$, $O^-(2, q)$, $SO^-(4, q)$. All of these were done via Pless power moment identity and by utilizing our previous results on explicit expressions of Gauss sums for the stated finite classical groups. Still, in all, we had only a handful of recursive formulas generating power moments of Kloosterman and 2-dimensional Kloosterman sums.

In this paper, we will be able to produce infinite families of recursive formulas generating power moments of Kloosterman and 2-dimensional Kloosterman sums. To do that, we construct two infinite families of binary linear codes $C(DC^-(n, q))$ ($n = 1, 3, 5, \dots$) and $C(DC^+(n, q))$ ($n = 2, 4, 6, \dots$), respectively associated with the double cosets $DC^-(n, q) = P\sigma_{n-1}P$ and $DC^+(n, q) = P\sigma_{n-2}P$ with respect to the maximal parabolic subgroup $P = P(2n, q)$ of the symplectic group $Sp(2n, q)$, and express those power moments in terms of the frequencies of weights in each code. Then, thanks to our previous results on the explicit expressions of exponential sums over those double cosets related to the evaluations of ‘‘Gauss sums’’ for the symplectic groups $Sp(2n, q)$ [8], we can express the weight of each codeword in the duals of the codes in terms of Kloosterman or 2-dimensional Kloosterman sums. Then our formulas will follow immediately from the Pless power moment identity.

Theorem 1.1 in the following (cf. (1.5), (1.6), (1.8)-(1.10)) is the main result of this paper. Henceforth, we agree that the binomial coefficient $\binom{b}{a} = 0$, if $a > b$ or $a < 0$. To simplify notations, we introduce the following ones which will be used throughout this paper at various places.

$$(1.1) \quad A^-(n, q) = q^{\frac{1}{4}(5n^2-1)} \prod_{j=1}^{(n-1)/2} \binom{n}{1}_q (q^{2j-1} - 1),$$

$$(1.2) \quad B^-(n, q) = q^{\frac{1}{4}(n-1)^2} (q^n - 1) \prod_{j=1}^{(n-1)/2} (q^{2j} - 1),$$

$$(1.3) \quad A^+(n, q) = q^{\frac{1}{4}(5n^2-2n)} [2]_q^{\lfloor \frac{n-2}{2} \rfloor} \prod_{j=1}^{(n-2)/2} (q^{2j-1} - 1),$$

$$(1.4) \quad B^+(n, q) = q^{\frac{1}{4}(n-2)^2} (q^n - 1)(q^{n-1} - 1) \prod_{j=1}^{(n-2)/2} (q^{2j} - 1).$$

Henceforth we agree that the binomial coefficient $\binom{b}{a} = 0$, if $a > b$ and $a < 0$.

Theorem 1.1. *Let $q = 2^r$. Then, with the notations in (1.1)-(1.4), we have the following. (a) For either each odd $n \geq 3$ and all q , or $n = 1$ and all $q \geq 8$, we have a recursive formula generating power moments of Kloosterman sums over \mathbb{F}_q*

$$(1.5) \quad \begin{aligned} MK^h &= \sum_{l=0}^{h-1} (-1)^{h+l+1} \binom{h}{l} B^-(n, q)^{h-l} MK^l \\ &+ qA^-(n, q)^{-h} \sum_{j=0}^{\min\{N^-(n, q), h\}} (-1)^{h+j} C_j^-(n, q) \\ &\times \sum_{t=j}^h t! S(h, t) 2^{h-t} \binom{N^-(n, q) - j}{N^-(n, q) - t} \\ &(h = 1, 2, \dots), \end{aligned}$$

where $N^-(n, q) = |DC^-(n, q)| = A^-(n, q)B^-(n, q)$, and $\{C_j^-(n, q)\}_{j=0}^{N^-(n, q)}$ is the weight distribution of $C(DC^-(n, q))$ given by

$$(1.6) \quad \begin{aligned} C_j^-(n, q) &= \sum_{\nu_0} \left(q^{-1}A^-(n, q)(B^-(n, q) + 1) \right) \\ &\times \prod_{\substack{\nu_\beta \\ \text{tr}(\beta^{-1})=0}} \left(q^{-1}A^-(n, q)(B^-(n, q) + q + 1) \right) \\ &\times \prod_{\substack{\nu_\beta \\ \text{tr}(\beta^{-1})=1}} \left(q^{-1}A^-(n, q)(B^-(n, q) - q + 1) \right). \end{aligned}$$

Here the sum is over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta \in \mathbb{F}_q}$ satisfying $\sum_{\beta \in \mathbb{F}_q} \nu_\beta = j$ and $\sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0$. In addition, $S(h, t)$ is the Stirling number of the second kind defined by

$$(1.7) \quad S(h, t) = \frac{1}{t!} \sum_{j=0}^t (-1)^{t-j} \binom{t}{j} j^h.$$

(b) For each even $n \geq 2$ and all $q \geq 4$, we have recursive formulas generating power moments of 2-dimensional Kloosterman sums over \mathbb{F}_q and even power moments of Kloosterman sums over \mathbb{F}_q

$$\begin{aligned}
(1.8) \quad MK_2^h &= \sum_{l=0}^{h-1} (-1)^{h+l+1} \binom{h}{l} (B^+(n, q) - q^2)^{h-l} MK_2^l \\
&\quad + qA^+(n, q)^{-h} \sum_{j=0}^{\min\{N^+(n, q), h\}} (-1)^{h+j} C_j^+(n, q) \\
&\quad \times \sum_{t=j}^h t! S(h, t) 2^{h-t} \binom{N^+(n, q) - j}{N^+(n, q) - t} \\
&\quad (h = 1, 2, \dots),
\end{aligned}$$

