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GENERAL SUMMATION FORMULAS FOR THE KAMPÉ DE FÉRIET FUNCTION

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ABSTRACT. Very recently by employing two well-known Euler's transformations for the hypergeometric function, Liu and Wang established numerous general transformation formulas for the Kampé de Fériet function and deduced many new summation formulas for the Kampé de Fériet function by using classical summation theorems for the series ${}_2F_1$ due to Kummer, Gauss and Bailey. Here, we aim to establish 176 interesting summation formulas for the Kampé de Fériet function in the form of 16 general summation formulas based on the transformation formulas due to Liu and Wang. The results are derived with the help of generalizations of Kummer's summation theorem, Gauss' second summation theorem and Bailey's summation theorem established earlier by Lavoie et al. The 176 formulas for the Kampé de Fériet function are pointed out to contain 16 known formulas, which are also recalled as corollaries.
Keywords: Gamma function, Pochhammer symbol, Gauss's hypergeometric function ${}_2F_1$, Generalized hypergeometric function ${}_pF_q$, Kampé de Fériet function, Generalization of Kummer's summation theorem, Generalization of Gauss' second summation theorem, Generalization of Bailey's summation theorem.
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1. Introduction and Preliminaries

The natural generalization of the Gauss's hypergeometric function ${}_2F_1$ is called the generalized hypergeometric series ${}_pF_q$ ($p,\, q \in \mathbb{N}_0$) defined by (see, e.g., [2], [29, p. 73] and [40, pp. 71-75]):

$$
(1.1)\qquad {}_pF_q\left[\begin{matrix}\alpha_1,\,\ldots,\,\alpha_p;\\ \beta_1,\,\ldots,\,\beta_q;\end{matrix}\, z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} = {}_pF_q(\alpha_1,\,\ldots,\,\alpha_p;\, \beta_1,\,\ldots,\,\beta_q;\, z),
$$

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where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [40, p. 2 and p. 5]):

$$(1.2)\qquad \begin{aligned}(\lambda)_n :&= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-)\\ &= \begin{cases}1 & (n=0)\\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N})\end{cases}\end{aligned}$$

and $\Gamma(\lambda)$ is the familiar Gamma function. Here an empty product is interpreted as 1, and we assume (for simplicity) that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that no zeros appear in the denominator of (1.1), that is,

$$(1.3)\qquad (\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ j = 1, \ldots, q).$$

Here and in the following, let $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, integers and positive integers, respectively, and let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad \text{and} \quad \mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}.$$

For more details of ${}_pF_q$ including its convergence, its various special and limiting cases, and its further diverse generalizations, one may be referred, for example, to [2,3,14,15,29,38,40,42,43].

It is worthy of note that whenever the generalized hypergeometirc function ${}_pF_q$ (including ${}_2F_1$) with its specified argument (for example, unit argument or $\frac{1}{2}$ argument) can be summed to be expressed in terms of the Gamma functions, the result may be very important from both theoretical and applicable points of view. Here, the classical summation theorems for the generalized hypergeometric series such as those of Gauss and Gauss second, Kummer, and Bailey for the series ${}_2F_1$; Watson's, Dixon's, Whipple's and Saalschütz's summation theorems for the series ${}_3F_2$ and others play important roles in theory and application. During 1992-1996, in a series of works, Lavoie et al. [24–26] have generalized the above mentioned classical summation theorems for ${}_3F_2$ of Watson, Dixon, and Whipple and presented a large number of special and limiting cases of their results, which have been further generalized and extended by Rakha and Rathie [31] and Kim et al. [22]. Those results have also been obtained and verified with the help of computer programs (for example, Mathematica).

The vast popularity and immense usefulness of the hypergeometric function and the generalized hypergeometric functions of one variable have inspired and stimulated a large number of researchers to introduce and investigate hypergeometric functions of two or more variables. A serious, significant and systematic study of the hypergeometric functions of two variables was initiated by Appell [1] who presented the so-called Appell functions F_1, F_2, F_3 and F_4 which are generalizations of the Gauss' hypergeometric function. The confluent forms of the Appell functions were studied by

Humbert [17]. A complete list of these functions can be seen in the standard literature, see, e.g., [14]. Later, the four Appell functions and their confluent forms were further generalized by Kampé de Fériet [19] who introduced more general hypergeometric functions of two variables. The notation defined and introduced by Kampé de Fériet for his double hypergeometric functions of superior order was subsequently abbreviated by Burchnall and Chaudndy [4, 5]. We recall here the definition of a more general double hypergeometric function (than one defined by Kampé de Fériet) in a slightly modified notation given by Srivastava and Panda [44, p. 423, Eq. (26)]. The convenient generalization of the Kampé de Fériet function is defined as follows:

$$F^{H:A;B}_{G:C;D}\left[\begin{array}{lll}(h_H): & (a_A); & (b_B);\\ (g_G): & (c_C); & (d_D);\end{array} x,\, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((h_H))_{m+n}\,((a_A))_m\,((b_B))_n}{((g_G))_{m+n}\,((c_C))_m\,((d_D))_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}, \tag{1.4}$$

where (h_H) denotes the sequence of parameters $(h_1, h_2, \ldots, h_H)$ and $((h_H))_n$ is defined by the following product of Pochhammer symbols

$$((h_H))_n := (h_1)_n\,(h_2)_n \cdots (h_H)_n \quad (n \in \mathbb{N}_0),$$

where, when $n = 0$, the product is to be interpreted as unity. For more details about the function (1.4) including its convergence, we refer, for example, to [42].

When some extensively generalized special functions like (1.4) were appeared, it has been an interesting and natural research subject to consider certain reducibilities of the functions. In this regard, the Kampé de Fériet function has attracted many mathematicians to investigate its reducibility and transformation formulas. In fact, there are numerous reduction formulas and transformation formulas of the Kampé de Fériet function in the literature, see, e.g., [6–13, 18, 20, 21, 23, 28, 30, 32–37, 39, 41, 45, 46, 48]. In the above-cited references, most of the reduction formulas were related to the cases $H + A = 3$ and $G + C = 2$. In 2010, by using Euler's transformation formula for ${}_2F_1$, Cvijoviȅ and Miller [13] established a reduction formula for the case $H + A = 2$ and $G + C = 1$. Motivated essentially by the work [13], recently, Liu and Wang [27] used Euler's first and second transformation formulas for ${}_2F_1$ and the above-mentioned classical summation theorems for ${}_pF_q$ to present a number of very interesting reduction formulas and then deduced summation formulas for the Kampé de Fériet function. Indeed, only a few summation formulas for the Kampé de Fériet function are available in the literature.

In this sequel, we aim to demonstrate how easily one can obtain as many as 176 new and interesting summation formulas for the Kampé de Fériet function, which contain 16 known formulas, in the forms of 16 general summation formulas based on the summation formulas obtained by Liu and Wang [27]. The results are established with the help of generalizations of

Kummer summation theorem, Gauss second summation theorem and Bailey summation theorem due to Lavoie et al. [26].

