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SOME ALGEBRAIC PROPERTIES OVER THE GENERALIZED GENERAL PRODUCT OBTAINED BY MONOIDS AND GROUPS

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Abstract. Suppose A and B be arbitrary monoids. In [2], it has been recently defined a new consequence of the general product $A^{\oplus B}$ ${}_{\delta}\bowtie_{\psi}$ $B^{\oplus A}$ under the name of the generalized general product, and then has been given a presentation for it. In this paper we give some algebraic properties of the generalized general product obtained by some certain monoids and groups.
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1. Introduction and Preliminaries

By defining an extension over any algebraic structure (such as groups, monoids and semigroups etc.) and then investigate some known algebraic properties over this extension have been recently taking so much interest (see, for instance, [15, 16]). Specially in [16], the authors defined a new semigroup extension and then investigated some fundamental algebraic properties (such as Green's relations, inverse property) and finiteness conditions for this new semigroup structure.

In [2], by bringing together the definitions of the general product (cf. [7, 17, 18]) and the (restricted) wreath product (cf. [1, 13]) of monoids $A^{\oplus B}$ and $B^{\oplus A}$, it has been recently defined a new product with its presentation and also in the same reference it has been investigated some other results of the main theories in terms of finite and infinite cases for this generalization. This new product named as generalized general products which can be thought as a generalization of the results in [6, 8]. This generalization denoted by $A^{\oplus B}$ ${}_{\delta}\bowtie_{\psi}$ $B^{\oplus A}$. With a similar manner as in [15, 16], the main target in this paper is to study some algebraic properties (left cancellative, equdivisible, unit, length function, length-preserving, regular semigroup) of this generalization $A^{\oplus B}$ ${}_{\delta}\bowtie_{\psi}$ $B^{\oplus A}$.

For convenience of the reader, we begin by recalling the definition of this new generalization. Let A and B be monoids. The generalized general product of the monoid $A^{\oplus B}$ by the monoid $B^{\oplus A}$ denoted by $A^{\oplus B}$ ${}_{\delta}\bowtie_{\psi}$

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$B^{\oplus A}$ is defined on the set $A^{\oplus B} \times B^{\oplus A}$ with a multiplication $(f,h)(f',h') = \left(f \ {}^{h}f', h^{f'}\ h'\right)$, where $f, f' \in A^{\oplus B}$ and $h, h' \in B^{\oplus A}$ as well as $\delta : B^{\oplus A} \to \tau\left(A^{\oplus B}\right)$, $(f')\,\delta_h =^h f'$ and $\psi : A^{\oplus B} \to \tau\left(B^{\oplus A}\right)$, $(h)\,\psi_{f'} = h^{f'}$ are defined by, for $a \in A$ and $b \in B$, ${}^{h}f' =^{(h^a)} f'$ and $h^{f'} = h^{\left({}^{b}f'\right)}$. Also, for $x \in A$ and $y \in B$, one can define $(x)\,h^a = (ax)\,h$ and $(y)^b\,f' = (yb)\,f'$ such that, for all $c \in A, d \in B$, $(d)^{(h^a)}\,f' = (dh^a)\,f'$ and $(c)\,h^{\left({}^{b}f'\right)} = \left({}^{b}f'c\right)h$, and further the following general axioms are satisfied for all $f, f' \in A^{\oplus B}$ and $h, h' \in B^{\oplus A}$:

$(GP1)$ ${}^{(hh')}f =^h ({}^{h'}f)$; $(GP2)$ ${}^{h}(ff') = \left({}^{h}f\right)\left({}^{h^f}f'\right)$;

$(GP3)$ $\left(h^f\right)^{f'} = h^{(ff')}$; $(GP4)$ $(hh')^f = h^{\left({}^{h'}f\right)}(h')^f$;

$(GP5)$ ${}^{h}\overline{1} = \overline{1}$; $(GP6)$ $h^{\overline{1}} = h$;

$(GP7)$ ${}^{\widetilde{1}}f = f$; $(GP8)$ $\widetilde{1}^f = \widetilde{1}$.

It is easy to show that the generalized general product $A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi} B^{\oplus A}$ is a monoid with the identity $\left(\overline{1}, \widetilde{1}\right)$, where $\overline{1} : B \to A, (b)\,\overline{1} = 1_A$ and $\widetilde{1} : A \to B, (a)\,\widetilde{1} = 1_B$, for all $a \in A$ and $b \in B$.

2. **Some properties over this new product**

In this section we study the structure of the generalized general products obtained by certain monoids and groups. In [10, 11], Lawson determined the structure of general product of free monoids and groups. We will determine the general structure of the generalized general product $A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi} B^{\oplus A}$ of left cancellative monoid $A^{\oplus B}$ by the group $B^{\oplus A}$ and for $A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi} B^{\oplus A}$ such that $A^{\oplus B}$ is a band and $B^{\oplus A}$ is a group. Recall from [5] that an arbitrary monoid M is said to be *left cancellative* if the equation $ab = ac$ implies that $b = c$ for all $a, b, c \in M$. An arbitrary monoid M is said to be *equidivisible* if $ab = cd$ implies that either $a = cu, ub = d$ for some $u \in M$ or $c = av, b = vd$ for some $v \in M$ for all $a, b, c, d \in M$.