and

$$\begin{aligned}
(1.9) \quad MK^{2h} &= \sum_{l=0}^{h-1} (-1)^{h+l+1} \binom{h}{l} (B^+(n, q) - q^2 + q)^{h-l} MK^{2l} \\
&\quad + qA^+(n, q)^{-h} \sum_{j=0}^{\min\{N^+(n, q), h\}} (-1)^{h+j} C_j^+(n, q) \\
&\quad \times \sum_{t=j}^h t! S(h, t) 2^{h-t} \binom{N^+(n, q) - j}{N^+(n, q) - t} \\
&\quad (h = 1, 2, \dots),
\end{aligned}$$

where $N^+(n, q) = |DC^+(n, q)| = A^+(n, q)B^+(n, q)$, and $\{C_j^+(n, q)\}_{j=0}^{N^+(n, q)}$ is the weight distribution of $C(DC^+(n, q))$ given by

$$\begin{aligned}
(1.10) \quad C_j^+(n, q) &= \sum_{\nu_0} \left(q^{-1} A^+(n, q) (B^+(n, q) + q^3 - q^2 - 1) \right) \prod_{\substack{|\tau| < 2\sqrt{q} \\ \tau \equiv -1(4)}} \\
&\quad \times \prod_{K(\lambda; \beta^{-1}) = \tau} \left(q^{-1} A^+(n, q) (B^+(n, q) + q\tau - q^2 - 1) \right)_{\nu_\beta}.
\end{aligned}$$

Here the sum is over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta \in \mathbb{F}_q}$ satisfying $\sum_{\beta \in \mathbb{F}_q} \nu_\beta = j$, and $\sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0$.

The following corollary is just $n = 1$ the case of (a) in the above.

Corollary 1.2. *Let $q \geq 8$. For $h = 1, 2, \dots$,*

$$\begin{aligned} MK^h &= \sum_{l=0}^{h-1} (-1)^{h+l+1} \binom{h}{l} (q-1)^{h-l} MK^l \\ &\quad + q^{1-h} \sum_{j=0}^{\min\{q(q-1), h\}} (-1)^{h+j} C_j^-(1, q) \\ &\quad \times \sum_{t=j}^h t! S(h, t) 2^{h-t} \binom{q(q-1)-j}{q(q-1)-t}, \end{aligned}$$

where $\{C_j^-(1, q)\}_{j=0}^{q(q-1)}$ is the weight distribution of $C(DC^-(1, q))$ given by

$$C_j^-(1, q) = \sum \binom{q}{\nu_0} \prod_{\text{tr}(\beta^{-1})=0} \binom{2q}{\nu_\beta}.$$

Here the sum is over all the sets of nonnegative integers $\{\nu_0\} \cup \{\nu_\beta\}_{\text{tr}(\beta^{-1})=0}$ satisfying $\nu_0 + \sum_{\text{tr}(\beta^{-1})=0} \nu_\beta = j$ and $\sum_{\text{tr}(\beta^{-1})=0} \nu_\beta \beta = 0$. In addition, $S(h, t)$ is the Stirling number of the second kind given in (1.7).

2. $Sp(2n, q)$

For more details about this section, one is referred to the paper [8]. Throughout this paper, the following notations will be used:

$$q = 2^r \quad (r \in \mathbb{Z}_{>0}),$$

\mathbb{F}_q = the finite field with q elements,

$\text{Tr}A$ = the trace of A for a square matrix A ,

tB = the transpose of B for any matrix B .

The symplectic group over the field is defined as:

$$Sp(2n, q) = \{w \in GL(2n, q) \mid {}^t w J w = J\},$$

with

$$J = \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix}.$$

$P = P(2n, q)$ is the maximal parabolic subgroup of $Sp(2n, q)$ defined by:

$$P(2n, q) = \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \mid A \in GL(n, q), {}^t B = B \right\}.$$

Then, with respect to $P = P(2n, q)$, the Bruhat decomposition of $Sp(2n, q)$ is given by

$$(2.1) \quad Sp(2n, q) = \prod_{r=0}^n P \sigma_r P,$$

where

$$\sigma_r = \begin{bmatrix} 0 & 0 & 1_r & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ 1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix} \in Sp(2n, q).$$

Put, for each r with $0 \leq r \leq n$,

$$A_r = \{w \in P(2n, q) \mid \sigma_r w \sigma_r^{-1} \in P(2n, q)\}.$$

Expressing $Sp(2n, q)$ as a disjoint union of right cosets of $P = P(2n, q)$, the Bruhat decomposition in (11) can be written as

$$Sp(2n, q) = \prod_{r=0}^n P \sigma_r (A_r \backslash P).$$

The order of the general linear group $GL(n, q)$ is given by

$$g_n = \prod_{j=0}^{n-1} (q^n - q^j) = q^{\binom{n}{2}} \prod_{j=1}^n (q^j - 1).$$

For integers n, r with $0 \leq r \leq n$, the q -binomial coefficients are defined as:

$$[n]_q = \prod_{j=0}^{r-1} (q^{n-j} - 1) / (q^{r-j} - 1).$$

Then, for integers n, r with $0 \leq r \leq n$, we have

$$(2.2) \quad \frac{g_n}{g_{n-r} g_r} = q^{r(n-r)} [n]_q.$$

In [8], it is shown that

$$(2.3) \quad |A_r| = g_r g_{n-r} q^{\binom{n+1}{2}} q^{r(2n-3r-1)/2}.$$

Also, it is immediate to see that

$$(2.4) \quad |P(2n, q)| = q^{\binom{n+1}{2}} g_n.$$

So, from (2.2)-(2.4), we get

$$(2.5) \quad |A_r \backslash P(2n, q)| = q^{\binom{r+1}{2}} [r]_q,$$

and

$$(2.6) \quad \begin{aligned} & |P(2n, q) \sigma_r P(2n, q)| \\ &= |P(2n, q)|^2 |A_r|^{-1} \\ &= q^{n^2} [n]_q q^{\binom{r}{2}} q^r \prod_{j=1}^n (q^j - 1). \end{aligned}$$