2. **Results required**

In order to make this paper self-contained, we recall the deduction formulas for the Kampé de Fériet function established by Liu and Wang [27].

$$
\begin{aligned}
&F^{1:1;2}_{1:0;1}\left[\begin{array}{ccc} \alpha: & \epsilon; & \beta-\epsilon,\,\gamma; \\ \beta: & -; & \gamma+\beta; \end{array} x,\, x\right] \\
&\qquad = (1-x)^{\beta-\epsilon-\alpha}\,{}_2F_1\left[\beta-\epsilon,\,\gamma+\beta-\alpha\,;\gamma+\beta\,;\; x\right];
\end{aligned} \tag{2.1}
$$

$$
\begin{aligned}
&F^{1:1;2}_{1:0;1}\left[\begin{array}{ccc} \alpha: & \epsilon; & \beta-\epsilon,\,1+\frac{1}{2}\alpha; \\ \beta: & -; & \frac{1}{2}\alpha; \end{array} x,\, x\right] \\
&\qquad = (1-x)^{\beta-\epsilon-\alpha}\,{}_2F_1\left[\beta-\epsilon,\,1+\frac{1}{2}\beta\,;\,\frac{1}{2}\beta\,;\; x\right];
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
&F^{1:1;3}_{1:0;2}\left[\begin{array}{ccc} \alpha: & \epsilon; & \beta-\epsilon,\,1+\frac{1}{2}\alpha,\,\frac{\alpha-\beta}{2}; \\ \beta: & -; & \frac{1}{2}\alpha,\,1+\frac{\alpha+\beta}{2}; \end{array} x,\, x\right] \\
&\qquad = (1-x)^{\beta-\epsilon-\alpha}\,{}_2F_1\left[\beta-\epsilon,\,\frac{\beta-\alpha}{2}\,;\,1+\frac{\alpha+\beta}{2}\,;\; x\right];
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
&F^{0:2;2}_{1:0;0}\left[\begin{array}{ccc} -: & \alpha,\,\epsilon; & \beta-\epsilon,\,\gamma; \\ \beta: & -; & -; \end{array} x,\, \frac{x}{x-1}\right] \\
&\qquad = F_3\left(\alpha,\,\beta-\epsilon\,:\,\epsilon,\,\gamma\,;\,\beta\,;\,x,\,\frac{x}{x-1}\right) \\
&\qquad = (1-x)^{-\alpha}\,{}_2F_1\left[\beta-\epsilon,\,\alpha+\gamma\,;\,\beta\,;\;\frac{x}{x-1}\right];
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
&F^{2:0;1}_{1:0;1}\left[\begin{array}{ccc} \alpha,\,\gamma: & -; & \epsilon; \\ \beta: & -; & \beta+\epsilon; \end{array} x,\, -x\right] \\
&\qquad = (1-x)^{-\alpha}\,{}_2F_1\left[\beta-\epsilon,\,\alpha+\gamma\,;\,\beta\,;\;\frac{x}{x-1}\right];
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
&F^{2:0;1}_{1:0;1}\left[\begin{array}{ccc} \alpha,\,\gamma: & -; & 1+\frac{1}{2}\gamma; \\ \beta: & -; & \frac{1}{2}\gamma; \end{array} x,\, -x\right] \\
&\qquad = (1-x)^{-\alpha}\,{}_2F_1\left[\alpha,\,1+\frac{1}{2}\beta\,;\,\frac{1}{2}\beta\,;\;\frac{x}{x-1}\right];
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
&F^{2:0;2}_{1:0;2}\left[\begin{array}{ccc} \alpha,\,\gamma: & -; & 1+\frac{1}{2}\gamma,\,\frac{\gamma-\beta}{2}; \\ \beta: & -; & \frac{1}{2}\gamma,\,1+\frac{\gamma+\beta}{2}; \end{array} x,\, -x\right] \\
&\qquad = (1-x)^{-\alpha}\,{}_2F_1\left[\alpha,\,\frac{\beta-\gamma}{2}\,;\,1+\frac{\gamma+\beta}{2}\,;\;\frac{x}{x-1}\right].
\end{aligned} \tag{2.7}
$$

In addition, we also need the following generalizations of Kummer summation theorem, Gauss second summation theorem, and Bailey's summation theorem due to Lavoie et al. [24–26]. Here and in the following, $[x]$ is the greatest integer less than or equal to x and $|x|$ is the absolute value (modulus) of x.

Generalization of Kummer's summation theorem is given as follows:

$$\begin{aligned} {}_2F_1\left[\begin{matrix} a,\, b\,; \\ 1+a-b+i\,; \end{matrix}\; -1\right] &= \frac{2^{-a}\,\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(1+a-b+i\right)\,\Gamma\left(1-b\right)}{\Gamma\left(1-b+\frac{1}{2}i+\frac{1}{2}|i|\right)} \\ &\times\left\{\frac{\mathcal{A}_i}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+1\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}+\frac{1}{2}i-\left[\frac{i+1}{2}\right]\right)}\right. \\ &\left.+\frac{\mathcal{B}_i}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\right\}, \end{aligned} \tag{2.8}$$

for $i = 0,\, \pm1,\, \pm2,\, \pm3,\, \pm4,\, \pm5$. The coefficients $\mathcal{A}_i$ and $\mathcal{B}_i$ are given in the Table 1.

Generalization of Gauss' second summation theorem is given as follows:

$$\begin{aligned} {}_2F_1\left[\begin{matrix} a,\, b\,; \\ \frac{1}{2}(a+b+i+1)\,; \end{matrix}\; \frac{1}{2}\right] &= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}|i|\right)} \\ &\times\left\{\frac{\mathcal{C}_i}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}-\left[\frac{i+1}{2}\right]\right)}+\frac{\mathcal{D}_i}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\right\}, \end{aligned} \tag{2.9}$$

for $i = 0,\, \pm1,\, \pm2,\, \pm3,\, \pm4,\, \pm5$. The coefficients $\mathcal{C}_i$ and $\mathcal{D}_i$ are given in the Table 2.

Generalization of Bailey's summation theorem is given as follows:

$$\begin{aligned} {}_2F_1\left[\begin{matrix} a,\, 1-a+i\,; \\ b\,; \end{matrix}\; \frac{1}{2}\right] &= 2^{1+i-b}\frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(b\right)\,\Gamma\left(1-a\right)}{\Gamma\left(1-a+\frac{1}{2}i+\frac{1}{2}|i|\right)} \\ &\times\left\{\frac{\mathcal{E}_i}{\Gamma\left(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}a-\left[\frac{i+1}{2}\right]\right)}\right. \\ &\left.+\frac{\mathcal{F}_i}{\Gamma\left(\frac{1}{2}b-\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}a-\frac{1}{2}-\left[\frac{i}{2}\right]\right)}\right\}, \end{aligned} \tag{2.10}$$

for $i = 0,\, \pm1,\, \pm2,\, \pm3,\, \pm4,\, \pm5$. The coefficients $\mathcal{E}_i$ and $\mathcal{F}_i$ are given in the Table 3.

3. General summation formulas for the Kampé de Fériet function

Here, we establish many summation formulas for the Kampé de Fériet function in the following theorems, each of which contains 11 results (except in Theorems 3.5, 3.8 and 3.26) in a single form.

Theorem 3.1. *Let $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. Then*

$$\begin{aligned} &F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \epsilon; & \beta-\epsilon, 1-\alpha-\epsilon+i;\ \frac{1}{2}, \frac{1}{2} \\ \beta: & -; & 1-\alpha-\epsilon+\beta+i;\ \end{array}\right] \\ &= 2^{\epsilon+\alpha-\beta}\, \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1-\epsilon-\alpha+\beta+i)\,\Gamma(\alpha-i)}{\Gamma\left(\alpha-\frac{1}{2}i+\frac{1}{2}|i|\right)} \\ &\times\left\{\frac{\mathcal{C}_i'}{\Gamma\left(\frac{1}{2}\beta-\frac{1}{2}\epsilon+\frac{1}{2}\right)\Gamma\left(1-\alpha-\frac{1}{2}\epsilon+\frac{1}{2}\beta+i-\left[\frac{i+1}{2}\right]\right)}\right. \\ &\left.+\frac{\mathcal{D}_i'}{\Gamma\left(\frac{1}{2}\beta-\frac{1}{2}\epsilon\right)\Gamma\left(\frac{1}{2}-\alpha-\frac{1}{2}\epsilon+\frac{1}{2}\beta+i-\left[\frac{i}{2}\right]\right)}\right\}, \end{aligned} \tag{3.1}$$

where the coefficients $\mathcal{C}_i'$ and $\mathcal{D}_i'$ are obtained from the Table 2 *by replacing a by $\beta-\epsilon$ and b by $1-2\alpha-\epsilon+\beta+i$ in $\mathcal{C}_i$ and $\mathcal{D}_i$, respectively.*

Proof. Setting $x = \frac{1}{2}$ and $\gamma = 1-\alpha-\epsilon+i$ ($i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$) in (2.1), we get

$$\begin{aligned} &F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \epsilon; & \beta-\epsilon, 1-\alpha-\epsilon+i;\ \frac{1}{2}, \frac{1}{2} \\ \beta: & -; & 1-\alpha-\epsilon+\beta+i;\ \end{array}\right] \\ &= 2^{\epsilon+\alpha-\beta}\, {}_2F_1\left[\begin{array}{c} \beta-\epsilon, 1-2\alpha-\epsilon+\beta+i;\ \frac{1}{2} \\ 1-\alpha-\epsilon+\beta+i;\ \end{array}\right]. \end{aligned} \tag{3.2}$$

Now, the ${}_2F_1$ in the right side of (3.2) can be evaluated with the help of the result (2.9) by taking $a = \beta-\epsilon$ and $b = 1-2\alpha-\epsilon+\beta+i$. After some simplification, we get the desired result. □

The particular case $i = 0$ in Theorem 3.1 yields a known result due to Lin and Wang [27, Corollary 5.1 (2)], which is recorded in the following corollary.