Proposition 2.1. *Let $P = A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi} B^{\oplus A}$ be the generalized general product of a monoid $A^{\oplus B}$ and a group $B^{\oplus A}$. Then P is a left cancellative if and only if $A^{\oplus B}$ is a left cancellative monoid.*

Proof. Assume that $ab = ac$ holds for $a, b, c \in P$ such that $a = (f_1, g_1)$, $b = (f_2, g_2)$, $c = (f_3, g_3)$, where $f_1, f_2, f_3 \in A^{\oplus B}$ and $g_1, g_2, g_3 \in B^{\oplus A}$. Then

$$(f_1, g_1)(f_2, g_2) = (f_1, g_1)(f_3, g_3)$$

which implies

$$\left(f_1\ {}^{g_1}f_2, f_2^{g_1} g_2\right) = \left(f_1\ {}^{g_1}f_3, f_3^{g_1} g_3\right).$$

Therefore

$$f_1\ {}^{g_1}f_2 = f_1\ {}^{g_1}f_3 \tag{2.1}$$

$$f_2^{g_1} g_2 = f_3^{g_1} g_3 \tag{2.2}$$

Since $A^{\oplus B}$ is a left cancellative monoid, from (2.1) we have ${}^{g_1}f_2 = {}^{g_1}f_3$, and since $B^{\oplus A}$ is a group and $\widetilde{1} \in B^{\oplus A}$ act trivially, it follows that $f_2 = f_3$. From (2.2), we have $f_2^{g_1}g_2 = f_2^{g_1}g_3$, since $B^{\oplus A}$ is a left cancellative. We now deduce that $g_2 = g_3$. Therefore $b = c$ and hence P is a left cancellative.

Conversely, clearly if P is a left cancellative monoid, then $A^{\oplus B} \subseteq P$ is a left cancellative. □

The same proof can be applied to the following more general result.

Proposition 2.2. *If $A^{\oplus B}$ and $B^{\oplus A}$ are left cancellative monoids such that $B^{\oplus A}$ acts on $A^{\oplus B}$ injectively and $\widetilde{1} \in B^{\oplus A}$ acts trivially, then $P = A^{\oplus B}\ {}_{\delta}\bowtie_{\psi} B^{\oplus A}$ is a left cancellative monoid.*

The original version of the following result presented in [11]. In fact we generalized this version as follows:

Proposition 2.3. *If $P = A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ is a left cancellative monoid, then the posets of principal right ideals of P and $A^{\oplus B}$ are isomorphic.*

Proof. Let $p \in P$. Then $p = fg$ uniquely written in which $f \in A^{\oplus B}$ and $g \in B^{\oplus A}$, since $B^{\oplus A}$ is a group then g is a unit. This tells us that p and f are associates each other. So $pP = fP$ in a left cancellative monoid. Hence we can assume that the generator of a principal right ideal is in $A^{\oplus B}$. Now we need to prove that $f'P \subseteq f''P$ if and only if $f'A^{\oplus B} \subseteq f''A^{\oplus B}$ with $f', f'' \in A^{\oplus B}$. Suppose $f'P \subseteq f''P$. Then $f' = f''b$ for some $b \in P$. But $b = zu$, where $z \in A^{\oplus B}$ and $u \in B^{\oplus A}$. Thus $f' = f''(zu) = (f''z)u$, since $f' = f'1$, where $f' \in A^{\oplus B}$ and $1 \in B^{\oplus A}$. Then $f' = f''z$ and $u = 1$. Therefore $f' \in f''M$. Hence $f'M \subseteq f''M$.

Conversely, let $f'M \subseteq f''M$. Since $A^{\oplus B} \cong \left\{\left(f, \widetilde{1}\right) : f \in A^{\oplus B}\right\} \subseteq P$, we have $f'P \subseteq f''P$. Then the principal right ideals posets of P and $A^{\oplus B}$ are isomorphic. □

Proposition 2.4. *Let $P = A^{\oplus B}\ {}_{\delta}\bowtie_{\psi} B^{\oplus A}$ be the generalized general product of a monoid $A^{\oplus B}$ and a group $B^{\oplus A}$. Then P is equdivisible if and only if $A^{\oplus B}$ is equidivisible.*