In particular, with

$$\begin{aligned} DC^-(n, q) &= P(2n, q)\sigma_{n-1}P(2n, q), \\ DC^+(n, q) &= P(2n, q)\sigma_{n-2}P(2n, q), \end{aligned}$$

$$(2.7) \quad |DC^-(n, q)| = q^{\frac{1}{2}n(3n-1)} [n]_q \prod_{j=1}^n (q^j - 1),$$

$$(2.8) \quad |DC^+(n, q)| = q^{\frac{1}{2}(3n^2-3n+2)} [n]_q \prod_{j=1}^n (q^j - 1).$$

Also, from (2.1), (2.6), we have

$$\begin{aligned} |Sp(2n, q)| &= \sum_{r=0}^n |P(2n, q)|^2 |A_r|^{-1} \\ &= q^{n^2} \prod_{j=1}^n (q^{2j} - 1), \end{aligned}$$

where one can apply the following q -binomial theorem with $x = -q$:

$$\sum_{r=0}^n [n]_q (-1)^r q^{\binom{r}{2}} x^r = (x; q)_n,$$

with $(x; q)_n = (1-x)(1-qx) \cdots (1-q^{n-1}x)$ (x an indeterminate, $n \in \mathbb{Z}_{>0}$).

3. Exponential sums over double cosets of $Sp(2n, q)$

The following notations will be used throughout this paper.

$$\begin{aligned} tr(x) &= x + x^2 + \cdots + x^{2^r-1} \text{ the trace function } \mathbb{F}_q \rightarrow \mathbb{F}_2, \\ \lambda(x) &= (-1)^{tr(x)} \text{ the canonical additive character of } \mathbb{F}_q. \end{aligned}$$

Then any nontrivial additive character ψ of \mathbb{F}_q is given by $\psi(x) = \lambda(ax)$, for a unique $a \in \mathbb{F}_q^*$.

For any nontrivial additive character ψ of \mathbb{F}_q and $a \in \mathbb{F}_q^*$, the Kloosterman sum $K_{GL(t,q)}(\psi; a)$ for $GL(t, q)$ is defined as

$$K_{GL(t,q)}(\psi; a) = \sum_{w \in GL(t,q)} \psi(Trw + a Trw^{-1}).$$

Notice that, for $t = 1$, $K_{GL(1,q)}(\psi; a)$ denotes the Kloosterman sum $K(\psi; a)$.

For the Kloosterman sum $K(\psi; a)$, we have the Weil bound(cf. [14])

$$(3.1) \quad |K(\psi; a)| \leq 2\sqrt{q}.$$

In [8], it is shown that $K_{GL(t,q)}(\psi; a)$ satisfies the following recursive relation: for integers $t \geq 2$, $a \in \mathbb{F}_q^*$,

$$(3.2) \quad K_{GL(t,q)}(\psi; a) = q^{t-1} K_{GL(t-1,q)}(\psi; a) K(\psi; a) \\ + q^{2t-2} (q^{t-1} - 1) K_{GL(t-2,q)}(\psi; a),$$

where we understand that $K_{GL(0,q)}(\psi; a) = 1$. From (3.2), in [8] an explicit expression of the Kloosterman sum for $GL(t, q)$ was derived.

Theorem 3.1 ([8]). *For integers $t \geq 1$, and $a \in \mathbb{F}_q^*$, the Kloosterman sum $K_{GL(t,q)}(\psi; a)$ is given by*

$$K_{GL(t,q)}(\psi; a) = q^{(t-2)(t+1)/2} \sum_{l=1}^{\lfloor (t+2)/2 \rfloor} q^l K(\psi; a)^{t+2-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu-2\nu} - 1),$$

where $K(\psi; a)$ is the Kloosterman sum and the inner sum is over all integers j_1, \dots, j_{l-1} satisfying $2l-1 \leq j_{l-1} \leq j_{l-2} \leq \dots \leq j_1 \leq t+1$. Here we agree that the inner sum is 1 for $l=1$.

In Section 5 of [8], it is shown that the Gauss sum for $Sp(2n, q)$ is given by:

$$(3.3) \quad \sum_{w \in Sp(2n,q)} \psi(Trw) \\ = \sum_{r=0}^n \sum_{w \in P\sigma_r P} \psi(Trw) \\ = \sum_{r=0}^n |A_r \backslash P| \sum_{w \in P} \psi(Trw\sigma_r) \\ = q^{\binom{n+1}{2}} \sum_{r=0}^n |A_r \backslash P| q^{r(n-r)} a_r K_{GL(n-r,q)}(\psi; 1).$$

Here ψ is any nontrivial additive character of \mathbb{F}_q , $a_0 = 1$, and, for $r \in \mathbb{Z}_{>0}$, a_r denotes the number of all $r \times r$ nonsingular alternating matrices over \mathbb{F}_q , which is given by

$$(3.4) \quad a_r = \begin{cases} 0, & \text{if } r \text{ is odd,} \\ q^{\frac{r}{2}(\frac{r}{2}-1)} \prod_{j=1}^{\frac{r}{2}} (q^{2j-1}), & \text{if } r \text{ is even,} \end{cases}$$

(cf. [8], Proposition 5.1).