Corollary 3.2. *The following summation formula holds.*

$$\begin{aligned} &F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \epsilon; & \beta-\epsilon, 1-\alpha+\epsilon;\ \frac{1}{2}, \frac{1}{2} \\ \beta: & -; & 1-\alpha-\epsilon+\beta;\ \end{array}\right] \\ &= 2^{\alpha+\epsilon-\beta}\, \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1+\beta-\alpha-\epsilon)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta-\frac{1}{2}\epsilon\right)\Gamma\left(1-\alpha-\frac{1}{2}\epsilon+\frac{1}{2}\beta\right)}. \end{aligned} \tag{3.3}$$

It is noted that setting $i = \pm1, \pm2, \pm3, \pm4, \pm5$ in (3.1) yields 10 new results.

Theorem 3.3. *Let $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. Then*

$$
\begin{aligned}
&F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \epsilon; & \beta-\epsilon, 1+\alpha-2\beta+\epsilon+i; \\ \beta: & -; & 1+\alpha-\beta+\epsilon+i; \end{array} \frac{1}{2}, \frac{1}{2}\right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\alpha-\beta+\epsilon+i\right)}{\Gamma\left(1-\beta+\epsilon+\frac{1}{2}i+\frac{1}{2}|i|\right)} \\
&\quad \times \left\{ \frac{\mathcal{E}_i'}{\Gamma\left(1+\frac{1}{2}\alpha-\beta+\epsilon+i\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}i-\left[\frac{i+1}{2}\right]\right)} \right. \\
&\qquad \left. + \frac{\mathcal{F}_i'}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha-\beta+\epsilon+\frac{1}{2}i\right)\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)} \right\},
\end{aligned} \tag{3.4}
$$

where the coefficients $\mathcal{E}_i'$ and $\mathcal{F}_i'$ are obtained from the Table 3 *by replacing a by $\beta-\epsilon$ and b by $1+\alpha-\beta+\epsilon-i$ in $\mathcal{E}_i$ and $\mathcal{F}_i$, respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x = \frac{1}{2}$ and $\gamma = 1+\alpha-2\beta+\epsilon+i$ $(i = 0, \pm1, \pm2, \pm3, \pm4, \pm5)$ in (2.1) with the help of the result (2.10). We omit the details. □

The particular case $i = 0$ in Theorem 3.3 yields a known result due to Lin and Wang [27, Corollary 5.1 (3)], which is recorded in the following corollary.

Corollary 3.4. *The following summation formula holds.*

$$
\begin{aligned}
&F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \epsilon; & \beta-\epsilon, 1+\alpha-2\beta+\epsilon; \\ \beta: & -; & 1+\alpha-\beta+\epsilon; \end{array} \frac{1}{2}, \frac{1}{2}\right] \\
&= 2^{\alpha+\epsilon-\beta}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{1}{2}\epsilon\right)\Gamma\left(1+\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{1}{2}\epsilon\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma\left(1+\frac{1}{2}\alpha-\beta+\epsilon\right)}.
\end{aligned} \tag{3.5}
$$

It is noted that setting $i = \pm1, \pm2, \pm3, \pm4, \pm5$ in (3.4) yields 10 new results.

Theorem 3.5. *Let $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. Then*

$$
\begin{aligned}
&F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \beta-2+i; & 2+i, \frac{1}{2}\alpha+1; \\ \beta: & -; & \frac{1}{2}\alpha; \end{array} -1, -1\right] \\
&= 2^{1-\alpha-\frac{1}{2}\beta+i}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta\right)\Gamma\left(-1-i\right)}{\Gamma\left(-1-\frac{1}{2}i+\frac{1}{2}|i|\right)} \\
&\quad \times \left\{ \frac{\mathcal{A}_i'}{\Gamma\left(\frac{1}{4}\beta-\frac{1}{2}-\frac{1}{2}i\right)\Gamma\left(\frac{1}{4}\beta+1+\frac{1}{2}i-\left[\frac{i+1}{2}\right]\right)} \right. \\
&\qquad \left. + \frac{\mathcal{B}_i'}{\Gamma\left(\frac{1}{4}\beta-1-\frac{1}{2}i\right)\Gamma\left(\frac{1}{4}\beta+\frac{1}{2}+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)} \right\},
\end{aligned} \tag{3.6}
$$

where the coefficients $\mathcal{A}'_i$ *and* $\mathcal{B}'_i$ *are obtained from the* Table 1 *by replacing* a *by* $1+\frac{1}{2}\beta$ *and* b *by* $2-i$ *in* $\mathcal{A}_i$ *and* $\mathcal{B}_i$*, respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x=-1$ and $\epsilon=\beta-2+i$ $(i=0,\,\pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5)$ in (2.2) with the help of the result (2.8). We omit the details. □

Remark 3.6. By using $\Gamma(z+1)=z\Gamma(z)$, the expression $\Gamma(-1-i)\,/\Gamma\left(-1-\frac{1}{2}i+\frac{1}{2}|i|\right)$ in Theorems 3.5, 3.8 and 3.26 should be evaluated as follows:

$$\frac{\Gamma(-1-i)}{\Gamma\left(-1-\frac{1}{2}i+\frac{1}{2}|i|\right)}=\frac{(-1)^i}{(1+i)!}\quad(i\in\mathbb{N})\,. \tag{3.7}$$

The particular case $i=0$ in Theorem 3.5 yields a known result due to Lin and Wang [27, Corollary 5.2 (1)], which is recorded in the following corollary.

Corollary 3.7. *The following summation formula holds.*

$$F^{1:1;2}_{1:0;1}\left[\begin{array}{lcccc}\alpha: & \beta-2; & 2,\,\frac{1}{2}\alpha+1\,; & -1,\,-1\\ \beta: & -; & \frac{1}{2}\alpha\,; & \end{array}\right]=\frac{2^{-\alpha}(\beta-2)}{\beta}. \tag{3.8}$$

It is noted that setting $i=\pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5$ in (3.6) yields 10 new results.

Theorem 3.8. *Let* $i=0,\,\pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5$*. Then*

$$\begin{aligned}&F^{1:1;2}_{1:0;1}\left[\begin{array}{lcccc}\alpha: & \frac{1}{2}\beta+2+i\,; & \frac{1}{2}\beta-2-i,\,1+\frac{1}{2}\alpha\,; & \frac{1}{2}, & \frac{1}{2}\\ \beta: & -; & \frac{1}{2}\alpha\,; & & \end{array}\right]\\ &\quad=2^{\alpha-\frac{1}{2}\beta+2+i}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta\right)\Gamma(-1-i)}{\Gamma\left(-1-\frac{1}{2}i+\frac{1}{2}|i|\right)}\\ &\qquad\times\left\{\frac{\mathcal{C}_i^{(2)}}{\Gamma\left(\frac{1}{4}\beta-\frac{1}{2}-\frac{1}{2}i\right)\Gamma\left(\frac{1}{4}\beta+1+\frac{1}{2}i-\left[\frac{i+1}{2}\right]\right)}\right.\\ &\qquad\qquad\left.+\frac{\mathcal{D}_i^{(2)}}{\Gamma\left(\frac{1}{4}\beta-1-\frac{1}{2}i\right)\Gamma\left(\frac{1}{4}\beta+\frac{1}{2}+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\right\},\end{aligned} \tag{3.9}$$

where the coefficients $\mathcal{C}_i^{(2)}$ *and* $\mathcal{D}_i^{(2)}$ *are obtained from the* Table 2 *by replacing* a *by* $\frac{1}{2}\beta-2-i$ *and* b *by* $1+\frac{1}{2}\beta$ *in* $\mathcal{C}_i$ *and* $\mathcal{D}_i$*, respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x=\frac{1}{2}$ and $\epsilon=\frac{1}{2}\beta+2+i$ $(i=0,\,-1,\,-2,\,-3,\,-4,\,-5)$ in (2.2) with the help of the result (2.9). We omit the details. □

The particular case $i = 0$ in Theorem 3.8 yields a known result due to Lin and Wang [27, Corollary 5.2 (2)], which is recorded in the following corollary.