Proof. Assume that $A^{\oplus B}$ is an equidivisible monoid. We want to prove $P = A^{\oplus B}\ {}_{\delta}\bowtie_{\psi} B^{\oplus A}$ is equidivisible as well. Let $a, b, c, d \in P$ and $ab = cd$ such that

$$a = (f_1, g_1)\,,\ b = (f_2, g_2)\,,\ c = (f_3, g_3)\,,\ d = (f_4, g_4)\,.$$

We have to prove that either $\lambda = (f, g) \in P$ such that $a = c\lambda, d = \lambda b$ or $\mu = (f', g') \in P$ such that $c = a\mu, b = \mu d$. Now we have the equality

$$(f_1, g_1)(f_2, g_2) = (f_3, g_3)(f_4, g_4)$$

which implies that

$$\left(f_1\left({}^{g_1}f_2\right), g_1^{f_2}g_2\right) = \left(f_3\left({}^{g_3}f_4\right), g_3^{f_4}g_4\right).$$

Therefore

$$f_1\left({}^{g_1}f_2\right) = f_3\left({}^{g_3}f_4\right) \tag{2.3}$$
$$g_1^{f_2}g_2 = g_3^{f_4}g_4. \tag{2.4}$$

From (2.3), since $A^{\oplus B}$ is equidivisible monoid, for $\gamma, \delta \in A^{\oplus B}$,

$$\text{either } f_1 = f_3\gamma,\ {}^{g_3}f_4 = \gamma^{g_1}f_2 \text{ or } f_3 = f_1\delta,\ {}^{g_1}f_2 = \delta^{g_3}f_4.$$

If $f_1 = f_3\gamma$, ${}^{g_3}f_4 = \gamma\left({}^{g_1}f_2\right)$ in $A^{\oplus B}$, we set $f = {}^{g_3^{-1}}\gamma$, so that ${}^{g_3}f = \gamma$ and $g = \left(g_3^f\right)^{-1} g_1 = \left(g_3^{\left({}^{g_3^{-1}}\gamma\right)}\right)^{-1} g_1 = \left(g_3^{-1}\right)^{\gamma} g_1$ since

$$1 = 1^{\gamma} = \left(g_3g_3^{-1}\right)^{\gamma} = g_3^{{}^{g_3^{-1}}\gamma}\left(g_3^{-1}\right)^{\gamma}.$$

Thus $\left(g_3^{-1}\right)^{\gamma} = \left(g_3^{\left(g_3^{-1}\right)_{\gamma}}\right)^{-1}$ which gives $g_1 = g_3^{f}g$. If $\lambda = (f,g) = \left({}^{g_3^{-1}}\gamma, \left(g_3^{-1}\right)^{\gamma} g_1\right) \in P$, then

$$(f_3, g_3)(f,g) = \left(f_3\left({}^{g_3}f\right), g_3^f g\right) = (f_3\gamma, g_1) = (f_1, g_1).$$

Thus $a = c\lambda$, and from (2.4) we have

$$\begin{aligned}
(f,g)(f_2,g_2) &= \left({}^{g_3^{-1}}\gamma, \left(g_3^{-1}\right)^{\gamma} g_1\right)(f_2,g_2)\\
&= \left(\left({}^{g_3^{-1}}\gamma\right)\left({}^{\left[\left(g_3^{-1}\right)^{\gamma}g_1\right]}f_2\right), \left[\left(g_3^{-1}\right)^{\gamma} g_1\right]^{f_2} g_2\right)\\
&= \left({}^{g_3^{-1}}\left[\gamma\left({}^{g_1}f_2\right)\right], \left[\left(g_3^{-1}\right)^{\gamma}\right]^{{}^{g_1}f_2}\left(g_1^{f_2}g_2\right)\right)\\
&= \left({}^{g_3^{-1}}\left({}^{g_3}f_4\right), \left(\left(g_3^{-1}\right)^{{}^{g_3}f_4}\right) g_3^{f_4}g_4\right)\\
&= \left(\left({}^{g_3^{-1}g_3}\right)f_4, \left(g_3^{-1}g_3\right)^{f_4} g_4\right)\\
&= (f_4, g_4).
\end{aligned}$$

Thus $d = \lambda b$. As a result of this $P = A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ is equidivisible.

Conversely, suppose that $P = A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ is equidivisible. To prove that $A^{\oplus B}$ is also equidivisible, it is suffices to show that $A^{\oplus B} \subseteq P$ is a unitary submonoid of P since if we take an equation $ab = cd$ in $A^{\oplus B}$, so equidivisiblity on P implies that either $a = c\gamma$, $d = \gamma b$ or $c = a\lambda$, $b = \lambda d$, where $\gamma, \lambda \in P$. Hence, if $A^{\oplus B}$ is unitary then it is equidivisible. Now

$A^{\oplus B} \subseteq A^{\oplus B}\ {}_{\delta}\bowtie_{\psi} B^{\oplus A}$ and

$$\left(f,\widetilde{1}\right)(k,g) = \left(f\left({}^{\widetilde{1}}k\right), \widetilde{1}^{k} g\right) = (fk,g).$$

Therefore, if $\left(f,\widetilde{1}\right)(k,g) \in A^{\oplus B}$, then $(k,g) = \left(k,\widetilde{1}\right) \in A^{\oplus B}$. So $A^{\oplus B}$ is equidivisible. □