Thus we see from (2.5), (3.3), and (3.4) that, for each r with $0 \leq r \leq n$,

$$(3.5) \quad \sum_{w \in P\sigma_r P} \psi(Trw) = \begin{cases} 0, & \text{if } r \text{ is odd,} \\ q^{\binom{n+1}{2}} q^{rn - \frac{1}{4}r^2} \begin{bmatrix} n \\ r \end{bmatrix}_q \\ \times \prod_{j=1}^{r/2} (q^{2j-1} - 1) & , \text{ if } r \text{ is even.} \\ \times K_{GL(n-r, q)}(\psi; 1) \end{cases}$$

For our purposes, we need two infinite families of exponential sums in (3.5) over $P(2n, q)\sigma_{n-1}P(2n, q) = DC^-(n, q)$ for $n = 1, 3, 5, \dots$ and over $P(2n, q)\sigma_{n-2}P(2n, q) = DC^+(n, q)$ for $n = 2, 4, 6, \dots$. So we state them separately as a theorem.

Theorem 3.2. *Let ψ be any nontrivial additive character of \mathbb{F}_q . Then, in the notations of (1) and (3), we have*

$$\sum_{w \in DC^-(n, q)} \psi(Trw) = A^-(n, q)K(\psi; 1), \text{ for } n = 1, 3, 5, \dots,$$

$$\begin{aligned} \sum_{w \in DC^+(n, q)} \psi(Trw) &= q^{-1}A^+(n, q)K_{GL(2, q)}(\psi; 1) \\ &= A^+(n, q)(K(\psi; 1)^2 + q^2 - q), \text{ for } n = 2, 4, 6, \dots \end{aligned}$$

(cf. (3.5), (3.2)).

Proposition 3.3 ([10]). *For $n = 2^s$ ($s \in \mathbb{Z}_{\geq 0}$), and ψ a nontrivial additive character of \mathbb{F}_q ,*

$$K(\psi; a^n) = K(\psi; a).$$

We need a result of Carlitz for the next corollary.

Theorem 3.4 ([2]). *For the canonical additive character λ of \mathbb{F}_q , and $a \in \mathbb{F}_q^*$,*

$$(3.6) \quad K_2(\lambda; a) = K(\lambda; a)^2 - q.$$

The next corollary follows from Theorem 4, Proposition 5, (3.6), and simple change of variables.

Corollary 3.5. *Let λ be the canonical additive character of \mathbb{F}_q , and let $a \in \mathbb{F}_q^*$. Then we have*

$$(3.7) \quad \sum_{w \in DC^-(n, q)} \lambda(aTrw) = A^-(n, q)K(\lambda; a),$$

for $n = 1, 3, 5, \dots$,

$$\begin{aligned}
(3.8) \quad \sum_{w \in DC^+(n,q)} \lambda(aTrw) &= A^+(n,q)(K(\lambda; a)^2 + q^2 - q) \\
&= A^+(n,q)(K_2(\lambda; a) + q^2), \\
&\text{for } n = 2, 4, 6, \dots
\end{aligned}$$

(cf. (1.1), (1.3)).

Proposition 3.6 ([10]). *Let λ be the canonical additive character of \mathbb{F}_q , $m \in \mathbb{Z}_{>0}$, $\beta \in \mathbb{F}_q$. Then*

$$\begin{aligned}
(3.9) \quad \sum_{a \in \mathbb{F}_q^*} \lambda(-a\beta)K_m(\lambda; a) \\
= \begin{cases} qK_{m-1}(\lambda; \beta^{-1}) + (-1)^{m+1}, & \text{if } \beta \neq 0, \\ (-1)^{m+1}, & \text{if } \beta = 0, \end{cases}
\end{aligned}$$

with the convention $K_0(\lambda; \beta^{-1}) = \lambda(\beta^{-1})$.

For any integer r with $0 \leq r \leq n$, and each $\beta \in \mathbb{F}_q$, we let

$$N_{P\sigma_r P}(\beta) = |\{w \in P\sigma_r P | Trw = \beta\}|.$$

Then it is easy to see that

$$(3.10) \quad qN_{P\sigma_r P}(\beta) = |P\sigma_r P| + \sum_{a \in \mathbb{F}_q^*} \lambda(-a\beta) \sum_{w \in P\sigma_r P} \lambda(aTrw).$$

Now, from (3.7)-(3.10), (2.7), and (2.8), we have the following result.

Proposition 3.7. (a) *For $n = 1, 3, 5, \dots$,*

$$\begin{aligned}
(3.11) \quad &N_{DC^-(n,q)}(\beta) \\
&= q^{-1}A^-(n,q)B^-(n,q) + q^{-1}A^-(n,q) \\
&\times \begin{cases} 1, & \beta = 0, \\ q+1, & tr(\beta^{-1}) = 0, \\ -q+1, & tr(\beta^{-1}) = 1. \end{cases}
\end{aligned}$$

(b) *For $n = 2, 4, 6, \dots$,*

$$\begin{aligned}
(3.12) \quad &N_{DC^+(n,q)}(\beta) \\
&= q^{-1}A^+(n,q)B^+(n,q) + q^{-1}A^+(n,q) \\
&\times \begin{cases} qK(\lambda; \beta^{-1}) - q^2 - 1, & \beta \neq 0, \\ q^3 - q^2 - 1, & \beta = 0. \end{cases}
\end{aligned}$$

(cf. (1.1)- (1.4)).