Corollary 3.9. *The following summation formula holds.*

$$F^{1:1;2}_{1:0;1}\left[\begin{array}{llll} \alpha: & \frac{1}{2}\beta+2; & \frac{1}{2}\beta-2,\, 1+\frac{1}{2}\alpha; & \frac{1}{2},\ \frac{1}{2} \\ \beta: & -; & \frac{1}{2}\alpha; & \end{array}\right] = 2^{\alpha-\frac{1}{2}\beta+2}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta\right)}{\Gamma\left(\frac{1}{4}\beta-\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\beta+1\right)}. \tag{3.10}$$

It is noted that setting $i = \pm1, \pm2, \pm3, \pm4, \pm5$ in (3.9) yields 10 new results.

Theorem 3.10. *Let $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. Then*

$$\begin{aligned} &F^{1:1;3}_{1:0;2}\left[\begin{array}{llll} \alpha: & \alpha+\beta-i; & i-\alpha,\, 1+\frac{1}{2}\alpha,\, \frac{\alpha-\beta}{2}; & -1,\, -1 \\ \beta: & -; & \frac{1}{2}\alpha,\, 1+\frac{\alpha+\beta}{2}; & \end{array}\right] \\ &= 2^{-\frac{1}{2}\beta-\frac{3}{2}\alpha-i}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\alpha+\frac{1}{2}\beta\right)\Gamma\left(1+\alpha-i\right)}{\Gamma\left(1+\alpha-\frac{1}{2}i+\frac{1}{2}|i|\right)} \\ &\times\left\{\frac{\mathcal{A}_i^{(2)}}{\Gamma\left(\frac{1}{4}\beta+\frac{3}{4}\alpha-\frac{1}{2}i+1\right)\Gamma\left(\frac{1}{4}\beta-\frac{1}{4}\alpha+\frac{1}{2}+\frac{1}{2}i-\left[\frac{i+1}{2}\right]\right)}\right. \\ &\left.+\frac{\mathcal{B}_i^{(2)}}{\Gamma\left(\frac{1}{4}\beta+\frac{3}{4}\alpha-\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\beta-\frac{1}{4}\alpha+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\right\}, \end{aligned} \tag{3.11}$$

where the coefficients $\mathcal{A}_i^{(2)}$ and $\mathcal{B}_i^{(2)}$ are obtained from the Table 1 *by replacing a by $\frac{1}{2}\beta-\frac{1}{2}\alpha$ and b by $i-\alpha$ in $\mathcal{A}_i$ and $\mathcal{B}_i$, respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x = -1$ and $\epsilon = \alpha+\beta-i$ ($i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$) in (2.3) with the help of the result (2.8). We omit the details. □

The particular case $i = 0$ in Theorem 3.10 yields a known result due to Lin and Wang [27, Corollary 5.3 (a)], which is recorded in the following corollary.

Corollary 3.11. *The following summation formula holds.*

$$F^{1:1;3}_{1:0;2}\left[\begin{array}{llll} \alpha: & \alpha+\beta; & -\alpha,\, 1+\frac{1}{2}\alpha,\, \frac{\alpha-\beta}{2}; & -1,\, -1 \\ \beta: & -; & \frac{1}{2}\alpha,\, 1+\frac{\alpha+\beta}{2}; & \end{array}\right] = 2^{-2\alpha}\,\frac{\Gamma\left(1+\frac{1}{2}\alpha+\frac{1}{2}\beta\right)\Gamma\left(1+\frac{1}{4}\beta-\frac{1}{4}\alpha\right)}{\Gamma\left(1+\frac{1}{2}\beta-\frac{1}{2}\alpha\right)\Gamma\left(1+\frac{1}{4}\beta+\frac{3}{4}\alpha\right)}. \tag{3.12}$$

It is noted that setting $i = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ in (3.11) yields 10 new results.

Theorem 3.12. *Let* $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. *Then*

(3.13)
$$
\begin{aligned}
&F^{1:1;3}_{1:0;2}\left[\begin{array}{ll} \alpha: & \frac{1}{2}\beta - \frac{3}{2}\alpha - 1 + i; \quad \frac{1}{2}\beta + \frac{3}{2}\alpha + 1 - i,\ 1 + \frac{1}{2}\alpha,\ \frac{\alpha-\beta}{2};\ \frac{1}{2},\ \frac{1}{2} \\ \beta: & \qquad -; \qquad\qquad\qquad \frac{1}{2}\alpha,\ 1 + \frac{\alpha+\beta}{2};\ \end{array}\right] \\
&\quad = 2^{-\frac{1}{2}\alpha - \frac{1}{2}\beta - 1}\, \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1 + \frac{1}{2}\alpha + \frac{1}{2}\beta\right)\Gamma\left(1 + \alpha - i\right)}{\Gamma\left(1 + \alpha - \frac{1}{2}i + \frac{1}{2}|i|\right)} \\
&\qquad \times \left\{ \frac{\mathcal{C}_i^{(3)}}{\Gamma\left(\frac{1}{4}\beta + \frac{3}{4}\alpha - \frac{1}{2}i + 1\right)\Gamma\left(\frac{1}{4}\beta - \frac{1}{4}\alpha + \frac{1}{2} + \frac{1}{2}i - \left[\frac{i+1}{2}\right]\right)} \right. \\
&\qquad\qquad \left. + \frac{\mathcal{D}_i^{(3)}}{\Gamma\left(\frac{1}{4}\beta + \frac{3}{4}\alpha - \frac{1}{2}i + \frac{1}{2}\right)\Gamma\left(\frac{1}{4}\beta - \frac{1}{4}\alpha + \frac{1}{2}i - \left[\frac{i}{2}\right]\right)} \right\},
\end{aligned}
$$

where the coefficients $\mathcal{C}_i^{(3)}$ *and* $\mathcal{D}_i^{(3)}$ *are obtained from the* Table 2 *by replacing* a *by* $\frac{1}{2}\beta + \frac{3}{2}\alpha + 1 - i$ *and* b *by* $\frac{1}{2}\beta - \frac{1}{2}\alpha$ *in* $\mathcal{C}_i$ *and* $\mathcal{D}_i$, *respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x = \frac{1}{2}$ and $\epsilon = \frac{1}{2}\beta - \frac{3}{2}\alpha - 1 + i$ $(i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5)$ in (2.3) with the help of the result (2.9). We omit the details. □

The particular case $i = 0$ in Theorem 3.12 yields a known result due to Lin and Wang [27, Corollary 5.3 (b)], which is recorded in the following corollary.

Corollary 3.13. *The following summation formula holds.*

(3.14)
$$
\begin{aligned}
&F^{1:1;3}_{1:0;2}\left[\begin{array}{ll} \alpha: & \frac{1}{2}\beta - \frac{3}{2}\alpha - 1; \quad \frac{1}{2}\beta + \frac{3}{2}\alpha + 1,\ 1 + \frac{1}{2}\alpha,\ \frac{\alpha-\beta}{2};\ \frac{1}{2},\ \frac{1}{2} \\ \beta: & \qquad -; \qquad\qquad \frac{1}{2}\alpha,\ 1 + \frac{\alpha+\beta}{2};\ \end{array}\right] \\
&\quad = 2^{-\frac{1}{2}\alpha - \frac{1}{2}\beta - 1}\, \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1 + \frac{\alpha+\beta}{2}\right)}{\Gamma\left(1 + \frac{1}{4}\beta + \frac{3}{4}\alpha\right)\Gamma\left(\frac{1}{2} + \frac{1}{4}\beta - \frac{1}{4}\alpha\right)}.
\end{aligned}
$$

It is noted that setting $i = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ in (3.13) yields 10 new results.

