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FORMULAS AND COMBINATORIAL SUMS INCLUDING SPECIAL NUMBERS ON p-ADIC INTEGRALS

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Abstract. The main motivation of this work is to give some formulas for the special numbers, which were recently introduced by Kilar and Simsek [6] with the aid of the p-adic integrals methods. These formulas are related to the some well-known families of special numbers and polynomials such as the negative order Bernoulli polynomials, the negative order Euler numbers and polynomials, the Stirling numbers, the array polynomials, the combinatorial numbers including the numbers $y_1(n,k;\lambda)$, the numbers $y_2(n,k;\lambda)$, the numbers $y_3(n,k;\lambda;a,b)$, the central factorial numbers, and combinatorial sums.
Keywords: Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers of the second kind, the array polynomials, the numbers $y_1(n,k;\lambda)$, the numbers $y_2(n,k;\lambda)$, the numbers $y_3(n,k;\lambda;a,b)$, the central factorial numbers.
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1. Introduction

In [24], Simsek gave many different formulas for the Volkenborn integral and the fermionic p-adic integral with (p-adic) distributions. By using these p-adic integrals formulas including Volkenborn integral and fermionic integral, we also derive some formulas for the special numbers including the negative order Bernoulli polynomials, the negative order Euler numbers and polynomials, the Stirling numbers, the array polynomials, the combinatorial numbers including the numbers $y_1(n,k;\lambda)$, the numbers $y_2(n,k;\lambda)$, the numbers $y_3(n,k;\lambda;a,b)$ and the central factorial numbers.

Before we give these formulas, let's give some notations that we used throughout this paper:

$$\mathbb{N}=\{1,2,3,...\},\ \mathbb{N}_0=\{0,1,2,3,...\}=\mathbb{N}\cup\{0\}$$

and let $\mathbb{Z}$ denote the set of integers, $\mathbb{R}^+$ denote the set of positive real numbers and $\mathbb{C}$ denote the set of complex numbers.
For $n\in\mathbb{N}$, the falling factorial is defined as

$$(\alpha)^{\underline{n}}=\alpha(\alpha-1)...(\alpha-n+1)=\binom{\alpha}{n}n!.$$

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Now, we give some very useful numbers and polynomials of generating functions, their recurrence relations and other well-known properties.

The higher order Bernoulli polynomials $B_n^{(k)}(x)$ are defined by

$$F_B(t,x;k) = \left(\frac{t}{e^t-1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}, \tag{1.1}$$

where $|t| < 2\pi$, $k \in \mathbb{N}_0$ (*cf.* [13], [14], [27]; and the references therein).

We observe that

$$B_n^{(k)}(0) = B_n^{(k)}$$

which denotes the Bernoulli numbers of order k (*cf.* [13], [14], [27]).

Substituting $k = 1$ and $x = 0$ into (1.1), we have the Bernoulli polynomials and numbers:

$$B_n^{(1)}(x) = B_n(x),$$

and

$$B_n = B_n(0) \quad (n \in \mathbb{N}_0)$$

(*cf.* [1], [4], [5], [27]; and the references therein).

The negative order Bernoulli polynomials of order $B_n^{(-k)}(x)$ are defined by

$$F_B(t,x;-k) = \left(\frac{e^t-1}{t}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(-k)}(x)\frac{t^n}{n!} \tag{1.2}$$

(*cf.* [14], [15], [26]; and the references therein).

We observe that

$$B_n^{(-k)}(0) = B_n^{(-k)}$$

which denote the Bernoulli numbers of the order $-k$, $k \in \mathbb{N}$ (*cf.* [14], [15], [26]; and the references therein).

The higher order Euler polynomials of the first kind $E_n^{(k)}(x)$ are defined by means of the following generating function:

$$F_E(t,x;k) = \left(\frac{2}{e^t+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!} \tag{1.3}$$

where $|t| < \pi$, $k \in \mathbb{N}_0$ (*cf.* [5], [13], [14], [27]; and the references therein).

We observe that

$$E_n^{(k)}(0) = E_n^{(k)}$$

which denote the Euler numbers of order k (*cf.* [5], [14], [27]; and the references therein).

