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SUMS OF INFINITE POWER SERIES WHOSE COEFFICIENTS INVOLVE PRODUCTS OF THE CATALAN–QI NUMBERS

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Abstract. In the paper, the authors discuss some relations between generalized hypergeometric functions and infinite power series whose coefficients are products of the Catalan–Qi numbers, and derive closed expressions for some special power series whose coefficients are products of the Catalan–Qi numbers.
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1. Preliminaries

For $z\in\mathbb{C}$ and $n\in\{0\}\cup\mathbb{N}$, the rising factorial $(z)_n$ is defined [18] by

$$(z)_n=\prod_{\ell=0}^{n-1}(z+\ell)=\frac{\Gamma(z+n)}{\Gamma(z)}=\begin{cases}z(z+1)\cdots(z+n-1), & n\ge1;\\ 1, & n=0,\end{cases}\tag{1.1}$$

where $\Gamma(z)$ is the classical Euler gamma function which can be defined [26, p. 51, (3.9)] by

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^{n}(z+k)},\quad z\ne0,-1,-2,\dots.$$

It is also called the Pochhammer symbol or shifted factorial.

The Gauss hypergeometric function $F(a,b;c;z)$ is defined [26, p. 108, (5.3)] by

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\quad |z|<1\tag{1.2}$$

for $a,b,c\in\mathbb{C}$ with $c\ne0,-1,-2,\dots$. It has the integral representation

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{b^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\,\mathrm{d}t$$

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for $\Re(c)>\Re(b)>0$ and $|\arg(1-z)|<\pi$, see [26, p. 110, (5.4)].

The generalized hypergeometric functions can be defined [26, Section 5.9] by

$$ {}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}, \tag{1.3}$$

where $p,q=0,1,2,\dots$ with $p\le q+1$ and $a_i,b_i\in\mathbb{C}$ with $b_i\ne0,-1,-2,\dots$. Specially, we assume ${}_0F_0(z)=e^z$. It is clear that ${}_2F_1(a,b;c;z)=F(a,b;c;z)$. If $p=q+1$ in (1.3), the radius of convergence of ${}_pF_q$ is unity; if $p<q+1$, the radius of convergence is ∞.

The Catalan numbers C_n for $n\ge0$ form a sequence of positive integers and can be represented [4, 9, 10, 25, 27] by

$$ C_n=\frac{1}{n+1}\binom{2n}{n}=\frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)}=F(1-n,-n;2;1). \tag{1.4}$$

If replacing $n\ge0$ by $x\ge0$, we call C_x the Catalan function, see [16, Sections 5.1.5 and 5.1.6] and closely related references therein. In the past decades, the Catalan numbers C_n have been extensively investigated in [4, 11, 12, 20, 23, 24, 25] and closely related references therein.

Motivated by the second expression in (1.4) for the Catalan numbers C_n, the Catalan numbers C_n was generalized in [14, 15, 19, 21, 22] as

$$ C(a,b;z)=\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z\frac{\Gamma(z+a)}{\Gamma(z+b)},\quad \Re(a),\Re(b)>0,\quad \Re(z)\ge0. \tag{1.5}$$

In the literature, the quantity $C(a,b;z)$ is called the Catalan–Qi function and, when taking $x=n\in\{0\}\cup\mathbb{N}$, the quantity $C(a,b;n)$ is called the Catalan–Qi number in a series of papers including [2, 7, 13, 14, 15, 17, 21, 28, 29] and closely related references therein.

Since the reciprocal $\frac{1}{\Gamma(z)}$ of the gamma function $\Gamma(z)$ is entire, see [8,p. 136, 5.2.1], the ranges $\Re(a),\Re(b)>0$ and $\Re(z)\ge0$ in (1.5) can be extended to $a\ne0$ and $b,z+a\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$. By the definition in (1.1), it is immediate that

$$ C(a,b;n)=\left(\frac{b}{a}\right)^n\frac{(a)_n}{(b)_n}. \tag{1.6}$$

It is easy to see that

$$ C\left(\frac12,2;n\right)=C_n\quad\text{and}\quad C(b,a;z)=\frac{1}{C(a,b;z)}. \tag{1.7}$$