Theorem 3.14. *Let* $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. *Then*

(3.15)
$$\begin{aligned} & F^{1:1;3}_{1:0;2}\left[\begin{array}{l} \alpha: \ \tfrac{3}{2}\beta - \tfrac{1}{2}\alpha - 1 - i; \quad \tfrac{1}{2}\alpha - \tfrac{1}{2}\beta + 1 + i,\, 1 + \tfrac{1}{2}\alpha,\, \tfrac{\alpha-\beta}{2};\ \dfrac{1}{2},\ \dfrac{1}{2} \\ \beta: \qquad\qquad - ; \qquad\qquad \tfrac{1}{2}\alpha,\, 1 + \tfrac{\alpha+\beta}{2}; \end{array}\right] \\ &= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(1 + \frac{1}{2}\alpha + \frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}\beta - \frac{1}{2}\alpha - i\right)}{\Gamma\left(\frac{1}{2}\beta - \frac{1}{2}\alpha - \frac{1}{2}i + \frac{1}{2}|i|\right)} \\ &\quad \times \left\{ \frac{\mathcal{E}_i^{(2)}}{\Gamma\left(1 + \frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2} + \frac{1}{2}\beta - \left[\frac{i+1}{2}\right]\right)} + \frac{\mathcal{F}_i^{(2)}}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}\beta - \left[\frac{i}{2}\right]\right)} \right\}, \end{aligned}$$

where the coefficients $\mathcal{E}_i^{(2)}$ *and* $\mathcal{F}_i^{(2)}$ *are obtained from the* Table 3 *by replacing* a *by* $\frac{1}{2}\beta - \frac{1}{2}\alpha$ *and* b *by* $1 + \frac{\alpha+\beta}{2}$ *in* $\mathcal{E}_i$ *and* $\mathcal{F}_i$, *respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x = \frac{1}{2}$ and $\epsilon = \frac{3}{2}\beta - \frac{1}{2}\alpha - 1 - i$ ($i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$) in (2.3) with the help of the result (2.10). We omit the details. □

The particular case $i = 0$ in Theorem 3.14 yields a known result due to Lin and Wang [27, Corollary 5.3 (c)], which is recorded in the following corollary.

Corollary 3.15. *The following summation formula holds.*

(3.16)
$$\begin{aligned} & F^{1:1;3}_{1:0;2}\left[\begin{array}{l} \alpha: \ \tfrac{3}{2}\beta - \tfrac{1}{2}\alpha - 1; \quad \tfrac{1}{2}\alpha - \tfrac{1}{2}\beta + 1,\, 1 + \tfrac{1}{2}\alpha,\, \tfrac{\alpha-\beta}{2};\ \dfrac{1}{2},\ \dfrac{1}{2} \\ \beta: \qquad\qquad - ; \qquad\qquad \tfrac{1}{2}\alpha,\, 1 + \tfrac{\alpha+\beta}{2}; \end{array}\right] \\ &= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(1 + \frac{\alpha+\beta}{2}\right)}{\Gamma\left(1 + \frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2} + \frac{1}{2}\beta\right)}. \end{aligned}$$

It is noted that setting $i = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ in (3.15) yields 10 new results.

Theorem 3.16. *Let* $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. *Then*

(3.17)
$$\begin{aligned} & F^{0:2;2}_{1:0;0}\left[\begin{array}{l} -: \ \alpha,\, \epsilon; \quad \beta - \epsilon,\, 1 - \alpha - \epsilon + i;\ \dfrac{1}{2},\ -1 \\ \beta: \quad - ; \qquad\qquad - ; \end{array}\right] \\ &= F_3\left(\alpha,\, \beta - \epsilon \,:\, \epsilon,\, 1 - \alpha - \epsilon + i\,;\, \beta\,;\, \frac{1}{2},\, -1\right) \\ &= 2^{\alpha-\beta+\epsilon}\, \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(\beta)\Gamma(\epsilon - i)}{\Gamma\left(\epsilon - \frac{1}{2}i + \frac{1}{2}|i|\right)} \\ &\quad \times \left\{ \frac{\mathcal{A}_i^{(3)}}{\Gamma\left(\frac{1}{2}\beta + \frac{1}{2}\epsilon - \frac{1}{2}i\right)\Gamma\left(\frac{1}{2}\beta - \frac{1}{2}\epsilon + \frac{1}{2} + \frac{1}{2}i - \left[\frac{i+1}{2}\right]\right)} \right. \\ &\qquad \left. + \frac{\mathcal{B}_i^{(3)}}{\Gamma\left(\frac{1}{2}\beta + \frac{1}{2}\epsilon - \frac{i}{2} - \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta - \frac{1}{2}\epsilon + \frac{i}{2} - \left[\frac{i}{2}\right]\right)} \right\}, \end{aligned}$$

where the coefficients $\mathcal{A}_i^{(3)}$ *and* $\mathcal{B}_i^{(3)}$ *are obtained from the* Table 1 *by replacing* a *by* $\beta-\epsilon$ *and* b *by* $1-\epsilon+i$ *in* $\mathcal{A}_i$ *and* $\mathcal{B}_i$, *respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x=\frac{1}{2}$ and $\gamma=1-\alpha-\epsilon+i$ $(i=0,\,\pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5)$ in (2.4) with the help of the result (2.9). We omit the details. □

The particular case $i=0$ in Theorem 3.16 yields a known result due to Lin and Wang [27, Corollary 5.4 (a)], which is recorded in the following corollary.

Corollary 3.17. *The following summation formula holds.*

$$\begin{aligned}
&F^{0:2;2}_{1:0;0}\left[\begin{matrix} -: & \alpha,\,\epsilon\,; & \beta-\epsilon,\,1-\alpha-\epsilon\,; \\ \beta: & -\,; & -\,; \end{matrix}\ \frac{1}{2},\,-1\right]\\
&= F_3\left(\alpha,\,\beta-\epsilon\,:\,\epsilon,\,1-\alpha-\epsilon\,;\,\beta\,;\,\frac{1}{2},\,-1\right)\\
&= 2^{\alpha}\,\frac{\Gamma(\beta)\,\Gamma\left(1+\frac{1}{2}\beta-\frac{1}{2}\epsilon\right)}{\Gamma(1+\beta-\epsilon)\,\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\epsilon\right)}.
\end{aligned} \tag{3.18}$$

in (3.17) yields 10 new results.

Theorem 3.18. *Let* $i=0,\,\pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5$. *Then*

$$\begin{aligned}
&F^{0:2;2}_{1:0;0}\left[\begin{matrix} -: & \alpha,\,\epsilon\,; & \beta-\epsilon,\,\beta+\epsilon-\alpha-i-1\,; \\ \beta: & -\,; & -\,; \end{matrix}\ -1,\,\frac{1}{2}\right]\\
&= F_3\left(\alpha,\,\beta-\epsilon\,:\,\epsilon,\,\beta+\epsilon-\alpha-i-1\,;\,\beta\,;\,-1,\,\frac{1}{2}\right)\\
&= 2^{-\alpha}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(\beta)\,\Gamma(1-\epsilon)}{\Gamma\left(1-\epsilon+\frac{1}{2}i+\frac{1}{2}|i|\right)}\\
&\quad\times\left\{\frac{\mathcal{C}_i^{(4)}}{\Gamma\left(\frac{1}{2}\beta-\frac{1}{2}\epsilon+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\epsilon-\left[\frac{i+1}{2}\right]\right)}\right.\\
&\qquad\left.+\frac{\mathcal{D}_i^{(4)}}{\Gamma\left(\frac{1}{2}\beta-\frac{1}{2}\epsilon+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\epsilon-\frac{1}{2}-\frac{i}{2}-\left[\frac{i}{2}\right]\right)}\right\},
\end{aligned} \tag{3.19}$$

where the coefficients $\mathcal{C}_i^{(4)}$ *and* $\mathcal{D}_i^{(4)}$ *are obtained from the* Table 2 *by replacing* a *by* $\beta-\epsilon$ *and* b *by* $\beta+\epsilon-i-1$ *in* $\mathcal{C}_i$ *and* $\mathcal{D}_i$, *respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x=-1$ and $\gamma=\beta+\epsilon-\alpha-i$ $(i=0,\,\pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5)$ in (2.4) with the help of the result (2.9). We omit the details. □

The particular case $i=0$ in Theorem 3.18 yields a known result due to Lin and Wang [27, Corollary 5.4 (b)], which is recorded in the following corollary.

