Montes Taurus J. Pure Appl. Math.
1 (2), 13-20, 2019
Article ID : MTJPAM-D-19-00008

A SIMPLE PROOF OF A BINOMIAL IDENTITY WITH APPLICATIONS

TING-TING BAI AND QIU-MING LUO*

Abstract. Peterson [Amer. Math. Monthly, 120 (2013), 558–562] gave a probabilistic proof of a binomial identity. In this paper, by using the partial fraction decomposition, we give a simple proof of this binomial identity. As some applications, we obtain some interesting harmonic number identities.
Keywords: Binomial identity, Harmonic number, Bell polynomial, Partial fraction decomposition.
MSC(2010): Primary 05A10; Secondary 05A19, 11B65.

1. **Introduction**

For $n, r \in \mathbb{N} = \{1, 2, \cdots\}$ and $z \in \mathbb{C}$.

The classical harmonic numbers H_n and higher order harmonic numbers $H_n^{(r)}$ are defined respectively (see [3]),

$$H_0 = 1, \qquad H_n = \sum_{k=1}^{n} \frac{1}{k}$$

and

$$H_0^{(r)} = 1, \qquad H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}.$$

Clearly,

$$H_n = H_n^{(1)}.$$

J. Choi, H. M. Srivastava, T. M. Rassias have studied some summation formulas and classes of infinite series and generalized harmonic numbers. They defined the generalized higher order harmonic numbers as follows (see [1], [2] and [6])

$$H_n^{(r)}(z) = \sum_{k=1, k\neq -z}^{n} \frac{1}{(k+z)^r}. \tag{1.1}$$

Date: Received: 5 December 2019, Accepted: 12 December 2019.
*Corresponding author.

$$H_0^{(r)}(z) := \begin{cases} 0, & \text{when } z \neq 0, \\ 1, & \text{when } z = 0. \end{cases}$$

Obviously,

$$H_n^{(r)}(0) = H_n^{(r)}.$$

The standard Bell polynomials are displayed in Comtet's book (see [3,p.133–134]). We below give a variant of the Bell polynomials, i.e., the *modifield Bell polynomials* $\mathbf{L}_n(x_1, x_2, \cdots)$ which are defined by

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k}\right) = 1 + \sum_{n=1}^{\infty} \mathbf{L}_n(x_1, x_2, \cdots) z^n. \tag{1.2}$$

The expansion above starts as

$$1 + x_1 z + \left(\frac{x_2}{2} + \frac{x_1^2}{2}\right) z^2 + \left(\frac{x_3}{3} + \frac{x_1 x_2}{2} + \frac{x_1^3}{6}\right) z^3$$
$$+ \left(\frac{x_4}{4} + \frac{x_1 x_3}{3} + \frac{x_2^2}{8} + \frac{x_2 x_1^2}{4} + \frac{x_1^4}{24}\right) z^4 + \cdots,$$

which fixes the first few values, the general formula being

(1.3)
$$\mathbf{L}_n(x_1, x_2, \cdots) = \sum_{m_1 + 2m_2 + 3m_3 + \cdots = n} \frac{1}{m_1! m_2! m_3! \cdots} \left(\frac{x_1}{1}\right)^{m_1} \left(\frac{x_2}{2}\right)^{m_2} \left(\frac{x_3}{3}\right)^{m_3} \cdots.$$

The following binomial identity is the well-known: for $x > 0$ and $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x}{x+k} = \prod_{k=1}^{n} \frac{k}{x+k}. \tag{1.4}$$

Recently, by the probabilistic method, Peterson [5] proved the following binomial, for $r, n \in \mathbb{N}$; $x > 0$,

(1.5)
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{x}{x+k}\right)^r =$$
$$\left(\prod_{k=1}^{n} \frac{k}{x+k}\right) \left(1 + \sum_{j=1}^{r-1} \sum_{1 \le k_1 \le \cdots \le k_j \le n} \frac{x^j}{(x+k_1)(x+k_2)\cdots(x+k_j)}\right).$$

The goal of this note is to give a very simple and elementary proof of the binomial identity above using the partial fraction decomposition. As some applications, we obtain some interesting binomial identity involving the harmonic numbers.

2. Theorem and proof

First we need the following lemma.