Corollary 3.8. (a) For all odd $n \geq 3$ and all q , $N_{DC^-(n,q)}(\beta) > 0$, for all β ; for $n = 1$ and all q ,

$$(3.13) \quad N_{DC^-(1,q)}(\beta) = \begin{cases} q, & \beta = 0, \\ 2q, & \text{tr}(\beta^{-1}) = 0, \\ 0, & \text{tr}(\beta^{-1}) = 1. \end{cases}$$

(b) For all even $n \geq 4$ and all q , or $n = 2$ and all $q \geq 4$, $N_{DC^+(n,q)}(\beta) > 0$, for all β ; for $n = 2$ and $q = 2$,

$$N_{DC^+(2,2)}(\beta) = \begin{cases} 0, & \beta = 1, \\ 48 = |P(4, 2)|, & \beta = 0. \end{cases}$$

Proof. (a) $n = 1$ case follows directly from (3.11). Let $n \geq 3$ be odd. Then, from (3.11), we see that, for any β ,

$$\begin{aligned} & N_{DC^-(n,q)}(\beta) \\ & \geq q^{\frac{1}{2}(3n^2-n-2)}(q^n - 1) \prod_{j=2}^n (q^j - 1) - q^{\frac{5}{4}(n^2-1)} \\ & \quad \times \prod_{j=1}^{(n+1)/2} (q^{2j-1} - 1) \\ & > q^{\frac{1}{2}(3n^2-n-2)}(q^n - 1) \prod_{j=2}^n (q^j - 1) - q^{\frac{5}{4}(n^2-1)} \prod_{j=1}^{(n+1)/2} q^{2j-1} \\ & = q^{\frac{1}{2}(3n^2-n-2)} \left\{ (q^n - 1) \left(\prod_{j=2}^n (q^j - 1) - 1 \right) - 1 \right\} > 0. \end{aligned}$$

(b) Let $n = 2$. Let $\beta \neq 0$. Then, from (3.12), we have

$$(3.14) \quad N_{DC^+(2,q)}(\beta) = q^4 \{ q^2 - 2q - 1 + K(\lambda; \beta^{-1}) \},$$

where $q^2 - 2q - 1 + K(\lambda; \beta^{-1}) \geq q^2 - 2q - 1 - 2\sqrt{q} > 0$, for $q \geq 4$, by invoking the Weil bound in (3.1). Also, observe from (3.14) that $N_{DC^+(2,2)}(1) = 0$.

On the other hand, if $\beta = 0$, then, from (3.12), we get

$$N_{DC^+(2,q)}(0) = q^4(2q^2 - 2q - 1) > 0, \text{ for all } q \geq 2.$$

In addition, we note that $N_{DC^+(2,2)}(0) = 48$.

Assume now that $n \geq 4$. If $\beta = 0$, then, from (3.12), we see that $N_{DC^+(n,q)}(0) > 0$, for all q . Let $\beta \neq 0$. Then, again by invoking the

Weil bound,

$$\begin{aligned} N_{DC^+(n,q)}(\beta) &\geq q^{-1}A^+(n,q) \\ &\times \{(q^n - 1)(q^{n-1} - 1)q^{\frac{n^2}{4}-n+1} \\ &\times \prod_{j=1}^{(n-2)/2} (q^{2j} - 1) - (q^2 + 2q^{\frac{3}{2}} + 1)\}. \end{aligned}$$

Clearly, $\prod_{j=1}^{(n-2)/2} (q^{2j} - 1) > 1$. So we only need to show, for all $q \geq 2$,

$$f(q) = (q^n - 1)(q^{n-1} - 1)q^{\frac{n^2}{4}-n+1} - (q^2 + 2q^{\frac{3}{2}} + 1) > 0.$$

But, as $n \geq 4$, $f(q) \geq q(q^4 - 1)(q^3 - 1) - (q^2 + 2q^{\frac{3}{2}} + 1) > 0$, for all $q \geq 2$. \square

4. Construction of codes

Let

$$(4.1) \quad \begin{aligned} N^-(n,q) &= |DC^-(n,q)| = A^-(n,q)B^-(n,q), \\ &\text{for } n = 1, 3, 5, \dots, \end{aligned}$$

$$(4.2) \quad \begin{aligned} N^+(n,q) &= |DC^+(n,q)| = A^+(n,q)B^+(n,q), \\ &\text{for } n = 2, 4, 6, \dots, \end{aligned}$$

(cf. (2.7), (2.8), (1.1)-(1.4)).

Here we will construct two infinite families of binary linear codes $C(DC^-(n,q))$ of length $N^-(n,q)$ for all positive odd integers n and all q , and $C(DC^+(n,q))$ of length $N^+(n,q)$ for all positive even integers n and all q , respectively associated with the double cosets $DC^-(n,q)$ and $DC^+(n,q)$.

Let $g_1, g_2, \dots, g_{N^-(n,q)}$ and $g_1, g_2, \dots, g_{N^+(n,q)}$ be respectively fixed orderings of the elements in $DC^-(n,q)$ ($n = 1, 3, 5, \dots$) and $DC^+(n,q)$ ($n = 2, 4, 6, \dots$), by abuse of notations. Then we put

$$\begin{aligned} v^-(n,q) &= (Trg_1, Trg_2, \dots, Trg_{N^-(n,q)}) \in \mathbb{F}_q^{N^-(n,q)}, \\ &\text{for } n = 1, 3, 5, \dots, \end{aligned}$$

$$\begin{aligned} v^+(n,q) &= (Trg_1, Trg_2, \dots, Trg_{N^+(n,q)}) \in \mathbb{F}_q^{N^+(n,q)}, \\ &\text{for } n = 2, 4, 6, \dots \end{aligned}$$

Now, the binary codes $C(DC^-(n,q))$ and $C(DC^+(n,q))$ are defined as:

$$(4.3) \quad \begin{aligned} C(DC^-(n,q)) &= \{u \in \mathbb{F}_2^{N^-(n,q)} \mid u \cdot v^-(n,q) = 0\}, \\ &\text{for } n = 1, 3, 5, \dots, \end{aligned}$$

$$(4.4) \quad \begin{aligned} C(DC^+(n,q)) &= \{u \in \mathbb{F}_2^{N^+(n,q)} \mid u \cdot v^+(n,q) = 0\}, \\ &\text{for } n = 2, 4, 6, \dots, \end{aligned}$$

where the dot denotes the usual inner product in $\mathbb{F}_q^{N^-(n,q)}$ and $\mathbb{F}_q^{N^+(n,q)}$, respectively.