Substituting $k = 1$ and $x = 0$ into (1.3), we have the Euler polynomials and numbers:

$$E_n^{(1)}(x) = E_n(x)$$

and

$$E_n = E_n(0) \quad (n \in \mathbb{N}_0)$$

(*cf.* [1], [4], [5], [27]; and the references therein).

The higher order Euler polynomials of the first kind $E_n^{(-k)}(x)$ are defined by

$$F_E(t,x;-k)=\left(\frac{e^t+1}{2}\right)^k e^{xt}=\sum_{n=0}^{\infty}E_n^{(-k)}(x)\frac{t^n}{n!}, \tag{1.4}$$

where $k\in\mathbb{N}_0$ (*cf.* [23], [27]; and the references therein).

We observe that

$$E_n^{(-k)}(0)=E_n^{(-k)}$$

which denotes the Euler numbers of the first kind order $-k$ (*cf.* [23], [27]).

The Stirling numbers of the second kind $S_2(n,m)$ are defined by

$$F_S(t,m)=\frac{\left(e^t-1\right)^m}{m!}=\sum_{n=0}^{\infty}S_2(n,m)\frac{t^n}{n!}. \tag{1.5}$$

By using (1.5), we have

$$S_2(n,m)=\frac{1}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(m-j)^n,$$

$$S_2(n,n)=1 \quad (n\in\mathbb{N}_0),$$
$$S_2(n,0)=0 \quad (n\in\mathbb{N})$$

and

$$S_2(n,m)=0 \quad m>n$$

(*cf.* [2], [4], [27]; and the references therein).

The λ-array polynomials $S_k^n(x;\lambda)$ are defined by

$$F_A(t,x,k;\lambda)=\frac{\left(\lambda e^t-1\right)^k}{k!}e^{xt}=\sum_{n=0}^{\infty}S_k^n(x;\lambda)\frac{t^n}{n!}, \tag{1.6}$$

where $k\in\mathbb{N}_0$, $\lambda\in\mathbb{C}$ (*cf.* [2], [22], [27]). Substituting $\lambda=1$ into the above equation, we have

$$S_m^n(x)=\frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(x+j)^n$$

with

$$S_0^0(x)=S_n^n(x)=1,\ S_0^n(x)=x^n$$

and $m>n$,

$$S_m^n(x)=0$$

(*cf.* [2], [19], [20]).

The Bernoulli numbers of the second kind (or the Cauchy numbers) $b_n(0)$ are defined by

$$F_C(t)=\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}b_n(0)\frac{t^n}{n!}$$

(*cf.* [4], [16]; and the references therein).

The combinatorial numbers $y_1(n,k;\lambda)$ are defined by means of the following generating function:

$$F_{y_1}(t,k;\lambda)=\frac{1}{k!}\left(\lambda e^t+1\right)^k=\sum_{n=0}^{\infty}y_1(n,k;\lambda)\frac{t^n}{n!}, \tag{1.7}$$

where $k\in\mathbb{N}_0$, $\lambda\in\mathbb{C}$, and the combinatorial numbers $y_1(n,k;\lambda)$ are given by the explicit formula:

$$y_1(n,k;\lambda)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}j^n\lambda^j$$

(*cf.* [23]).

In [23,Eq-(28)], substituting $\lambda=1$, the following relation was given:

$$E_n^{(-k)}=k!2^{-k}y_1(n,k;1). \tag{1.8}$$

The combinatorial numbers $y_2(n,k;\lambda)$ are defined by means of the following generating function:

$$F_{y_2}(t,k;\lambda)=\frac{1}{(2k)!}\left(\lambda e^t+\lambda^{-1}e^{-t}+2\right)^k=\sum_{n=0}^{\infty}y_2(n,k;\lambda)\frac{t^n}{n!}, \tag{1.9}$$

where $k\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$ (*cf.* [23]).

The combinatorial numbers $y_2(n,k;\lambda)$ are given by the explicit formula:

$$y_2(n,k;\lambda)=\frac{1}{(2k)!}\sum_{j=0}^{k}\binom{k}{j}2^{k-j}\sum_{l=0}^{j}\binom{j}{l}(2l-j)^n\lambda^{2l-j}$$

where $n,k\in\mathbb{N}$ (*cf.* [23]).