Lemma 2.5. *Let $P = A^{\oplus B}\ {}_{\delta}\bowtie_{\psi} B^{\oplus A}$ be the generalized general product of a monoid $A^{\oplus B}$ and a group $B^{\oplus A}$. Then $(f,g) \in P$ is a unit if and only if f is a unit in $A^{\oplus B}$.*

Proof. Suppose that $f \in A^{\oplus B}$ is a unit. Then there exists $f^{-1} \in A^{\oplus B}$ such that $ff^{-1} = f^{-1}f = \overline{1}$. Define $(f,g)^{-1} = \left({}^{g^{-1}}f^{-1}, (g^{-1})^{f^{-1}}\right) \in P$. Then, by $(GP1)$ and $(GP4)$,

$$\begin{aligned}(f,g)\left({}^{g^{-1}}f^{-1}, (g^{-1})^{f^{-1}}\right) &= \left(f({}^{g}\left({}^{g^{-1}}f^{-1}\right)), g^{{}^{g^{-1}}f^{-1}}(g^{-1})^{f^{-1}}\right) \\ &= \left(ff^{-1}, (gg^{-1})^{f^{-1}}\right) \\ &= \left(\overline{1},\widetilde{1}\right)\end{aligned}$$

and, by $(GP2)$ and $(GP3)$, we have

$$\begin{aligned}\left({}^{g^{-1}}f^{-1}, (g^{-1})^{f^{-1}}\right)(f,g) &= \left(({}^{g^{-1}}f^{-1})\left({}^{(g^{-1})^{f^{-1}}}f\right), \left((g^{-1})^{f^{-1}}\right)^{f} g\right) \\ &= \left({}^{g^{-1}}\left(f^{-1}f\right), (g^{-1})^{f^{-1}f} g\right) \\ &= \left(\overline{1},\widetilde{1}\right).\end{aligned}$$

Thus (f,g) is a unit in P.

Conversely, suppose (f,g) is a unit in P. Then there exists $(k,h) \in P$ such that $(f,g)(k,h) = \left(\overline{1},\widetilde{1}\right)$. Thus, we have

$$f\left({}^{g}k\right) = \overline{1} \text{ and } g^{x}h = \widetilde{1} \tag{2.5}$$

and $(x,y)(m,g) = (1_M, 1_G)$, and additionally

$$k\left({}^{h}f\right) = \overline{1} \text{ and } h^{f}g = \widetilde{1}. \tag{2.6}$$

By (2.6),

$${}^{g}\left(k\left({}^{h}f\right)\right) = {}^{g}\,\overline{1} \Longrightarrow ({}^{g}k)\left({}^{g^{k}}\left({}^{h}f\right)\right) = \overline{1} \Longrightarrow ({}^{g}k)\left({}^{\left({}^{g^{k}}h\right)}f\right) = \overline{1}$$

which implies $({}^{g}k)\left({}^{\widetilde{1}}f\right) = \overline{1}$ by (2.5), and hence $({}^{g}k)f = \overline{1}$ whereupon f is a unit in $A^{\oplus B}$. □

The action of a group $B^{\oplus A}$ on the monoid $A^{\oplus B}$ is *length-preserving* if for all $g \in B^{\oplus A}$ and $f \in A^{\oplus B}$, we have $\lambda({}^{g}f) = \lambda(f)$ and *prefix-preserving* if $f = hl$ implies that $({}^{g}f) = ({}^{g}h)\,l$ for some $l \in A^{\oplus B}$.

Lemma 2.6. *If $A^{\oplus B}$ is a monoid that has a length function and the action of $B^{\oplus A}$ on $A^{\oplus B}$ is length-preserving, then $P = A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi} B^{\oplus A}$ has a length function.*

Proof. Let $P = A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi} B^{\oplus A}$ be the generalized general product of the monoid $A^{\oplus B}$ and the group $B^{\oplus A}$ such that the action of $B^{\oplus A}$ on $A^{\oplus B}$ is length preserving and such that $A^{\oplus B}$ has a length function λ. To prove P has a length function, we define $\tau : P \to \mathbb{N}$ by $\tau(f,g) = \lambda(f)$. Now τ is a homomorphism since λ is a homomorphism. Also the action of $B^{\oplus A}$ on $A^{\oplus B}$ is length preserving:

$$\begin{aligned}
\tau\left((f_1,g_1)(f_2,g_2)\right) &= \tau\left(f_1\left({}^{g_1}f_2\right), g_1^{f_2}g_2\right)\\
&= \lambda\left(f_1\left({}^{g_1}f_2\right)\right)\\
&= \lambda(f_1) + \lambda\left({}^{g_1}f_2\right)\\
&= \tau(f_1,g_1) + \tau(f_2,g_2).
\end{aligned}$$