The following Delsarte's theorem is well-known.

Theorem 4.1 ([16]). *Let B be a linear code over \mathbb{F}_q . Then*

$$(B|_{\mathbb{F}_2})^\perp = \text{tr}(B^\perp).$$

In view of this theorem, the duals $C(DC^-(n, q))^\perp$ and $C(DC^+(n, q))^\perp$ of the respective codes $C(DC^-(n, q))$ and $C(DC^+(n, q))$ are given by

$$(4.5) \quad \begin{aligned} & C(DC^-(n, q))^\perp \\ &= \{c^-(a) = c^-(a; n, q) \\ & \quad = (\text{tr}(a\text{Tr}g_1), \dots, \text{tr}(a\text{Tr}g_{N^-(n,q)})) | a \in \mathbb{F}_q\} \\ & (n = 1, 3, 5, \dots), \end{aligned}$$

$$(4.6) \quad \begin{aligned} & C(DC^+(n, q))^\perp \\ &= \{c^+(a) = c^+(a; n, q) \\ & \quad = (\text{tr}(a\text{Tr}g_1), \dots, \text{tr}(a\text{Tr}g_{N^+(n,q)})) | a \in \mathbb{F}_q\} \\ & (n = 2, 4, 6, \dots). \end{aligned}$$

Let \mathbb{F}_2^+ , \mathbb{F}_q^+ denote the additive groups of the fields $\mathbb{F}_2, \mathbb{F}_q$, respectively. Then we have the following exact sequence of groups:

$$0 \rightarrow \mathbb{F}_2^+ \rightarrow \mathbb{F}_q^+ \rightarrow \Theta(\mathbb{F}_q) \rightarrow 0,$$

where the first map is the inclusion and the second one is the Artin-Schreier operator in characteristic two given by $x \mapsto \Theta(x) = x^2 + x$. So

$$(4.7) \quad \Theta(\mathbb{F}_q) = \{\alpha^2 + \alpha \mid \alpha \in \mathbb{F}_q\}, \text{ and } [\mathbb{F}_q^+ : \Theta(\mathbb{F}_q)] = 2.$$

Theorem 4.2 ([10]). *Let λ be the canonical additive character of \mathbb{F}_q , and let $\beta \in \mathbb{F}_q^*$. Then*

$$(4.8) \quad (a) \quad \sum_{\alpha \in \mathbb{F}_q - \{0,1\}} \lambda\left(\frac{\beta}{\alpha^2 + \alpha}\right) = K(\lambda; \beta) - 1,$$

$$(b) \quad \sum_{\alpha \in \mathbb{F}_q} \lambda\left(\frac{\beta}{\alpha^2 + \alpha + b}\right) = -K(\lambda; \beta) - 1,$$

if $x^2 + x + b$ ($b \in \mathbb{F}_q$) is irreducible over \mathbb{F}_q , or equivalently if $b \in \mathbb{F}_q \setminus \Theta(\mathbb{F}_q)$ (cf. (4.7)).

Theorem 4.3. (a) *The map $\mathbb{F}_q \rightarrow C(DC^-(n, q))^\perp$ ($a \mapsto c^-(a)$) is an \mathbb{F}_2 -linear isomorphism for $n \geq 3$ odd and all q , or $n = 1$ and $q \geq 8$.*

(b) The map $\mathbb{F}_q \rightarrow C(DC^+(n, q))^\perp (a \mapsto c^+(a))$ is an \mathbb{F}_2 -linear isomorphism for $n \geq 4$ even and all q , or $n = 2$ and $q \geq 4$.

Proof. (a) The map is clearly \mathbb{F}_2 -linear and surjective. Let a be in the kernel of map. Then $tr(aTrg) = 0$, for all $g \in DC^-(n, q)$. If $n \geq 3$ is odd, then, by Corollary 10 (a), $Tr : DC^-(n, q) \rightarrow \mathbb{F}_q$ is surjective and hence $tr(a\alpha) = 0$, for all $\alpha \in \mathbb{F}_q$. This implies that $a = 0$, since otherwise $tr : \mathbb{F}_q \rightarrow \mathbb{F}_2$ would be the zero map. Now, assume that $n = 1$ and $q \geq 8$. Then, by (31), $tr(a\beta) = 0$, for all $\beta \in \mathbb{F}_q^*$, with $tr(\beta^{-1}) = 0$. Hilbert's theorem 90 says that $tr(\gamma) = 0 \Leftrightarrow \gamma = \alpha^2 + \alpha$, for some $\alpha \in \mathbb{F}_q$. This implies that $\sum_{\alpha \in \mathbb{F}_q - \{0,1\}} \lambda(\frac{a}{\alpha^2 + \alpha}) = q - 2$. If $a \neq 0$, then, using (40) and the Weil bound (19), we would have

$$q - 2 = \sum_{\alpha \in \mathbb{F}_q - \{0,1\}} \lambda(\frac{a}{\alpha^2 + \alpha}) = K(\lambda; a) - 1 \leq 2\sqrt{q} - 1.$$

But this is impossible, since $x > 2\sqrt{x} + 1$, for $x \geq 8$.