The combinatorial numbers $y_3(n,k;\lambda;a,b)$ are defined by means of the following generating function:

$$F_{y_3}(t,k;\lambda;a,b)=\frac{e^{bkt}}{k!}\left(\lambda e^{(a-b)t}+1\right)^k=\sum_{n=0}^{\infty}y_3(n,k;\lambda;a,b)\frac{t^n}{n!}, \tag{1.10}$$

where $k\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$ (*cf.* [22]).

The central factorial numbers of the second kind $T(n,k)$ are defined by

$$F_T(t,k)=\frac{1}{(2k)!}\left(e^t+e^{-t}-2\right)^k=\sum_{n=0}^{\infty}T(n,k)\frac{t^{2n}}{(2n)!} \tag{1.11}$$

(*cf.* [3], [20], [23], [25]).

1.1. **Identities and relations**

Here, we give the following identities and relations. By applying p-adic integrals to these identities, in Section 2, we derive some formulas including special numbers and combinatorial sums.

Theorem 1.1. *(cf. [6]) Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}(1.12)\qquad & \sum_{v=0}^{m} (x)^{\underline{v}} S_2(m,v)\, n^m \\ &= x^m \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{m-j} \binom{m-j}{k} (m)^{\underline{j}}\, 2^{m-j} \\ &\quad \times E_{m-j-k}^{(j-n)}\left(\frac{j-n}{2}\right) B_k^{(-j)}\left(-\frac{j}{2}\right).\end{aligned}$$

Theorem 1.2. *(cf. [6]) Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}(1.13)\qquad & \sum_{v=0}^{m} (x)^{\underline{v}} S_2(m,v)\, n^m \\ &= 2^{m-n} n! x^m \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{m}{k} y_1(k, n-j; 1)\, S_j^{m-k}\left(-\frac{n}{2}; 1\right).\end{aligned}$$

Theorem 1.3. *(cf. [6]) Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}(1.14)\qquad & \sum_{v=0}^{m} (x)^{\underline{v}} S_2(m,v)\, n^m \\ &= x^m \sum_{j=0}^{n} \binom{n}{j} j! 2^{m-j} \sum_{k=0}^{m} \binom{m}{k} E_k^{(j-n)} S_j^{m-k}\left(-\frac{n}{2}; 1\right).\end{aligned}$$

Theorem 1.4. *(cf. [6]) Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}(1.15)\qquad & \sum_{v=0}^{m} (x)^{\underline{v}} S_2(m,v)\, n^m \\ &= x^m \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \\ &\quad \times \sum_{l=0}^{m} \binom{m}{l} \frac{j!\,(2k)!\,(-1)^{n-j-k}}{2^{j+k+l-m}} S_j^{m-l}\left(-\frac{j}{2}; 1\right) y_2(l, k; 1).\end{aligned}$$

Theorem 1.5. *(cf. [6]) Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}(1.16)\qquad & \sum_{v=0}^{m} (x)^{\underline{v}} S_2(m,v)\, n^m \\ &= 2^{m-n} n! x^m \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{m}{k} y_3\left(k, n-j; 1; \frac{1}{2}, \frac{-1}{2}\right) S_j^{m-k}\left(-\frac{j}{2}; 1\right).\end{aligned}$$

Theorem 1.6. *(cf. [6]) Let $m,n\in\mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{m}(x)^{\underline{v}}S_2(m,v)\,n^m \tag{1.17}$$
$$= x^m\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{n-j}\binom{n-j}{k}$$
$$\times\sum_{l=0}^{\left[\frac{m}{2}\right]}\binom{m}{2l}\frac{j!\,(2k)!}{2^{j+k+2l-m}}S_j^{m-2l}\left(-\frac{j}{2};1\right)T(l,k).$$

1.2. p-adic integrals including Volkenborn integral and fermionic integral

Here, we give some definitions and notations for p-adic integrals. Let $\mathbb{K}$ be a field with a complete valuation and $C^1(\mathbb{Z}_p\to\mathbb{K})$ be a set of continuous derivative functions. That is $C^1(\mathbb{Z}_p\to\mathbb{K})$ is contained in the following set

$$\left\{f:\mathbb{X}\to\mathbb{K}: f(x)\ \text{is differentiable and}\ \frac{d}{dx}f(x)\ \text{is continuous}\right\}.$$

The Volkenborn integral (p-adic bosonic integral) of function $f\in(\mathbb{Z}_p\to\mathbb{K})$ is given by

$$\int_{\mathbb{Z}_p} f(x)\,d\mu_1(x)=\lim_{n\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1}f(x), \tag{1.18}$$

where

$$\mu_1(x)=\mu_1\left(x+p^N\mathbb{Z}_p\right)=\frac{1}{p^N}$$

(*cf.* [8], [9], [17], [21], [24]; and the references therein).