In combinatorics and statistics, the Fuss–Catalan numbers $A_n(p,r)$ are defined [3] as numbers of the form

$$ A_n(p,r)=\frac{r}{np+r}\binom{np+r}{n}=\frac{r\Gamma(np+r)}{\Gamma(n+1)\Gamma(n(p-1)+r+1)},$$

where $n \ge 0$, $p > 1$, and $r > 0$ are nonnegative integers. In [12, Theorem 1], the product-ratio expression

$$A_n(p,r) = r^n \frac{\prod_{k=0}^{p-1} C\bigl(\frac{k+r}{p}, 1; n\bigr)}{\prod_{k=0}^{p-2} C\bigl(\frac{k+r+1}{p-1}, 1; n\bigr)}, \tag{1.8}$$

which represents the Fuss–Catalan numbers $A_n(p,r)$ in terms of a series of the Catalan–Qi numbers $C(a,b;n)$, was discovered.

In [19, Section 2], the notion

$$Q(a,b;p,q;z) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^{(q-p+1)z} [\Gamma(z+1)]^{q-p} \frac{\Gamma(pz+a)}{\Gamma(qz+b)}$$

for $\Re(a), \Re(b) > 0$, $\Re(p), \Re(q) > 0$, and $\Re(z) \ge 0$ was invented. In [19, Theorem 1], when $p,q \in \mathbb{N}$, the product-ratio expression

$$Q(a,b;p,q;z) = \left[\left(\frac{b}{a}\right)^{q-p+1} \frac{\Gamma(b)\Gamma(p+a)}{\Gamma(a)\Gamma(q+b)}\right]^z \frac{\prod_{k=0}^{p-1} C\bigl(\frac{k+a}{p}, 1; z\bigr)}{\prod_{k=0}^{q-1} C\bigl(\frac{k+b}{q}, 1; z\bigr)} \tag{1.9}$$

for $\Re(a), \Re(b) > 0$ and $\Re(z) \ge 0$ was created.

The product-ratio expressions (1.8) and (1.9) are vey interesting and seem to imply something in mathematics.

2. Catalan–Qi numbers and hypergeometric functions

Now we start off to discuss how generalized hypergeometric functions ${}_mF_{m-1}$ are expressed in terms of the Catalan–Qi numbers $C(a,b;n)$.

Since $(1)_n = n!$, we can write (1.2) in the form

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad |z| < 1. \tag{2.1}$$

Making use of the identity in (1.6), we can further write (2.1) as

$$F(a,b;c;z) = \sum_{n=0}^{\infty} C(a,c;n) C(b,1;n) \left(\frac{a \cdot b}{c \cdot 1}\right)^n z^n, \quad |z| < 1. \tag{2.2}$$

Consequently, we naturally consider the function

$$Q(a,b;c,d;z) = \sum_{n=0}^{\infty} C(a,c;n) C(b,d;n) z^n, \quad |z| < \frac{|ab|}{|cd|}. \tag{2.3}$$

By the above argument, it is ready to see that

$$Q\left(a,b;c,1;\frac{ab}{c}z\right) = F(a,b;c;z)$$
$$\text{and} \quad F\left(a,b;c;\frac{c}{ab}z\right) = Q(a,b;c,1;z).$$

More generally, for $1\le k\le m$ and $a_k,b_k\in\mathbb{C}$ with $b_k\ne0,-1,-2,\dots$, we can consider the function

$$Q_m(a_1,a_2,\dots,a_m;b_1,b_2,\dots,b_m;z)=\sum_{n=0}^\infty\prod_{k=1}^m C(a_k,b_k;n)z^n, \tag{2.4}$$

where $|z|<\prod_{k=1}^m\frac{|a_k|}{|b_k|}$. This is a generalization of the function $Q(a,b;c,d;z)$ in (2.3). It is clear that