Corollary 3.19. *The following summation formula holds.*

$$\begin{aligned} &F^{0:2;2}_{1:0;0}\left[\begin{matrix} -: & \alpha,\,\epsilon\,; & \beta-\epsilon,\,\beta+\epsilon-\alpha-1\,; \\ \beta: & -; & -; \end{matrix}\;-1,\,\frac{1}{2}\right] \\ &= F_3\left(\alpha,\,\beta-\epsilon\,:\,\epsilon,\,\beta+\epsilon-\alpha-1\,;\,\beta\,;\,-1,\,\frac{1}{2}\right) \\ &= 2^{-\alpha}\,\frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\beta\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta-\frac{1}{2}\epsilon\right)\,\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\epsilon\right)}. \end{aligned} \tag{3.20}$$

It is noted that setting $i = \pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5$ in (3.19) yields 10 new results.

Theorem 3.20. *Let $i = 0,\,\pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5$. Then*

$$\begin{aligned} &F^{0:2;2}_{1:0;0}\left[\begin{matrix} -: & \alpha,\,\epsilon\,; & \beta-\epsilon,\,1-\alpha-\beta+\epsilon+i\,; \\ \beta: & -; & -; \end{matrix}\;-1,\,\frac{1}{2}\right] \\ &= F_3\left(\alpha,\,\beta-\epsilon\,:\,\epsilon,\,1-\alpha-\beta+\epsilon+i\,;\,\beta\,;\,-1,\,\frac{1}{2}\right) \\ &= 2^{1-\alpha-\beta+i}\,\frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\beta\right)\,\Gamma\left(1-\beta+\epsilon\right)}{\Gamma\left(1-\beta+\epsilon+\frac{1}{2}i+\frac{1}{2}|i|\right)} \\ &\quad\times\left\{\frac{\mathcal{E}^{(3)}_i}{\Gamma\left(\frac{1}{2}\epsilon+\frac{1}{2}\right)\,\Gamma\left(\beta-\frac{1}{2}\epsilon-\left[\frac{1+i}{2}\right]\right)}+\frac{\mathcal{F}^{(3)}_i}{\Gamma\left(\frac{1}{2}\epsilon\right)\,\Gamma\left(\beta-\frac{1}{2}\epsilon-\frac{1}{2}-\left[\frac{i}{2}\right]\right)}\right\}, \end{aligned} \tag{3.21}$$

where the coefficients $\mathcal{E}^{(3)}_i$ and $\mathcal{F}^{(3)}_i$ are obtained from the Table 3 *by replacing a by $\beta-\epsilon$ and b by β in $\mathcal{E}_i$ and $\mathcal{F}_i$, respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x = -1$ and $\gamma = 1-\alpha-\beta+\epsilon+i$ ($i = 0,\,\pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5$) in (2.4) with the help of the result (2.8). We omit the details. □

The particular case $i = 0$ in Theorem 3.20 yields a known result due to Lin and Wang [27, Corollary 5.4 (c)], which is recorded in the following corollary.

Corollary 3.21. *The following summation formula holds.*

$$\begin{aligned} &F^{0:2;2}_{1:0;0}\left[\begin{matrix} -: & \alpha,\,\epsilon\,; & \beta-\epsilon,\,1-\alpha-\beta+\epsilon\,; \\ \beta: & -; & -; \end{matrix}\;-1,\,\frac{1}{2}\right] \\ &= F_3\left(\alpha,\,\beta-\epsilon\,:\,\epsilon,\,1-\alpha-\beta+\epsilon\,;\,\beta\,;\,-1,\,\frac{1}{2}\right) \\ &= 2^{-\alpha}\,\frac{\Gamma\left(\frac{1}{2}\beta\right)\,\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}{\Gamma\left(\beta-\frac{1}{2}\epsilon\right)\,\Gamma\left(\frac{1}{2}\epsilon+\frac{1}{2}\right)}. \end{aligned} \tag{3.22}$$

It is noted that setting $i = \pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5$ in (3.21) yields 10 new results.

Theorem 3.22. *Let* $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. *Then*

$$(3.23)\quad \begin{aligned} &F^{2:0;1}_{1:0;1}\left[\begin{array}{ccc} \alpha, \gamma: & -; & \alpha-\beta-\gamma+1+i\,; \\ \beta: & -; & \alpha-\gamma+1+i\,; \end{array} -1,\, 1\right] \\ &= 2^{-\alpha}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\alpha-\gamma+i\right)\Gamma\left(\gamma-i\right)}{\Gamma\left(\gamma-\frac{1}{2}i+\frac{1}{2}|i|\right)} \\ &\quad \times\left\{\frac{\mathcal{C}_i^{(5)}}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha-\gamma+1+i-\left[\frac{1+i}{2}\right]\right)} \right. \\ &\qquad \left. +\frac{\mathcal{D}_i^{(5)}}{\Gamma\left(\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}\alpha-\gamma+i+\frac{1}{2}-\left[\frac{i}{2}\right]\right)}\right\}, \end{aligned}$$

where the coefficients $\mathcal{C}_i^{(5)}$ *and* $\mathcal{D}_i^{(5)}$ *are obtained from the* Table 2 *by replacing* a *by* α *and* b *by* $1+\alpha-2\gamma+i$ *in* $\mathcal{C}_i$ *and* $\mathcal{D}_i$, *respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x = -1$ and $\epsilon = \alpha-\beta-\gamma+1+i$ ($i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$) in (2.5) with the help of the result (2.8). We omit the details. □

The particular case $i = 0$ in Theorem 3.22 yields a known result due to Lin and Wang [27, Corollary 5.7 (b)], which is recorded in the following corollary.

Corollary 3.23. *The following summation formula holds.*

$$(3.24)\quad F^{2:0;1}_{1:0;1}\left[\begin{array}{ccc} \alpha, \gamma: & -; & \alpha-\beta-\gamma+1\,; \\ \beta: & -; & \alpha-\gamma+1\,; \end{array} -1,\, 1\right] = 2^{-\alpha}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\alpha-\gamma\right)}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\alpha-\gamma\right)}.$$

It is noted that setting $i = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ in (3.23) yields 10 new results.

Theorem 3.24. *Let* $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. *Then*

$$(3.25)\quad \begin{aligned} &F^{2:0;1}_{1:0;1}\left[\begin{array}{ccc} \alpha, \gamma: & -; & 1-\alpha-\beta+\gamma+i\,; \\ \beta: & -; & 1-\alpha+\gamma+i\,; \end{array} -1,\, 1\right] \\ &= 2^{-\gamma}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\alpha+\gamma+i\right)\Gamma\left(1-\alpha\right)}{\Gamma\left(1-\alpha+\frac{1}{2}i+\frac{1}{2}|i|\right)} \\ &\quad \times\left\{\frac{\mathcal{E}_i^{(4)}}{\Gamma\left(1-\alpha+\frac{1}{2}\gamma+\frac{1}{2}i\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\gamma+\frac{1}{2}i-\left[\frac{1+i}{2}\right]\right)} \right. \\ &\qquad \left. +\frac{\mathcal{F}_i^{(4)}}{\Gamma\left(\frac{1}{2}-\alpha+\frac{1}{2}\gamma+\frac{1}{2}i\right)\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\right\}, \end{aligned}$$

where the coefficients $\mathcal{E}_i^{(4)}$ *and* $\mathcal{F}_i^{(4)}$ *are obtained from the* Table 3 *by replacing* a *by* α *and* b *by* $1-\alpha+\gamma+i$ *in* $\mathcal{E}_i$ *and* $\mathcal{F}_i$, *respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x = -1$ and $\epsilon = 1 - \alpha - \beta + \gamma + i$ ($i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$) in (2.5) with the help of the result (2.10). We omit the details. □

The particular case $i = 0$ in Theorem 3.24 yields a known result due to Lin and Wang [27, Corollary 5.7 (c)], which is recorded in the following corollary.