By Lemma 2.5, we have $(f,g) \in P$ is a unit if and only if f is a unit in $A^{\oplus B}$. We now have that $\tau(f,g) = 0 \Leftrightarrow \lambda(f) = 0 \Leftrightarrow f$ is a unit in $A^{\oplus B} \Leftrightarrow (f,g)$ is a unit in P. Thus P has a length function. If $B^{\oplus A}$ is not the trivial group, then the length function defined on $P = A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi} B^{\oplus A}$ cannot be a proper length function. We have $\tau\left(\overline{1},g\right) = \lambda(1) = 0$, but $\left(\overline{1},g\right) \neq \left(\overline{1},\widetilde{1}\right)$. ☐

An element $m \in M$ is called *an irreducible element* if $m = ab$ $(a,b \in M)$ then a or b is a unit in M. The following result is a generalization of [10,Lemma 3.3].

Proposition 2.7. *If $P = A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi} B^{\oplus A}$ such that $A^{\oplus B}$ has a length function λ and if $\lambda(f) > 1$, then f is not irreducible in $A^{\oplus B}$ and also the action of $B^{\oplus A}$ on $A^{\oplus B}$ is length-preserving and prefix-preserving.*

Proof. Since ${}^{g}(fh) = ({}^{g}f)\left({}^{g^f}h\right)$, it follows that ${}^{g}f$ is a prefix of ${}^{g}(fh)$. Thus the action of $B^{\oplus A}$ on $A^{\oplus B}$ is prefix-preserving. We now prove that the action of $B^{\oplus A}$ on $A^{\oplus B}$ is length-preserving. If $f = \overline{1}$, this implies that $\lambda(f) = 0$ and ${}^{g}f =^{g} \overline{1} = \overline{1}$. Therefore $\lambda({}^{g}f) = \lambda\left(\overline{1}\right) = 0$. Conversely, if ${}^{g}f = \overline{1}$ then $\lambda({}^{g}f) = 0$ and $f =^{g^{-1}} \overline{1} = \overline{1}$. Hence $\lambda(f) = \lambda\left(\overline{1}\right) = 0$. So $f = \overline{1}$ if and only if ${}^{g}f = \overline{1}$. Thus an action of $B^{\oplus A}$ on $A^{\oplus B}$ preserves elements of length 0. If f is a unit, then $fh = \overline{1}$ for some $h \in A^{\oplus B}$. Therefore $\lambda(fh) = \lambda\left(\overline{1}\right) = 0$ but $\lambda(fh) = \lambda(f) + \lambda(h)$ so $\lambda(f) + \lambda(h) = 0$. Hence $\lambda(f) = 0$ and

$$\overline{1} =^{g} \overline{1} =^{g} fh = ({}^{g}f)\left({}^{g^f}h\right).$$

So

$$\lambda\left({}^{g}f\right)+\lambda\left({}^{g^{f}}h\right)=\lambda\left(\overline{1}\right)=0.$$

Thus ${}^{g}f$ is a unit, and so the action of $B^{\oplus A}$ on $A^{\oplus B}$ preserves units.

Now to prove that the action of $B^{\oplus A}$ on $A^{\oplus B}$ preserves the length of non-units elements of $A^{\oplus B}$, we first show that the action of $B^{\oplus A}$ on $A^{\oplus B}$ preserves length if the length if this length is its minimum value >0. In fact this minimum value is 1. Suppose that if x is non-unit then $\lambda(x)>1$ and let $f\in A^{\oplus B}$ be a non-unit of minimal length. So $\lambda(f)>1$ and therefore m is not irreducible. Then $f=ab$ with a and b are non-units which gives $\lambda(a),\lambda(b)>0$ but $\lambda(f)=\lambda(a)+\lambda(b)$ in particular $0<\lambda(a)<\lambda(f)$ that is a contradiction. Therefore there exists a non-unit $f\in A^{\oplus B}$ with $\lambda(f)=1$.

To prove that the action of $B^{\oplus A}$ on $A^{\oplus B}$ preserves the length of non-units elements of $A^{\oplus B}$, we use induction steps. We prove if $\lambda(h)=1$, then $\lambda(g\cdot h)=1$. Suppose $\lambda(h)=1$ but $\lambda({}^{g}h)>1$. Then ${}^{g}h$ is not irreducible in $A^{\oplus B}$, and so ${}^{g}h=ab$ where a,b are non-units. Then

$$h={}^{g^{-1}}(ab)=({}^{g}a)({}^{g^{a}}b).$$

Thus

$$\lambda(h)=\lambda\left({}^{g}a\right)+\lambda\left({}^{g^{a}}b\right)\geq 1+1=2$$

which is a contradiction. Hence if $\lambda(h)=1$ then $\lambda({}^{g}h)=1$. Assume for $n>1$ that the $B^{\oplus A}$-action satisfies $\lambda({}^{g}h)=\lambda(h)$ if $\lambda(h)<n$. Now, suppose that $h\in A^{\oplus B}$ such that $\lambda(h)=n>1$. Then, by the assumption h is not irreducible, $h=ab$ where a,b are non-units and $0<\lambda(a),\lambda(b)<n$. Since ${}^{g}h={}^{g}(ab)=({}^{g}a)({}^{g^{a}}b)$, we have

$$\lambda\left({}^{g}h\right)=\lambda\left({}^{g}a\right)+\lambda\left({}^{g^{a}}b\right)=\lambda(a)+\lambda(b)=\lambda(h).$$