$$Q_2(a,b;c,d;z)=Q(a,b;c,d;z).$$

Combining (1.6) and (2.4) yields

$$\begin{aligned}
Q_m(a_1,a_2,\dots,a_m;b_1,b_2,\dots,b_m;z)
&=\sum_{n=0}^\infty\Biggl[\prod_{k=1}^m\frac{(a_k)_n}{(b_k)_n}\Biggr]\Biggl(\prod_{k=1}^m\frac{b_k}{a_k}\Biggr)^n z^n\\
&=\sum_{n=0}^\infty\Biggl[\prod_{k=1}^m\frac{(a_k)_n}{(b_k)_n}\Biggr](1)_n\Biggl(\prod_{k=1}^m\frac{b_k}{a_k}\Biggr)^n \frac{z^n}{n!}\\
&=\sum_{n=0}^\infty\frac{(a_1)_n(a_2)_n\cdots(a_m)_n(1)_n}{(b_1)_n(b_2)_n\cdots(b_m)_n}\frac1{n!}\Biggl[\Biggl(\prod_{k=1}^m\frac{b_k}{a_k}\Biggr)z\Biggr]^n\\
&={}_{m+1}F_m\biggl(a_1,a_2,\dots,a_m,1;b_1,b_2,\dots,b_m;\frac{b_1b_2\cdots b_m}{a_1a_2\cdots a_m}z\biggr)
\end{aligned}$$

and

$$\begin{aligned}
Q_m(a_1,a_2,\dots,a_m;b_1,b_2,\dots,b_{m-1},1;z)
&=\sum_{n=0}^\infty\Biggl[\prod_{k=1}^{m-1}\frac{(a_k)_n}{(b_k)_n}\Biggr]\Biggl(\prod_{k=1}^{m-1}\frac{b_k}{a_k}\Biggr)^n\frac{(a_m)_n}{a_m^n}\frac{z^n}{n!}\\
&=\sum_{n=0}^\infty\frac{(a_1)_n(a_2)_n\cdots(a_m)_n}{(b_1)_n(b_2)_n\cdots(b_{m-1})_n}\biggl(\frac{b_1b_2\cdots b_{m-1}}{a_1a_2\cdots a_{m-1}a_m}\biggr)^n\frac{z^n}{n!}\\
&={}_mF_{m-1}\biggl(a_1,a_2,\dots,a_m;b_1,b_2,\dots,b_{m-1};\frac{b_1b_2\cdots b_{m-1}}{a_1a_2\cdots a_{m-1}a_m}z\biggr).
\end{aligned}$$

We can write these two equalities as

$$\begin{aligned}
&Q_m(a_1,a_2,\dots,a_m;b_1,b_2,\dots,b_m;z)\\
&\quad={}_{m+1}F_m\biggl(a_1,a_2,\dots,a_m,1;b_1,b_2,\dots,b_m;\frac{b_1b_2\cdots b_m}{a_1a_2\cdots a_m}z\biggr)
\end{aligned}\tag{2.5}$$

for $m\ge1$ and

$$\begin{aligned}
&{}_mF_{m-1}(a_1,a_2,\dots,a_m;b_1,b_2,\dots,b_{m-1};z)\\
&\quad=Q_m\biggl(a_1,a_2,\dots,a_m;b_1,b_2,\dots,b_{m-1},1;\frac{a_1a_2\cdots a_{m-1}a_m}{b_1b_2\cdots b_{m-1}}z\biggr)
\end{aligned}\tag{2.6}$$

for $m \ge 2$.

Since

$$C(a,b;z)C(b,c;z)=C(a,c;z),$$

it is easy to see that

$$\begin{aligned}&Q_m(a_1,a_2,\dots,a_{\ell-1},c,a_{\ell+1}\dots,a_m;b_1,b_2,\dots,a_{k-1},c,a_{k+1}\dots,b_m;z)\\&=Q_{m-1}(a_1,a_2,\dots,a_{\ell-1},a_{\ell+1}\dots,a_m;b_1,b_2,\dots,a_{k-1},a_{k+1}\dots,b_m;z),\end{aligned}$$

where $1\le \ell,k\le m$ and $c\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$.

3. Sums of infinite series and hypergeometric functions

Now we recover or derive some closed forms for the Gauss hypergeometric functions ${}_2F_1$ and for sums of some infinite power series whose coefficients involve the Catalan numbers C_n or the Catalan–Qi numbers $C(a,b;n)$.

3.1.