(b) This can be proved in exactly the same manner as in the $n \geq 3$ odd case of (a) (cf. Corollary 10 (b)). □

Remark: One can show that the kernel of the map $\mathbb{F}_q \rightarrow C(DC^-(1, q))^\perp (a \mapsto c^-(a))$, for $q = 2, 4$ and of the map $\mathbb{F}_q \rightarrow C(DC^+(2, 2))^\perp (a \mapsto c^+(a))$ are all equal to \mathbb{F}_2 .

5. Recursive formulas for power moments of Kloosterman sums

Here we will be able to find, via Pless power moment identity, infinite families of recursive formulas generating power moments of Kloosterman and 2-dimensional Kloosterman sums over all \mathbb{F}_q (with three exceptions) in terms of the frequencies of weights in $C(DC^-(n, q))$ and $C(DC^+(n, q))$, respectively.

Theorem 5.1. (Pless power moment identity, [16]): Let B be an q -ary $[n, k]$ code, and let B_i (resp. B_i^\perp) denote the number of codewords of weight i in B (resp. in B^\perp). Then, for $h = 0, 1, 2, \dots$,

$$(5.1) \quad \sum_{j=0}^n j^h B_j = \sum_{j=0}^{\min\{n,h\}} (-1)^j B_j^\perp \sum_{t=j}^h t! S(h, t) q^{k-t} (q-1)^{t-j} \binom{n-j}{n-t},$$

where $S(h, t)$ is the Stirling number of the second kind defined in (1.7).

Lemma 5.2. Let

$$c^-(a) = (tr(aTrg_1), \dots, tr(aTrg_{N^-(n,q)})) \in C(DC^-(n, q))^\perp$$

$(n = 1, 3, 5, \dots)$, and let

$$c^+(a) = (tr(aTrg_1), \dots, tr(aTrg_{N^+(n,q)})) \in C(DC^+(n, q))^\perp$$

($n = 2, 4, 6, \dots$), for $a \in \mathbb{F}_q^*$. Then the Hamming weights $w(c^-(a))$ and $w(c^+(a))$ are expressed as follows:

$$(5.2) \quad (a) w(c^-(a)) = \frac{1}{2} A^-(n, q)(B^-(n, q) - K(\lambda; a)),$$

$$(5.3) \quad (b) w(c^+(a)) = \frac{1}{2} A^+(n, q)(B^+(n, q) - q^2 + q - K(\lambda; a)^2)$$

$$(5.4) \quad = \frac{1}{2} A^+(n, q)(B^+(n, q) - q^2 - K_2(\lambda; a)),$$

(cf. (1)–(4)).

Proof. $w(c^\mp(a)) = \frac{1}{2} \sum_{j=1}^{N^\mp(n, q)} (1 - (-1)^{\text{tr}(a \text{Tr} g_j)}) = \frac{1}{2} (N^\mp(n, q) - \sum_{w \in DC^\mp(n, q)} \lambda(a \text{Tr} w))$. Our results now follow from (4.1), (4.2), (3.7), and (3.8). \square

Let $u = (u_1, \dots, u_{N_{N^\mp(n, q)}}) \in \mathbb{F}_2^{N^\mp(n, q)}$, with ν_β 1's in the coordinate places where $\text{Tr}(g_j) = \beta$, for each $\beta \in \mathbb{F}_q$. Then from the definition of the codes $C(DC^\mp(n, q))$ (cf. (35), (36)) that u is a codeword with weight j if and only if $\sum_{\beta \in \mathbb{F}_q} \nu_\beta = j$ and $\sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0$ (an identity in \mathbb{F}_q). As there are $\prod_{\beta \in \mathbb{F}_q} \binom{N_{DC^\mp(n, q)}(\beta)}{\nu_\beta}$ many such codewords with weight j , we obtain the following result.

Proposition 5.3. Let $\{C_j^-(n, q)\}_{j=0}^{N^-(n, q)}$ be the weight distribution of $C(DC^-(n, q))$ ($n = 1, 3, 5, \dots$), and let $\{C_j^+(n, q)\}_{j=0}^{N^+(n, q)}$ be that of $C(DC^+(n, q))$ ($n = 2, 4, 6, \dots$). Then

$$(5.5) \quad C_j^\mp(n, q) = \sum \prod_{\beta \in \mathbb{F}_q} \binom{N_{DC^\mp(n, q)}(\beta)}{\nu_\beta},$$

where the sum is over all the sets of integers $\{\nu_\beta\}_{\beta \in \mathbb{F}_q}$ ($0 \leq \nu_\beta \leq N_{DC^\mp(n, q)}(\beta)$), satisfying

$$(5.6) \quad \sum_{\beta \in \mathbb{F}_q} \nu_\beta = j, \quad \text{and} \quad \sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0.$$

Corollary 5.4. Let $\{C_j^-(n, q)\}_{j=0}^{N^-(n, q)}$ ($n = 1, 3, 5, \dots$), $\{C_j^+(n, q)\}_{j=0}^{N^+(n, q)}$ ($n = 2, 4, 6, \dots$) be as above. Then we have

$$C_j^\mp(n, q) = C_{N^\mp(n, q) - j}^\mp(n, q),$$

for all j , with $0 \leq j \leq N^\mp(n, q)$.