Hence the action of $B^{\oplus A}$ on $A^{\oplus B}$ preserves the length of non-units elements of $A^{\oplus B}$. Thus the action of $B^{\oplus A}$ on $A^{\oplus B}$ is length-preserving. □

In [3], the authors considered Zappa-Szép products of bands and groups. In here, we give generalization results in the case of generalized general product. We will consider $P=A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ the generalized of the general product of a band $A^{\oplus B}$ and a group $B^{\oplus A}$ such that there exists a group action of $B^{\oplus A}$ on $A^{\oplus B}$. So we assume the additional axioms for the identity element $\widetilde{1}\in B^{\oplus A}$. We have ${}^{\widetilde{1}}f=f$ and for the identity $\overline{1}\in A^{\oplus B}$, for all $g\in B^{\oplus A}$, we get ${}^{g}\overline{1}=\overline{1}$ and $g^{\overline{1}}=g$. Recall that a monoid S is *unit regular* if, for all $s\in S$, there exists a unit $s'\in S$ such that $s=ss's$. An *orthodox* semigroup is a regular semigroup in which the set of idempotents forms a subsemigroup ([4, 12]). Moreover, $\mathcal{L}$-*unipotent semigroup* is a regular semigroup such that there is a unique generator idempotent in each principle left ideal.

Theorem 2.8. *For a generalized general product $P=A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ of a band $A^{\oplus B}$ and a group $B^{\oplus A}$, the following conditions satisfy:*

(1) *P is a regular semigroup with* $E(P) = \left\{ (f,g) : g^f = \widetilde{1} \right\}$,
(2) $V((f,g)) = \left\{ \left({}^{g^{-1}}f, h \right) : h^f = \left(g^{-1}\right)^f \right\}$,
(3) *If $A^{\oplus B}$ has an identity, then P is a unit-regular,*
(4) *If $A^{\oplus B}$ is a semilattice, then P is orthodox and $\mathcal{L}$-unipotent,*
(5) *If $A^{\oplus B}$ is a semilattice with identity, then P is covered by $E(P) \rtimes B^{\oplus A}$.*

Proof. (1). We prove that for each $(f,g) \in P$, the element $\left({}^{g^{-1}}f, g^{-f} \right)$ is an inverse of (f,g). Thus

$$\begin{aligned}
(f,g)\left({}^{g^{-1}}f, g^{-f} \right)(f,g) &= \left(f\left({}^{g({}^{g^{-1}}f)} \right), (g^{{}^{g^{-1}}f}) g^{-f} \right)(f,g) \\
&= \left(f\left({}^{gg^{-1}}f \right), \left(gg^{-1}\right)^f \right)(f,g) \\
&= \left(f, \widetilde{1} \right)(f,g) \text{ by } (GP7) \qquad (2.7) \\
&= \left(f\left({}^{\widetilde{1}}f \right), \widetilde{1}^f g \right) = (f,g).
\end{aligned}$$

On the other hand, since $(f,g)\left({}^{g^{-1}}f, g^{-f} \right) = \left(f, \widetilde{1} \right)$, we get

$$\begin{aligned}
\left({}^{g^{-1}}f, g^{-f} \right)(f,g)\left({}^{g^{-1}}f, g^{-f} \right) &= \left({}^{g^{-1}}f, g^{-f} \right)\left(f, \widetilde{1} \right) \\
&= \left(\left({}^{g^{-1}}f \right)\left({}^{g^{-f}}f \right), (g^{-f})^f \widetilde{1} \right) \\
&= ({}^{g^{-1}}f^2, g^{-f^2}) \text{ by } (GP2), (GP3) \\
&= \left({}^{g^{-1}}f, g^{-f} \right).
\end{aligned}$$

Then (f,g) is a regular element and $\left({}^{g^{-1}}f, g^{-f} \right) \in V(f,g)$. Suppose that (f,g) is an idempotent. Therefore $(f,g)^2 = \left(f\left({}^g f\right), g^f g \right) = (f,g)$ and so $(f,g) \in E(P)$ if and only if $f({}^g f) = f$ and $g^f = \widetilde{1}$ which implies that $f({}^g f) = f$. Hence $E(P) = \left\{ (f,g) : g^f = \widetilde{1} \right\}$.