By virtue of equalities in (1.7) and the identity (2.5), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{x^n}{C_n}&=\sum_{n=0}^{\infty}\frac{x^n}{C\left(\frac12,2;n\right)}=\sum_{n=0}^{\infty}C\left(2,\frac12;n\right)x^n\\&=Q_1\left(2;\frac12;x\right)={}_2F_1\left(2,1;\frac12;\frac{x}{4}\right).\end{aligned}\tag{3.1}$$

In [5], it was obtained that

$$\sum_{n=0}^{\infty}\frac{x^n}{C_n}=\begin{cases}1+\dfrac{\begin{bmatrix}x(4-x)^{3/2}+6x(4-x)^{1/2}\\+24\sqrt{x}\arcsin\dfrac{\sqrt{x}}{2}\end{bmatrix}}{(4-x)^{5/2}}, & 0\le x<4;\\[2ex] 1-\dfrac{\begin{bmatrix}|x|(4-x)^{3/2}+6\sqrt{|x|(4-x)}\\+24\sqrt{|x|}\ln\dfrac{\sqrt{-x}+\sqrt{4-x}}{2}\end{bmatrix}}{(4-x)^{5/2}}, & -4<x\le 0.\end{cases}$$

Consequently, we recover that

$${}_2F_1\left(1,2;\frac12;x\right)=\begin{cases}1+\dfrac{\begin{bmatrix}2x(1-x)^{3/2}+3x(1-x)^{1/2}\\+3\sqrt{x}\arcsin\sqrt{x}\end{bmatrix}}{2(1-x)^{5/2}}, & 0\le x<1;\\[2ex] 1-\dfrac{\begin{bmatrix}4|x|(1-x)^{3/2}+3\sqrt{|x|(1-x)}\\+6\sqrt{|x|}\ln\left(\sqrt{-x}+\sqrt{1-x}\right)\end{bmatrix}}{4(1-x)^{5/2}}, & -1<x\le 0.\end{cases}\tag{3.2}$$

In [1], by several methods, it was proved that

$$ {}_2F_1\left(1,2;\frac{1}{2};x\right)=\frac{x+2}{2(1-x)^2}+\frac{3\sqrt{x}}{2(1-x)^{5/2}}\arcsin\sqrt{x}\,, \quad |x|<1. \tag{3.3} $$

We remark that the above two expressions (3.2) and (3.3) for the Gauss hypergeometric function ${}_2F_1\left(1,2;\frac{1}{2};z\right)$ are equivalent to each other. For more information on derivation of closed expressions for ${}_2F_1\left(1,2;\frac{1}{2};z\right)$ and related power series involving the Catalan numbers C_n, please refer to [13,Sections 6 and 7] and closely related references therein.

3.2.

In [6, p. 452, Theorem] and [13, Section 8], it was obtained that

$$ \frac{2x\arcsin x}{\sqrt{1-x^2}}=\sum_{m=1}^{\infty}\frac{(2x)^{2m}}{m\binom{2m}{m}}, \quad |x|<1. \tag{3.4} $$

By the substitution $(2x)^2=t$, the formula (3.4) can be rewritten as

$$ \frac{t\sqrt{t}\,\arcsin\frac{\sqrt{t}}{2}}{\sqrt{1-\frac{t}{4}}}=\sum_{m=1}^{\infty}\frac{t^{m+1}}{m(m+1)\frac{1}{m+1}\binom{2m}{m}}=\sum_{m=1}^{\infty}\frac{t^{m+1}}{m(m+1)C_m}. $$

Differentiating twice with respect to t gives

$$ \left(\frac{t\sqrt{t}\,\arcsin\frac{\sqrt{t}}{2}}{\sqrt{1-\frac{t}{4}}}\right)''=\sum_{m=1}^{\infty}\frac{t^{m-1}}{C_m}=\frac{1}{t}\sum_{m=1}^{\infty}\frac{t^m}{C_m}=\frac{1}{t}\left(\sum_{m=0}^{\infty}\frac{t^m}{C_m}-1\right) $$

which is equivalent to

$$ \sum_{m=0}^{\infty}\frac{t^m}{C_m}=1+t\left(\frac{t\sqrt{t}\,\arcsin\frac{\sqrt{t}}{2}}{\sqrt{1-\frac{t}{4}}}\right)''=\frac{2(t+8)}{(t-4)^2}+\frac{24\sqrt{t}\,\arcsin\frac{\sqrt{t}}{2}}{(4-t)^{5/2}}. $$

Further considering (3.1) recovers (3.3).

3.3.