Corollary 3.25. *The following summation formula holds.*

$$F^{2:0;1}_{1:0;1}\left[\begin{array}{rrr} \alpha,\,\gamma: & -; & 1-\alpha-\beta+\gamma\,; \\ \beta: & -; & 1-\alpha+\gamma\,; \end{array}\ -1,\,1\right] = 2^{-\alpha}\,\frac{\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha+\frac{1}{2}\gamma\right)\,\Gamma\left(1-\frac{1}{2}\alpha+\frac{1}{2}\gamma\right)}{\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}\right)\,\Gamma\left(1-\alpha+\frac{1}{2}\gamma\right)}. \tag{3.26}$$

It is noted that setting $i = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ in (3.25) yields 10 new results.

Theorem 3.26. *Let* $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. *Then*

$$\begin{aligned} &F^{2:0;1}_{1:0;1}\left[\begin{array}{rrr} \alpha,\,\gamma: & -; & 1+\frac{1}{2}\gamma\,; \\ 2\alpha+4+2i: & -; & \frac{1}{2}\gamma\,; \end{array}\ -1,\,1\right] \\ &\quad = 2^{-\alpha}\,\frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\alpha+2+i\right)\,\Gamma\left(-1-i\right)}{\Gamma\left(-1-\frac{1}{2}i+\frac{1}{2}|i|\right)} \\ &\qquad \times\left\{\frac{\mathcal{C}_i^{(6)}}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}\alpha+2+i-\left[\frac{1+i}{2}\right]\right)}+\frac{\mathcal{D}_i^{(6)}}{\Gamma\left(\frac{1}{2}\alpha\right)\,\Gamma\left(\frac{1}{2}\alpha+\frac{3}{2}+i-\left[\frac{i}{2}\right]\right)}\right\}, \end{aligned} \tag{3.27}$$

where the coefficients $\mathcal{C}_i^{(6)}$ *and* $\mathcal{D}_i^{(6)}$ *are obtained from the* Table 2 *by replacing* a *by* α *and* b *by* $\alpha + 3 + i$ *in* $\mathcal{C}_i$ *and* $\mathcal{D}_i$, *respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x = -1$ and $\beta = 2\alpha + 4 + 2i$ ($i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$) in (2.6) with the help of the result (2.9). We omit the details. □

The particular case $i = 0$ in Theorem 3.26 yields a known result due to Lin and Wang [27, Corollary 5.8], which is recorded in the following corollary.

Corollary 3.27. *The following summation formula holds.*

$$F^{2:0;1}_{1:0;1}\left[\begin{array}{rrr} \alpha,\,\gamma: & -; & \frac{1}{2}\gamma+1\,; \\ 2\alpha+4: & -; & \frac{1}{2}\gamma\,; \end{array}\ -1,\,1\right] = \frac{2(\alpha+1)}{\alpha+2}. \tag{3.28}$$

It is noted that setting $i = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ in (3.27) yields 10 new results. It is also interesting that the right side of the summation formula (3.27) is *independent* of the parameter γ.

Theorem 3.28. *Let $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. Then*

$$F^{2:0;2}_{1:0;2}\left[\begin{array}{rcl} \alpha,\, -\alpha+i: & -; & 1-\frac{1}{2}\alpha+\frac{1}{2}i,\ \frac{i-\alpha-\beta}{2};\ \frac{1}{2},\ -\frac{1}{2} \\ \beta: & -; & -\frac{1}{2}\alpha+\frac{1}{2}i,\ 1+\frac{i-\alpha+\beta}{2}; \end{array}\right]$$

$$= 2^{\frac{1}{2}(\alpha-\beta+i)}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\beta-\frac{1}{2}\alpha+\frac{1}{2}i\right)\Gamma(1-\alpha)}{\Gamma\left(1-\alpha+\frac{1}{2}i+\frac{1}{2}|i|\right)}$$

$$\times\left\{\frac{\mathcal{A}_i^{(4)}}{\Gamma\left(\frac{1}{4}\beta-\frac{3}{4}\alpha+\frac{1}{4}i+1\right)\Gamma\left(\frac{1}{4}\alpha+\frac{1}{4}\beta+\frac{1}{4}i+\frac{1}{2}-\left[\frac{1+i}{2}\right]\right)}\right.$$

$$\left.+\frac{\mathcal{B}_i^{(4)}}{\Gamma\left(\frac{1}{4}\beta-\frac{3}{4}\alpha+\frac{1}{4}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\alpha+\frac{1}{4}\beta+\frac{1}{4}i-\left[\frac{i}{2}\right]\right)}\right\}, \tag{3.29}$$

where the coefficients $\mathcal{A}_i^{(4)}$ and $\mathcal{B}_i^{(4)}$ are obtained from the Table 1 *by replacing a by $\frac{1}{2}\alpha+\frac{1}{2}\beta-\frac{1}{2}i$ and b by α in $\mathcal{A}_i$ and $\mathcal{B}_i$, respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x=\frac{1}{2}$ and $\gamma=-\alpha+i$ ($i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$) in (2.7) with the help of the result (2.8). We omit the details.

□

The particular case $i = 0$ in Theorem 3.28 yields a known result due to Lin and Wang [27, Corollary 5.9 (a)], which is recorded in the following corollary.

Corollary 3.29. *The following summation formula holds.*

$$F^{2:0;2}_{1:0;2}\left[\begin{array}{rcl} \alpha,\, -\alpha: & -; & 1-\frac{1}{2}\alpha,\ -\frac{\alpha+\beta}{2};\ \frac{1}{2},\ -\frac{1}{2} \\ \beta: & -; & -\frac{1}{2}\alpha,\ 1+\frac{\beta-\alpha}{2}; \end{array}\right] = 2^{\alpha}\,\frac{\Gamma\left(1+\frac{1}{4}\alpha+\frac{1}{4}\beta\right)\Gamma\left(1+\frac{1}{2}\beta-\frac{1}{2}\alpha\right)}{\Gamma\left(1+\frac{1}{2}\alpha+\frac{1}{2}\beta\right)\Gamma\left(1+\frac{1}{4}\beta-\frac{3}{4}\alpha\right)}. \tag{3.30}$$

It is noted that setting $i = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ in (3.29) yields 10 new results.

Theorem 3.30. *Let $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. Then*

$$F^{2:0;2}_{1:0;2}\left[\begin{array}{rcl} \frac{1}{2}\beta+\frac{3}{2}\gamma+1-i,\, \gamma: & -; & 1+\frac{1}{2}\gamma,\ \frac{\gamma-\beta}{2};\ -1,\ 1 \\ \beta: & -; & \frac{1}{2}\gamma,\ 1+\frac{\gamma+\beta}{2}; \end{array}\right]$$

$$= 2^{-\alpha}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\gamma+\frac{1}{2}\beta\right)\Gamma(\gamma+1-i)}{\Gamma\left(\gamma+1-\frac{1}{2}i+\frac{1}{2}|i|\right)}$$

$$\times\left\{\frac{\mathcal{C}_i^{(7)}}{\Gamma\left(\frac{1}{4}\beta+\frac{3}{4}\gamma+1-\frac{1}{2}i\right)\Gamma\left(\frac{1}{4}\beta-\frac{1}{4}\gamma+\frac{1}{2}i+\frac{1}{2}-\left[\frac{1+i}{2}\right]\right)}\right.$$

$$\left.+\frac{\mathcal{D}_i^{(7)}}{\Gamma\left(\frac{1}{4}\beta+\frac{3}{4}\gamma+\frac{1}{2}-\frac{1}{2}i\right)\Gamma\left(\frac{1}{4}\beta-\frac{1}{4}\gamma+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\right\}, \tag{3.31}$$

where the coefficients $\mathcal{C}_i^{(7)}$ *and* $\mathcal{D}_i^{(7)}$ *are obtained from the* Table 2 *by replacing* a *by* $\frac{1}{2}\beta + \frac{3}{2}\gamma + 1 - i$ *and* b *by* $\frac{1}{2}\beta - \frac{1}{2}\gamma$ *in* $\mathcal{C}_i$ *and* $\mathcal{D}_i$*, respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x = -1$ and $\alpha = \frac{1}{2}\beta + \frac{3}{2}\gamma + 1 - i$ ($i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$) in (2.7) with the help of the result (2.9). We omit the details. ∎

The particular case $i = 0$ in Theorem 3.30 yields a known result due to Lin and Wang [27, Corollary 5.9 (b)], which is recorded in the following corollary.