(2). Let $(k,h) \in V((f,g))$. Then

$$(f,g) = (f,g)(k,h)(f,g) = \left(f\left({}^g k\right)\left({}^{g^k h}f \right), \left(g^k h\right)^f g \right).$$

Therefore $f = f\left({}^g k\right)\left({}^{g^k h}f \right)$ and $g = \left(g^k h\right)^f g$. Then $\left(g^k h\right)^f = \widetilde{1}$, and so $f = f\left({}^g k\right)$. Similarly

$$(k,h) = (k,h)(f,g)(k,h) = \left(k\left({}^h f\right)\left({}^{h^f g}k \right), \left(h^f g\right)^k h \right).$$

Therefore $k = k\left({}^h f\right)\left({}^{h^f g}k \right)$ and $h = \left(h^f g\right)^k h$. Then $\left(h^f g\right)^k = \widetilde{1}$ and $k = k\left({}^h f\right)$. Now we compute $\widetilde{1} = \left(g^k h\right)^f = g^{k\left({}^h f\right)} h^f = g^k h^f$. So $\left(g^k\right)^{-1} = h^f$, and this implies that

$$\left(k^f g\right)^k = h^{f({}^g k)} g^k = h^f g^k = \widetilde{1}.$$

Therefore $(k,h) \in V((f,g))$ if and only if $f = f({}^g k), k = k\left({}^h f\right), \left(g^k\right)^{-1} = h^f$. We have

$${}^{g^{-1}}f = {}^{g^{-1}}\left(f\left({}^g k\right)\right) = {}^{g^{-1}}\left(\left({}^g k\right) f\right) = k\left({}^{\left(g^k\right)^{-1}} f\right) = k\,.$$

Then $h^f = \left(g^k\right)^{-1} = \left(g^{-1}\right)^{gk} = g^{-f}$. Hence

$$V((f,g)) \subseteq \left\{\left({}^{g^{-1}}f, h\right) : h^f = g^{-f}\right\}.$$

Now to prove the second inclusion, we first notice that ${}^h f = h^f f = {}^{\left(g^{-1}\right)^f} f = {}^{g^{-1}} f$. Also $\widetilde{1} = \left(gg^{-1}\right)^f = g^{{}^{g^{-1}}f}\left(g^{-1}\right)^f = g^{{}^{g^{-1}}f} h^f$. Now

$$\left({}^{g^{-1}}f, h\right)(f,g) = \left(\left({}^{g^{-1}}f\right)\left({}^h f\right), h^f g\right) = \left({}^{g^{-1}}f, h^f g\right)$$

so that $(f,g)\left({}^{g^{-1}}f, h\right)(f,g) = \left(f, g^{{}^{g^{-1}}f} h^f g\right) = (e,g)$ from above. Also

$$\begin{aligned}
\left({}^{g^{-1}}f, h\right)(f,g)\left({}^{g^{-1}}f, h\right) &= \left({}^{g^{-1}}f, h^f g\right)\left({}^{g^{-1}}f, h\right) \\
&= \left(\left({}^{g^{-1}}f\right)\left(h^f f\right), h^f g^{{}^{g^{-1}}f} h\right) \\
&= \left({}^{g^{-1}}f, h\right),
\end{aligned}$$

as required. Hence $V((e,g)) = \left\{\left({}^{g^{-1}}f, h\right) : h^f = g^{-f}\right\}$.

(3). We first compute the group of units $U(P)$. If (f,g) is a unit with inverse (k,h) then $(f,g)(k,h) = (k,h)(f,g) = \left(\overline{1}, \widetilde{1}\right)$. Therefore

$$\left(f\left({}^g k\right), g^k h\right) = \left(k\left({}^h f\right), h^f g\right) = \left(\overline{1}, \widetilde{1}\right)$$

and so $f({}^g k) = k\left({}^h f\right) = \overline{1}$ and $g^k h = h^f g = \widetilde{1}$. Therefore $f = \overline{1}, {}^g k = \widetilde{1}$ which implies $gh = \widetilde{1}$ and so $h = g^{-1}$. Thus

$$U(P) \subseteq \left\{\left(\overline{1}, g\right) : g \in B^{\oplus A}\right\}.$$

Conversely, $\left\{\left(\overline{1}, g\right) : g \in B^{\oplus A}\right\} \subseteq U(P)$ since the element $\left(\overline{1}, g^{-1}\right)$ is in the set $\left\{\left(\overline{1}, g\right) : g \in B^{\oplus A}\right\}$ such that $\left(\overline{1}, g\right)\left(\overline{1}, g^{-1}\right) = \left(\overline{1}, g^{-1}\right)\left(\overline{1}, g\right) = \left(\overline{1}, \widetilde{1}\right)$ for each $\left(\overline{1}, g\right)$. Thus $U(P) = \left\{\left(\overline{1}, g\right) : g \in B^{\oplus A}\right\}$. Now let $(f,g) \in P$. Then there exists a unit $\left(\overline{1}, g^{-1}\right) \in U(P)$ such that

$$(f,g)\left(\overline{1}, g^{-1}\right)(f,g) = \left(f\left({}^g\overline{1}\right), g^1 g^{-1}\right)(f,g) = \left(f, \widetilde{1}\right)(f,g) = (f,g)\,.$$

Thus $P = E \bowtie G$ is a unit-regular monoid.