Example 5.1 in [26, p. 109] reads that, for $|z|<1$,

$$ F(a,b;b;z)=\frac{1}{(1-z)^a}, \qquad F(1,1;2;z)=-\frac{\ln(1-z)}{z}, $$
$$ F\left(\frac{1}{2},1;\frac{3}{2};z^2\right)=\frac{1}{2z}\ln\frac{1+z}{1-z}, \qquad F\left(\frac{1}{2},1;\frac{3}{2};-z^2\right)=\frac{\arctan z}{z}, $$
$$ F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2\right)=\frac{\arcsin z}{z}, \qquad F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};-z^2\right)=\frac{\ln\left(z+\sqrt{1+z^2}\,\right)}{z}. $$

From (2.2) or (2.6), it follows that

$$F(a,b;b;z)=\sum_{n=0}^{\infty}C(a,b;n)C(b,1;n)\left(\frac{ab}{b\cdot 1}\right)^{n}z^{n}$$

$$=\sum_{n=0}^{\infty}C(a,1;n)(az)^{n}=\frac{1}{(1-z)^{a}}.$$

Similarly, we obtain

$$F(1,1;2;z)=\sum_{n=0}^{\infty}C(1,2;n)C(1,1;n)\left(\frac{z}{2}\right)^{n}$$

$$=\sum_{n=0}^{\infty}C(1,2;n)\left(\frac{z}{2}\right)^{n}=-\frac{\ln(1-z)}{z},$$

$$F\left(\frac{1}{2},1;\frac{3}{2};z^{2}\right)=\sum_{n=0}^{\infty}C\left(\frac{1}{2},\frac{1}{3};n\right)C(1,1;n)\frac{z^{2n}}{3^{n}}$$

$$=\sum_{n=0}^{\infty}C\left(\frac{1}{2},\frac{1}{3};n\right)\frac{z^{2n}}{3^{n}}=\frac{1}{2z}\ln\frac{1+z}{1-z},$$

$$F\left(\frac{1}{2},1;\frac{3}{2};-z^{2}\right)=\sum_{n=0}^{\infty}C\left(\frac{1}{2},\frac{1}{3};n\right)C(1,1;n)\frac{\left(-z^{2}\right)^{n}}{3^{n}}$$

$$=\sum_{n=0}^{\infty}(-1)^{n}C\left(\frac{1}{2},\frac{1}{3};n\right)\frac{z^{2n}}{3^{n}}=\frac{\arctan z}{z},$$

$$F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^{2}\right)=\sum_{n=0}^{\infty}C\left(\frac{1}{2},\frac{3}{2};n\right)C\left(\frac{1}{2},1;n\right)\frac{z^{2n}}{6^{n}}=\frac{\arcsin z}{z},$$

$$F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};-z^{2}\right)=\sum_{n=0}^{\infty}C\left(\frac{1}{2},\frac{3}{2};n\right)C\left(\frac{1}{2},1;n\right)\frac{\left(-z^{2}\right)^{n}}{6^{n}}$$

$$=\sum_{n=0}^{\infty}(-1)^{n}C\left(\frac{1}{2},\frac{3}{2};n\right)C\left(\frac{1}{2},1;n\right)\frac{z^{2n}}{6^{n}}=\frac{\ln(z+\sqrt{1+z^{2}}\,)}{z}.$$

These equalities can be further arranged as

$$\sum_{n=0}^{\infty} C(a,1;n)z^n = \frac{a^a}{(a-z)^a},$$

$$\sum_{n=0}^{\infty} C(1,2;n)z^n = -\frac{\ln(1-2z)}{2z},$$

$$\sum_{n=0}^{\infty} C\left(\frac{1}{2},\frac{1}{3};n\right)z^n = \frac{1}{2\sqrt{3z}}\ln\frac{1+\sqrt{3z}}{1-\sqrt{3z}},$$

$$\sum_{n=0}^{\infty} (-1)^n C\left(\frac{1}{2},\frac{1}{3};n\right)z^n = \frac{\arctan\sqrt{3z}}{\sqrt{3z}},$$

$$\sum_{n=0}^{\infty} C\left(\frac{1}{2},\frac{3}{2};n\right)C\left(\frac{1}{2},1;n\right)z^n = \frac{\arcsin\sqrt{6z}}{\sqrt{6z}},$$

$$\sum_{n=0}^{\infty} (-1)^n C\left(\frac{1}{2},\frac{3}{2};n\right)C\left(\frac{1}{2},1;n\right)z^n = \frac{\ln\left(\sqrt{6z}+\sqrt{1+6z}\right)}{\sqrt{6z}}.$$

3.4.