Proof. Under the replacements $\nu_\beta \rightarrow N_{DC^\mp(n,q)}(\beta) - \nu_\beta$, for each $\beta \in \mathbb{F}_q$, the first equation in (5.6) is changed to $N^\mp(n, q) - j$, while the second one in there and the summands in (5.5) are left unchanged. Here the second sum in (5.6) is left unchanged, since $\sum_{\beta \in \mathbb{F}_q} N_{DC^\mp(n,q)}(\beta)\beta = 0$, as one can see by using the explicit expressions of $N_{DC^\mp(n,q)}(\beta)$ in (3.11) and (3.12). \square

Theorem 5.5 ([13]). *Let $q = 2^r$, with $r \geq 2$. Then the range R of $K(\lambda; a)$, as a varies over \mathbb{F}_q^* , is given by:*

$$R = \{\tau \in \mathbb{Z} \mid |\tau| < 2\sqrt{q}, \tau \equiv -1 \pmod{4}\}.$$

In addition, each value $\tau \in R$ is attained exactly $H(\tau^2 - q)$ times, where $H(d)$ is the Kronecker class number of d .

The formulas appearing in the next theorem and stated in (1.6) and (1.10) follow by applying the formula in (5.5) to each $C(DC^\mp(n, q))$, using the explicit values of $N_{DC^\mp(n,q)}(\beta)$ in (3.11) and (3.12), and taking Theorem 5.5 into consideration.

Theorem 5.6. *Let $\{C_j^-(n, q)\}_{j=0}^{N^-(n,q)}$ be the weight distribution of $C(DC^-(n, q))$ ($n = 1, 3, 5, \dots$), and let $\{C_j^+(n, q)\}_{j=0}^{N^+(n,q)}$ be that of $C(DC^+(n, q))$ ($n = 2, 4, 6, \dots$). Then (a) For $j = 0, \dots, N^-(n, q)$,*

$$\begin{aligned} C_j^-(n, q) &= \sum \binom{q^{-1}A^-(n, q)(B^-(n, q) + 1)}{\nu_0} \\ &\times \prod_{\text{tr}(\beta^{-1})=0} \binom{q^{-1}A^-(n, q)(B^-(n, q) + q + 1)}{\nu_\beta} \\ &\times \prod_{\text{tr}(\beta^{-1})=1} \binom{q^{-1}A^-(n, q)(B^-(n, q) - q + 1)}{\nu_\beta}, \end{aligned}$$

where the sum is over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta \in \mathbb{F}_q}$ satisfying $\sum_{\beta \in \mathbb{F}_q} \nu_\beta = j$ and $\sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0$.

(b) For $j = 0, \dots, N^+(n, q)$,

$$\begin{aligned} C_j^+(n, q) &= \sum \binom{q^{-1}A^+(n, q)(B^+(n, q) + q^3 - q^2 - 1)}{\nu_0} \prod_{\substack{|\tau| < 2\sqrt{q} \\ \tau \equiv -1 \pmod{4}}} \\ &\times \prod_{K(\lambda; \beta^{-1})=\tau} \binom{q^{-1}A^+(A, q)(B^+(n, q) + q\tau - q^2 - 1)}{\nu_\beta}, \end{aligned}$$

where the sum is over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta \in \mathbb{F}_q}$ satisfying $\sum_{\beta \in \mathbb{F}_q} \nu_\beta = j$, and $\sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0$, and we assume $q \geq 4$.

From now on, we will assume that, for $C(DC^-(n, q))^\perp$, either $n \geq 3$ odd and all q or $n = 1$ and $q \geq 8$, and that, for $C(DC^+(n, q))^\perp$, $n \geq 2$ even and $q \geq 4$. Under these assumptions, each codeword in $C(DC^\mp(n, q))^\perp$ can be written as $c^\mp(a)$, for a unique $a \in \mathbb{F}_q$ (cf. Theorem 4.3, (4.5), (4.6)).

Now, we apply the Pless power moment identity in (5.1) to $C(DC^\mp(n, q))^\perp$ for those values of n and q , in order to get the results in Theorem 1.1 (cf. (1.5), (1.8), (1.9)) about recursive formulas.

The left hand side of that identity in (5.1) is equal to

$$\sum_{a \in \mathbb{F}_q^*} w(c^\mp(a))^h,$$

with $w(c^\mp(a))$ given by (5.2)-(5.4). We have

$$\begin{aligned} \sum_{a \in \mathbb{F}_q^*} w(c^-(a))^h &= \frac{1}{2^h} A^-(n, q)^h \sum_{a \in \mathbb{F}_q^*} (B^-(n, q) - K(\lambda; a))^h \\ (5.7) \quad &= \frac{1}{2^h} A^-(n, q)^h \sum_{l=0}^h (-1)^l \binom{h}{l} B^-(n, q)^{h-l} M K^l. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \sum_{a \in \mathbb{F}_q^*} w(c^+(a))^h &= \frac{1}{2^h} A^+(n, q)^h \\ (5.8) \quad &\times \sum_{l=0}^h (-1)^l \binom{h}{l} (B^+(n, q) - q^2 + q)^{h-l} M K^{2l} \\ (5.9) \quad &= \frac{1}{2^h} A^+(n, q)^h \sum_{l=0}^h (-1)^l \binom{h}{l} (B^+(n, q) - q^2)^{h-l} M K^l. \end{aligned}$$

Note here that, in view of (3.6), obtaining power moments of 2-dimensional Kloosterman sums is equivalent to getting even power moments of Kloosterman sums. Also, one has to separate the term corresponding to $l = h$ in (5.7)-(5.9), and notes $\dim_{\mathbb{F}_2} C(DC^\mp(n, q))^\perp = r$.

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