(4). We have to prove that $E(P)$ is a subsemigroup of P. Suppose that $(f,g),(k,h)$ are idempotents in P. $(f,g)(k,h) = \left(f\left({}^{g}k\right), g^k h\right)$ is an idempotent if and only if $\left(g^k h\right)^{f({}^{g}k)} = \widetilde{1}$. We have

$$\left(g^k h g^{-1}\right)^f = \left(g^k h\right)^{{}^{g^{-1}}f} \left(g^{-1}\right)^f = \left(g^k h\right)^f.$$

We now have that $\left(g^{-1}\right)^f = \left(g^{-1}\right)^{{}^{g^{-1}}f} = \left(g^f\right)^{-1} = \widetilde{1}^{-1} = \widetilde{1}$. Then

$$\left(g^k h g^{-1}\right)^{{}^{g}k} = \left(g^k h\right)^{\left({}^{g^{-1}}g\right)k} \left(g^{-1}\right)^{{}^{g}k} = \left(g^k h\right)^k \left(g^{-1}\right)^{{}^{g}k} = g^k h^k \left(g^{-1}\right)^{{}^{g}k} = \widetilde{1}.$$

Therefore

$$\left(g^k h\right)^{f({}^{g}k)} = \left(\left(g^k h\right)^f\right)^{{}^{g}k} = \left(g^k h g^{-1}\right)^{f({}^{g}k)} = \left(\left(g^k h g^{-1}\right)^{{}^{g}k}\right)^f = \widetilde{1}.$$

Thus P is an orthodox monoid.

Now let $(f,g) \in P$, and let $\left({}^{g^{-1}}f, h\right), \left({}^{g^{-1}}f, l\right) \in V((f,g))$. Then $h^f = \left(g^{-1}\right)^f = l^f$. We compute

$$\left({}^{g^{-1}}f, h\right)(f,g) = \left(\left({}^{g^{-1}}f\right)\left({}^{h}f\right), h^f g\right) = \left(\left({}^{g^{-1}}f\right)\left({}^{\left(g^{-1}\right)^f}f\right), \left(g^{-1}\right)^f g\right).$$

Thus $\left({}^{g^{-1}}f, h\right)(f,g) = \left({}^{g^{-1}}f, l\right)(f,g)$. Therefore P is $\mathcal{L}$-unipotent.

(5). We have $E(P) = \left\{(f,g) : g^f = \widetilde{1}\right\}$ and $B^{\oplus A}$ acts on $E(P)$ such that for each $h \in B^{\oplus A}$ and $(f,g) \in E(P)$,

$$\begin{aligned} h * (f,g) &= \left(\overline{1}, h\right)(f,g)\left(\overline{1}, h^{-1}\right) \in P \\ &= \left({}^{h}f, h^f g\right)\left(\overline{1}, h^{-1}\right) = \left({}^{h}f, h^f g h^{-1}\right). \end{aligned}$$

This is in the set of $E(P)$ since

$$\left(h^f g h^{-1}\right)^{{}^{h}f} = \left(h^f g\right)^{\left(h^{-1}\right)h f} h^{-\left({}^{h}f\right)} = h^f h^{-\left({}^{h}f\right)} = \widetilde{1}.$$

So we now form the semidirect product $E(P) \rtimes B^{\oplus A}$ of the semigroup $E(P)$ and the group $B^{\oplus A}$ in which

$$\begin{aligned} ((f,g),x)((k,h),y) &= ((f,g)(x * (k,h)), xy) \\ &= \left(\left(f\left({}^{gx}k\right), g^{{}^{x}k} x^k h x^{-1}\right), xy\right). \end{aligned}$$

Suppose that $\theta : E(P) \rtimes B^{\oplus A} \to E \bowtie B^{\oplus A}$ is defined by: $\theta((f,g),x) \mapsto (f, gx)$. Then θ is a covering map. It is clear that θ is a surjective map and the idempotents in $E(P) \rtimes B^{\oplus A}$ are of the form $\left((f,g), \widetilde{1}\right)$; so $\left((f,g), \widetilde{1}\right) \mapsto$

(f,g) clearly θ is bijective on idempotents. Moreover, θ is a homomorphism of semigroups since

$$\begin{aligned}\theta\left(\left(f,g\right),x\right)\left(\left(k,h\right),y\right) &= \theta\left(\left(f\left({}^{gx}k\right), g^{x^k}x^k h x^{-1}\right), xy\right)\\ &= f\left({}^{gx}k\right), g^{x^k}x^k h y\\ &= \theta\left(\left(e,g\right),x\right)\theta\left(\left(f,h\right),y\right)\end{aligned}$$

which is the required result. □

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