Exercise 5.2 in [26, p. 128] states that, for $|x|<1$,

$$F\left(\frac{1}{2},\frac{1}{2};1;x^2\right) = \frac{2}{\pi}\int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1-x^2\sin^2\phi}}$$

and

$$F\left(-\frac{1}{2},\frac{1}{2};1;x^2\right) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{1-x^2\sin^2\phi}\,\mathrm{d}\phi.$$

From (2.2) or (2.6), it follows that, for $|x|<\frac{1}{2}$,

$$\sum_{n=0}^{\infty}\left[C\left(\frac{1}{2},1;n\right)\right]^2 x^{2n} = \frac{2}{\pi}\int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1-4x^2\sin^2\phi}}$$

and

$$\sum_{n=0}^{\infty} (-1)^n C\left(-\frac{1}{2},1;n\right)C\left(\frac{1}{2},1;n\right)x^{2n} = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{1-4x^2\sin^2\phi}\,\mathrm{d}\phi.$$

3.5.

Exercise 5.4 in [26, p. 129] gives

$$F\left(a,-a;\frac{1}{2};\sin^2 t\right) = \cos(2at)$$

and

$$F\left(a,1-a;\frac{1}{2};-\sinh^2 t\right) = \frac{\cosh[(2a-1)t]}{\cosh t}.$$

From (2.2) or (2.6), it follows that

$$\sum_{n=0}^{\infty}(-2)^n C\biggl(a,\frac{1}{2};n\biggr)C(-a,1;n)(a\sin t)^{2n}=\cos(2at)$$

and

$$\sum_{n=0}^{\infty}C\biggl(a,\frac{1}{2};n\biggr)C(1-a,1;n)[2a(a-1)]^n\sinh^{2n}t=\frac{\cosh[(2a-1)t]}{\cosh t}.$$

3.6.

Exercise 5.5 in [26, p. 129] lists

$$F(b,a;a-b+1;-1)=\frac{\sqrt{\pi}}{2^a}\frac{\Gamma(1+a-b)}{\Gamma\bigl(1+\frac{a}{2}-b\bigr)\Gamma\bigl(\frac{1}{2}+\frac{a}{2}\bigr)}$$

From (2.2) or (2.6), it follows that

$$\sum_{n=0}^{\infty}(-1)^n C(b,a-b+1;n)C(a,1;n)\biggl(\frac{ab}{a-b+1}\biggr)^n =\frac{\sqrt{\pi}}{2^a}\frac{\Gamma(1+a-b)}{\Gamma\bigl(1+\frac{a}{2}-b\bigr)\Gamma\bigl(\frac{1}{2}+\frac{a}{2}\bigr)}. \tag{3.5}$$

It is very interesting that, when letting $a=b=1$ in the above equality (3.5), we can deduce a contradiction

$$\sum_{n=0}^{\infty}(-1)^n=\frac{1}{2}.$$

3.7.

Exercise 5.10 in [26, p. 131] shows

$$F\biggl(a,a+\frac{1}{2};2a;\zeta\biggr)=\frac{2^{2a-1}}{\sqrt{1-\zeta}\,\bigl(1+\sqrt{1-\zeta}\,\bigr)^{2a-1}}$$

and

$$F\biggl(a,a+\frac{1}{2};2a+1;\zeta\biggr)=\frac{2^{2a}}{\bigl(1+\sqrt{1-\zeta}\,\bigr)^{2a}}.$$

From (2.2) or (2.6), it follows that

$$\sum_{n=0}^{\infty}C(a,2a;n)C\biggl(a+\frac{1}{2},1;n\biggr)\zeta^n=\frac{2^{2a-1}}{\sqrt{1-\frac{4\zeta}{2a+1}}\,\Bigl(1+\sqrt{1-\frac{4\zeta}{2a+1}}\,\Bigr)^{2a-1}}$$

and

$$\sum_{n=0}^{\infty}C(a,2a+1;n)C\biggl(a+\frac{1}{2},1;n\biggr)\zeta^n=\frac{2^{2a}}{\Bigl(1+\sqrt{1-\frac{2\zeta}{a}}\,\Bigr)^{2a}}.$$