By using (1.18), the Bernoulli numbers B_n are given by

$$\int_{\mathbb{Z}_p} x^n d_{\mu_1}(x)=B_n \tag{1.19}$$

(*cf.* [8], [9], [17], [21], [24]; and the references therein).

The fermionic p-adic integral on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x)=\lim_{n\to\infty}\sum_{x=0}^{p^N-1}(-1)^x f(x), \tag{1.20}$$

where

$$\mu_{-1}(x)=\mu_{-1}\left(x+p^N\mathbb{Z}_p\right)=\frac{(-1)^x}{p^N}$$

(*cf.* [8], [9], [17], [21], [24]; and the references therein).

By using (1.20), the Euler numbers E_n are also given by

$$\int_{\mathbb{Z}_p} x^n d_{\mu_{-1}}(x) = E_n \tag{1.21}$$

(*cf.* [8], [9], [17], [21], [24]; and the references therein).

Let $n \in \mathbb{N}_0$. The Daehee numbers are also given by

$$\int_{\mathbb{Z}_p} (x)^{\underline{n}} d_{\mu_1}(x) = D_n \tag{1.22}$$

(*cf.* [11]).

Let $n \in \mathbb{N}_0$. The Changhee numbers are also given by

$$\int_{\mathbb{Z}_p} (x)^{\underline{n}} d_{\mu_{-1}}(x) = Ch_n \tag{1.23}$$

(*cf.* [10]).

In section 2, by applying p-adic integrals methods to identities and formulas which are given in Section 1, we give some new formulas including specials numbers and polynomials, and combinatorial sums.

2. Identities and relations using p-adic integrals

In this section, by applying p-adic integrals, related to the Volkenborn integral and fermionic integral, to the identities and formulas in Section 1, we derive combinatorial sums, some new formulas, relations, and identities including specials numbers and polynomials, and combinatorial sums. These identities are related to the negative order Bernoulli polynomials, the negative order Euler numbers and polynomials, the Stirling numbers, the array polynomials, the central factorial numbers, the Daehee numbers, the Changhee numbers, the combinatorial numbers such as the numbers $y_1(n,k;\lambda)$, the numbers $y_2(n,k;\lambda)$, the numbers $y_3(n,k;\lambda;a,b)$.

By applying the Volkenborn integral to equation (1.12) and combining with (1.22) and (1.19), we arrive at the following theorem:

Theorem 2.1. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{v=0}^{m} D_v S_2(m,v)\, n^m \\ = \quad & B_m \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{m-j} \binom{m-j}{k} (m)^{\underline{j}}\, 2^{m-j} E_{m-j-k}^{(j-n)}\left(\frac{j-n}{2}\right) B_k^{(-j)}\left(-\frac{j}{2}\right).\end{aligned}$$

By applying the fermionic integral to equation (1.12) and combining with (1.23) and (1.21), we arrive at the following theorem:

Theorem 2.2. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{v=0}^{m} Ch_v S_2(m,v)\, n^m \\ = \;& E_m \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{m-j} \binom{m-j}{k} (m)^{\underline{j}}\, 2^{m-j} E_{m-j-k}^{(j-n)} \left(\frac{j-n}{2}\right) B_k^{(-j)} \left(-\frac{j}{2}\right).\end{aligned}$$

By applying the Volkenborn integral to equation (1.13) and combining with (1.22) with (1.19), we arrive at the following theorem:

Theorem 2.3. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{v=0}^{m} D_v S_2(m,v)\, n^m \\ = \;& 2^{m-n} n! B_m \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{m}{k} y_1(k, n-j; 1)\, S_j^{m-k}\left(-\frac{n}{2}; 1\right).\end{aligned}$$

By applying the fermionic integral to equation (1.13) and combining with (1.23) and (1.21), we arrive at the following theorem:

Theorem 2.4. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{v=0}^{m} Ch_v S_2(m,v)\, n^m \\ = \;& 2^{m-n} n! E_m \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{m}{k} y_1(k, n-j; 1)\, S_j^{m-k}\left(-\frac{n}{2}; 1\right).\end{aligned}$$

By applying the Volkenborn integral to equation (1.14) and combining with (1.22) and (1.19), we arrive at the following theorem:

Theorem 2.5. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{v=0}^{m} D_v S_2(m,v)\, n^m \\ = \;& B_m \sum_{j=0}^{n} \binom{n}{j} j! 2^{m-j} \sum_{k=0}^{m} \binom{m}{k} E_k^{(j-n)} S_j^{m-k}\left(-\frac{n}{2}; 1\right).\end{aligned}$$

By applying the fermionic integral to equation (1.14) and combining with (1.23) and (1.21), we arrive at the following theorem:

Theorem 2.6. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{v=0}^{m} Ch_v S_2(m,v)\, n^m \\ = \;& E_m \sum_{j=0}^{n} \binom{n}{j} j! 2^{m-j} \sum_{k=0}^{m} \binom{m}{k} E_k^{(j-n)} S_j^{m-k}\left(-\frac{n}{2}; 1\right).\end{aligned}$$

By applying the Volkenborn integral to equation (1.15) and combining with (1.22) and (1.19), we arrive at the following theorem:

Theorem 2.7. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{v=0}^{m} D_v S_2(m,v)\, n^m \\ = \;& B_m \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \sum_{l=0}^{m} \binom{m}{l} \frac{j!\,(2k)!\,(-1)^{n-j-k}}{2^{j+k+l-m}} \\ & \times S_j^{m-l}\left(-\frac{j}{2};1\right) y_2(l,k;1).\end{aligned}$$

By applying the fermionic integral to equation (1.15) and combining with (1.23) and (1.21), we arrive at the following theorem:

Theorem 2.8. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{v=0}^{m} Ch_v S_2(m,v)\, n^m \\ = \;& E_m \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \sum_{l=0}^{m} \binom{m}{l} \frac{j!\,(2k)!\,(-1)^{n-j-k}}{2^{j+k+l-m}} \\ & \times S_j^{m-l}\left(-\frac{j}{2};1\right) y_2(l,k;1).\end{aligned}$$

By applying the Volkenborn integral to equation (1.16) and combining with (1.22) and (1.19), we arrive at the following theorem:

Theorem 2.9. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{v=0}^{m} D_v S_2(m,v)\, n^m \\ = \;& 2^{m-n} n! B_m \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{m}{k} y_3\left(k, n-j; 1; \frac{1}{2}, \frac{-1}{2}\right) S_j^{m-k}\left(-\frac{j}{2};1\right).\end{aligned}$$

By applying the fermionic integral to equation (1.16) and combining with (1.23) and (1.21), we arrive at the following theorem:

Theorem 2.10. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{v=0}^{m} Ch_v S_2(m,v)\, n^m \\ = \;& 2^{m-n} n! E_m \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{m}{k} y_3\left(k, n-j; 1; \frac{1}{2}, \frac{-1}{2}\right) S_j^{m-k}\left(-\frac{j}{2};1\right).\end{aligned}$$

By applying the Volkenborn integral to equation (1.17) and combining with (1.22) and (1.19), we arrive at the following theorem:

Theorem 2.11. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{v=0}^{m} D_v S_2(m,v)\, n^m \\ = \;& B_m \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \sum_{l=0}^{\left[\frac{m}{2}\right]} \binom{m}{2l} \frac{j!\,(2k)!}{2^{j+k+2l-m}} \\ & \times S_j^{m-2l}\left(-\frac{j}{2};1\right) T(l,k).\end{aligned}$$

By applying the fermionic integral to equation (1.17) and combining with (1.23) and (1.21), we arrive at the following theorem:

Theorem 2.12. *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{v=0}^{m} Ch_v S_2(m,v)\, n^m \\ = \;& E_m \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} \sum_{l=0}^{\left[\frac{m}{2}\right]} \binom{m}{2l} \frac{j!\,(2k)!}{2^{j+k+2l-m}} \\ & \times S_j^{m-2l}\left(-\frac{j}{2};1\right) T(l,k).\end{aligned}$$

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