Corollary 3.31. *The following summation formula holds.*

$$
\begin{aligned}
&F^{2:0;2}_{1:0;2}\left[\begin{array}{rrr} \frac{1}{2}\beta + \frac{3}{2}\gamma + 1,\, \gamma: & -; & 1 + \frac{1}{2}\gamma,\, \frac{\gamma-\beta}{2}; \\ \beta: & -; & \frac{1}{2}\gamma,\, 1 + \frac{\gamma+\beta}{2}; \end{array} -1,\, 1\right] \\
&\qquad = 2^{-\alpha} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1 + \frac{1}{2}\beta + \frac{1}{2}\gamma\right)}{\Gamma\left(\frac{1}{4}\beta + \frac{3}{4}\gamma + 1\right)\Gamma\left(\frac{1}{4}\beta - \frac{1}{4}\gamma + \frac{1}{2}\right)}.
\end{aligned} \tag{3.32}
$$

It is noted that setting $i = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ in (3.31) yields 10 new results.

Theorem 3.32. *Let* $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$*. Then*

$$
\begin{aligned}
&F^{2:0;2}_{1:0;2}\left[\begin{array}{rrr} 1 - \frac{1}{2}\beta + \frac{1}{2}\gamma + i,\, \gamma: & -; & 1 + \frac{1}{2}\gamma,\, \frac{\gamma-\beta}{2}; \\ \beta: & -; & \frac{1}{2}\gamma,\, 1 + \frac{\gamma+\beta}{2}; \end{array} -1,\, 1\right] \\
&\quad = 2^{i-\alpha-\frac{1}{2}\beta-\frac{1}{2}\gamma} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1 + \frac{1}{2}\beta + \frac{1}{2}\gamma\right)\Gamma\left(1 - \frac{1}{2}\beta + \frac{1}{2}\gamma\right)}{\Gamma\left(1 - \frac{1}{2}\beta + \frac{1}{2}\gamma + \frac{1}{2}i + \frac{1}{2}|i|\right)} \\
&\qquad \times \left\{ \frac{\mathcal{E}_i^{(5)}}{\Gamma\left(1 + \frac{1}{2}\gamma\right)\Gamma\left(\frac{1}{2}\beta + \frac{1}{2} - \left[\frac{1+i}{2}\right]\right)} + \frac{\mathcal{F}_i^{(5)}}{\Gamma\left(\frac{1}{2}\gamma + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta - \left[\frac{i}{2}\right]\right)} \right\},
\end{aligned} \tag{3.33}
$$

where the coefficients $\mathcal{E}_i^{(5)}$ *and* $\mathcal{F}_i^{(5)}$ *are obtained from the* Table 3 *by replacing* a *by* $\frac{1}{2}\beta - \frac{1}{2}\gamma$ *and* b *by* $\frac{1}{2}\beta + \frac{1}{2}\gamma + 1$ *in* $\mathcal{E}_i$ *and* $\mathcal{F}_i$*, respectively.*

Proof. The proof would run parallel to that of Theorem 3.1, here, by setting $x = -1$ and $\alpha = 1 - \frac{1}{2}\beta + \frac{1}{2}\gamma + i$ ($i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$) in (2.7) with the help of the result (2.10). We omit the details. ∎

The particular case $i = 0$ in Theorem 3.32 yields a known result due to Lin and Wang [27, Corollary 5.9 (c)], which is recorded in the following corollary.

Corollary 3.33. *The following summation formula holds.*

$$F^{2:0;2}_{1:0;2}\left[\begin{array}{r} 1-\frac{1}{2}\beta+\frac{1}{2}\gamma,\,\gamma: \quad -; \quad 1+\frac{1}{2}\gamma,\,\frac{\gamma-\beta}{2}\,; \\ \beta: \quad -; \quad \frac{1}{2}\gamma,\,1+\frac{\gamma+\beta}{2}\,; \end{array} -1,\,1\right] = 2^{-\alpha-\frac{1}{2}\beta-\frac{1}{2}\gamma}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\beta+\frac{1}{2}\gamma\right)}{\Gamma\left(1+\frac{1}{2}\gamma\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}. \tag{3.34}$$

It is noted that setting $i=\pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5$ in (3.33) yields 10 new results.

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TABLE 1. Table for $\mathcal{A}_i$ and $\mathcal{B}_i$

i	$\mathcal{A}_i$	$\mathcal{B}_i$
-5	$4(a-b-4)^2-2b(a-b-4)-b^2$ $-8(a-b-4)-7b$	$4(a-b-4)^2+2b(a-b-4)-b^2$ $+16(a-b-4)-b+12$
-4	$2(a-b-3)(a-b-1)-b(b+3)$	$4(a-b-2)$
-3	$2a-3b-4$	$2a-b-2$
-2	$a-b-1$	2
-1	1	1
0	1	0
1	-1	1
2	$1+a-b$	-2
3	$3b-2a-5$	$2a-b+1$
4	$2(a-b+3)(1+a-b)-(b-1)(b-4)$	$-4(a-b+2)$
5	$-4(6+a-b)^2+2b(6+a-b)+b^2$ $+22(6+a-b)-13b-22$	$4(6+a-b)^2+2b(6+a-b)-b^2$ $-34(6+a-b)-b+62$

TABLE 2. Table for $\mathcal{C}_i$ and $\mathcal{D}_i$

i	$\mathcal{C}_i$	$\mathcal{D}_i$
-5	$(b+a-4)^2 - \frac{1}{4}(b-a-4)^2$ $-\frac{1}{2}(b+a-4)(b-a-4)$ $+4(b+a-4) - \frac{7}{2}(b-a-4)$	$(b+a-4)^2 - \frac{1}{4}(b-a-4)^2$ $+\frac{1}{2}(b+a-4)(b-a-4)$ $+8(b+a-4) - \frac{1}{2}(b-a-4) + 12$
-4	$\frac{1}{2}(b+a-3)(b+a+1)$ $-\frac{1}{4}(b-a-3)(b-a+3)$	$2(b+a-1)$
-3	$\frac{1}{2}(3a+b-2)$	$\frac{1}{2}(3b+a-2)$
-2	$\frac{1}{2}(b+a-1)$	2
-1	1	1
0	1	0
1	-1	1
2	$\frac{1}{2}(b+a-1)$	-2
3	$-\frac{1}{2}(3a+b-2)$	$\frac{1}{2}(a+3b-2)$
4	$\frac{1}{2}(b+a-3)(b+a+1)$ $-\frac{1}{4}(b-a+3)(b-a-3)$	$2(b+a-1)$
5	$-(b+a+6)^2 + \frac{1}{4}(b-a+6)^2$ $+\frac{1}{2}(b-a+6)(b+a+6)$ $+11(b+a+6) - \frac{13}{2}(b-a+6) - 20$	$(b+a+6)^2 - \frac{1}{4}(b-a+6)^2$ $+\frac{1}{2}(b+a+6)(b-a+6)$ $-17(b+a+6) - \frac{1}{2}(b-a+6) + 62$

TABLE 3. Table for $\mathcal{E}_i$ and $\mathcal{F}_i$

i	$\mathcal{E}_i$	$\mathcal{F}_i$
-5	$4b^2 - 2ab - a^2 + 8b - 7a$	$4b^2 + 2ab - a^2 + 16b - a + 12$
-4	$2b^2 - a^2 + 4b - 6a$	$4(b+1)$
-3	$2b - a$	$a + 2b + 2$
-2	b	2
-1	1	1
0	1	0
1	-1	1
2	$b - 2$	-2
3	$a - 2b - 3$	$a + 2b - 7$
4	$2b^2 - a^2 - 12b + 5a + 12$	$-4b + 12$
5	$-4b^2 + 2ab + a^2 + 22b - 13a - 20$	$4b^2 + 2ab - a^2 - 34b - a + 62$

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