In particular, taking $a=\frac{1}{2}$ arrives at

$$\sum_{n=0}^{\infty} C\left(\frac{1}{2},1;n\right)\zeta^n=\frac{1}{\sqrt{1-2\zeta}}$$

and

$$\sum_{n=0}^{\infty} C_n\zeta^n=\frac{2}{1+\sqrt{1-4\zeta}}.$$

This means that the generating function of the numbers $C\left(\frac{1}{2},1;n\right)$ is $\frac{1}{\sqrt{1-2\zeta}}$ and that the last equation recovers the generating function of the Catalan numbers C_n.

3.8.

Exercise 5.11 in [26, p. 131] verifies

$$F(-n,b;c;1)=\frac{(c-b)_n}{(c)_n}, \quad n\in\mathbb{N}.$$

From (2.2) or (2.6), it follows that

$$\sum_{k=0}^{\infty}(-1)^k C(-n,c;k)C(b,1;k)\left(\frac{bn}{c}\right)^k=\frac{(c-b)_n}{(c)_n}, \quad n\in\mathbb{N}.$$

3.9.

In [24, Lemma 2.6], it was obtained that, for $0\ne|t|<1$ and $k\in\mathbb{N}$,

$$F\left(\frac{1-k}{2},\frac{2-k}{2};1-k;\frac{1}{t^2}\right) =\frac{t}{2^k\sqrt{t^2-1}}\left[\left(1+\frac{\sqrt{t^2-1}}{t}\right)^k-\left(1-\frac{\sqrt{t^2-1}}{t}\right)^k\right], \tag{3.6}$$

where we take the principal branch of the complex function $\sqrt{t^2-1}$ for $|t|<1$. In [24, Remark 6.6], it was conjectured that the range of $k\in\mathbb{N}$ in (3.6) can be extended to $k\in\mathbb{R}$. This conjecture still keeps open.

References

[1] T. Amdeberhan, X. Guan, L. Jiu, V. H. Moll, C. Vignat, *A series involving Catalan numbers: Proofs and demonstrations*, Elem. Math. **71 (3)**, 109-121, 2016. http://dx.doi.org/10.4171/EM/306

[2] W. Chammam, *Several formulas and identities related to Catalan–Qi and q-Catalan–Qi numbers*, Indian J. Pure Appl. Math. **50 (4)**, 1039-1048, 2019. https://doi.org/10.1007/s13226-019-0372-1

[3] N. I. Fuss, *Solutio quaestionis, quot modis polygonum n laterum in polygona m laterum, per diagonales resolvi queat*, Nova Acta Academiae Sci. Petropolitanae **9**, 243-251, 1791.

[4] T. Koshy, *Catalan Numbers with Applications*, Oxford University Press, Oxford, 2009.

[5] T. Koshy, Z.-G. Gao, *Convergence of a Catalan series*, College Math. J. **43 (2)**, 141-146, 2012. http://dx.doi.org/10.4169/college.math.j.43.2.141

[6] D. H. Lehmer, *Interesting series involving the central binomial coefficient*, Amer. Math. Monthly **92 (7)**, 449-457, 1985. http://dx.doi.org/10.2307/2322496
[7] M. Mahmoud, F. Qi, *Three identities of the Catalan–Qi numbers*, Mathematics **4 (2)**, 35 (7 pages), 2016 https://doi.org/10.3390/math4020035
[8] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010. http://dlmf.nist.gov/
[9] F. Qi, *An improper integral, the beta function, the Wallis ratio, and the Catalan numbers*, Probl. Anal. Issues Anal. **7 (25)**, 104-115, 2018. https://doi.org/10.15393/j3.art.2018.4370
[10] F. Qi, *Parametric integrals, the Catalan numbers, and the beta function*, Elem. Math. **72 (3)**, 103-110, 2017. https://doi.org/10.4171/EM/332.
[11] F. Qi, *Some properties of the Catalan numbers*, Ars Combin. **160**, 2022, in press. https://doi.org/10.13140/RG.2.1.4371.6321
[12] F. Qi, P. Cerone, *Some properties of the Fuss–Catalan numbers*, Mathematics **6 (12)**, 277 (12 pages), 2018. https://doi.org/10.3390/math6120277
[13] F. Qi, B.-N. Guo, *Integral representations of the Catalan numbers and their applications*, Mathematics **5 (3)**, 40 (31 pages), 2017. https://doi.org/10.3390/math5030040
[14] F. Qi, B.-N. Guo, *Logarithmically complete monotonicity of a function related to the Catalan–Qi function*, Acta Univ. Sapientiae Math. **8 (1)**, 93-102, 2016. http://dx.doi.org/10.1515/ausm-2016-0006
[15] F. Qi, B.-N. Guo, *Logarithmically complete monotonicity of Catalan–Qi function related to Catalan numbers*, Cogent Math. **3**, 1179379 (6 pages), 2016. http://dx.doi.org/10.1080/23311835.2016.1179379
[16] F. Qi, B.-N. Guo, *Mathematical Analysis and Applications: Selected Topics, Some properties and generalizations of the Catalan, Fuss, and Fuss–Catalan numbers (Chapter 5) First Edition, Published by John Wiley & Sons, Inc. , Edited by Michael Ruzhansky, Hemen Dutta, and Ravi P. Agarwal*, 101-133, 2018. https://doi.org/10.1002/9781119414421.ch5
[17] F. Qi, M. Mahmoud, X.-T. Shi, F.-F. Liu, *Some properties of the Catalan–Qi function related to the Catalan numbers*, SpringerPlus **5**, 1126 (20 pages), 2016. https://doi.org/10.1186/s40064-016-2793-1
[18] F. Qi, D.-W. Niu, D. Lim, B.-N. Guo, *Closed formulas and identities for the Bell polynomials and falling factorials*, Contrib. Discrete Math. **14 (1)**, in press, 2019. https://doi.org/10.13140/RG.2.2.34679.52640
[19] F. Qi, X.-T. Shi, P. Cerone, *A unified generalization of the Catalan, Fuss, and Fuss–Catalan numbers*, Math. Comput. Appl. **24 (2)**, 49 (16 pages), 2019. https://doi.org/10.3390/mca24020049
[20] F. Qi, X.-T. Shi, F.-F. Liu, *An integral representation, complete monotonicity, and inequalities of the Catalan numbers*, Filomat **32 (2)**, 575-587, 2018. https://doi.org/10.2298/FIL1802575Q
[21] F. Qi, X.-T. Shi, M. Mahmoud, F.-F. Liu, *Schur-convexity of the Catalan–Qi function related to the Catalan numbers*, Tbilisi Math. J. **9 (2)**, 141-150, 2016. https://doi.org/10.1515/tmj-2016-0026
[22] F. Qi, X.-T. Shi, M. Mahmoud, and F.-F. Liu, *The Catalan numbers: a generalization, an exponential representation, and some properties*, J. Comput. Anal. Appl. **23 (5)**, 937-944, 2017.
[23] F. Qi, Y.-H. Yao, *Simplifying coefficients in differential equations for generating function of Catalan numbers*, J. Taibah Univ. Sci. **13 (1)**, 947-950, 2019. https://doi.org/10.1080/16583655.2019.1663782

[24] F. Qi, Q. Zou, B.-N. Guo, *The inverse of a triangular matrix and several identities of the Catalan numbers*, Appl. Anal. Discrete Math. **13 (2)**, 518-541, 2019. https://doi.org/10.2298/AADM190118018Q

[25] R. P. Stanley, *Catalan Numbers*, Cambridge University Press, New York, 2015. https://doi.org/10.1017/CBO9781139871495

[26] N. M. Temme, *Special Functions: An Introduction to Classical Functions of Mathematical Physics*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996. http://dx.doi.org/10.1002/9781118032572

[27] L. Yin, F. Qi, *Several series identities involving the Catalan numbers*, Trans. A. Razmadze Math. Inst. **172 (3)**, 466-474, 2018. https://doi.org/10.1016/j.trmi.2018.07.001

[28] Q. Zou, *Analogues of several identities and supercongruences for the Catalan–Qi numbers*, J. Inequal. Spec. Funct. **7 (4)**, 235-241, 2016.

[29] Q. Zou, *The q-binomial inverse formula and a recurrence relation for the q-Catalan–Qi numbers*, J. Math. Anal. **8 (1)**, 176-182, 2017.

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