Lemma 2.1. *For* $n,\ r\in\mathbb{N}_0,\ x>0$, *we have*

$$\frac{n!}{z(z-1)\cdots(z-n)}\left(\frac{x}{z+x}\right)^r$$

$$=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\left(\frac{x}{x+k}\right)^r\frac{1}{z-k}+\frac{\lambda}{(x+z)^r}+\cdots+\frac{\mu}{x+z}. \tag{2.1}$$

Proof. By means of the standard partial fraction decomposition, we have

$$f(z)=\frac{n!}{z(z-1)\cdots(z-n)}\left(\frac{x}{z+x}\right)^r=\sum_{k=0}^{n}\frac{A_k}{z-k}+\frac{\lambda}{(x+z)^r}+\cdots+\frac{\mu}{x+z},$$

where the coefficients A_k remain to be determined.

$$\begin{aligned}A_k&=\lim_{z\to k}(z-k)f(z)\\&=\lim_{z\to k}(z-k)\frac{n!}{z(z-1)\cdots(z-n)}\left(\frac{x}{z+x}\right)^r\\&=\lim_{z\to k}\frac{n!}{z\cdots(z-k+1)(z-k-1)\cdots(z-n)}\left(\frac{x}{z+x}\right)^r\\&=(-1)^{n-k}\binom{n}{k}\left(\frac{x}{x+k}\right)^r.\end{aligned}$$

This completes the proof. □

Theorem 2.2. *For* $n,\ r\in\mathbb{N}_0,\ x>0$, *we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{x}{x+k}\right)^r=$$

$$\left(\prod_{k=1}^{n}\frac{k}{x+k}\right)\sum_{m_1+2m_2+3m_3+\cdots=r-1}\frac{x^{r-1}}{m_1!m_2!m_3!\cdots} \tag{2.2}$$

$$\times\left(\frac{H_{n+1}^{(1)}(x-1)}{1}\right)^{m_1}\left(\frac{H_{n+1}^{(2)}(x-1)}{2}\right)^{m_2}\left(\frac{H_{n+1}^{(3)}(x-1)}{3}\right)^{m_3}\cdots,$$

where $H_n^{(r)}(z)$ *is the generalized higher order harmonic numbers.*

Proof. Multiplying the both sides of (2.1) by z, and then let $z\to\infty$, we obtain

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\left(\frac{x}{x+k}\right)^r+\mu=0. \tag{2.3}$$

By (1.2), (1.3) and (2.1), we have

$$\begin{aligned}
\mu =&[(x+z)^{-1}]\frac{n!}{z(z-1)\cdots(z-n)}\left(\frac{x}{z+x}\right)^r\\
=&n![(x+z)^{r-1}]\frac{x^r}{z(z-1)\cdots(z-n)}\\
=&(-1)^{n+1}n!\left(\prod_{k=1}^{n}\frac{1}{x+k}\right)x^{r-1}[z^{r-1}]\frac{1}{\left(1-\frac{z}{x}\right)\left(1-\frac{z}{x+1}\right)\cdots\left(1-\frac{z}{x+n}\right)}\\
=&(-1)^{n+1}n!\left(\prod_{k=1}^{n}\frac{1}{x+k}\right)x^{r-1}[z^{r-1}]\exp\Bigg(-\log\left(1-\frac{z}{x}\right)-\\
&\log\left(1-\frac{z}{x+1}\right)-\cdots-\log\left(1-\frac{z}{x+n}\right)\Bigg)\\
=&(-1)^{n+1}n!\left(\prod_{k=1}^{n}\frac{1}{x+k}\right)x^{r-1}[z^{r-1}]\times\\
&\exp\left\{\sum_{k\geq1}\left[\left(\frac{1}{x^k}+\frac{1}{(x+1)^k}+\cdots+\frac{1}{(x+n)^k}\right)\right]\frac{z^k}{k}\right\}\\
=&(-1)^{n+1}\left(\prod_{k=1}^{n}\frac{k}{x+k}\right)x^{r-1}[z^{r-1}]\exp\left(\sum_{k\geq1}H_{n+1}^{(k)}(x-1)\frac{z^k}{k}\right)\\
=&(-1)^{n+1}\left(\prod_{k=1}^{n}\frac{k}{x+k}\right)x^{r-1}\sum_{m_1+2m_2+3m_3+\cdots=r-1}\frac{1}{m_1!m_2!m_3!\cdots}\\
&\times\left(\frac{H_{n+1}^{(1)}(x-1)}{1}\right)^{m_1}\left(\frac{H_{n+1}^{(2)}(x-1)}{2}\right)^{m_2}\left(\frac{H_{n+1}^{(3)}(x-1)}{3}\right)^{m_3}\cdots.
\end{aligned}$$

This proof is complete. □

3. Applications

Below we obtain some interesting binomial identities including the harmonic numbers by applying the formulas (2.2).

Theorem 3.1. *For $n, r, M \in \mathbb{N},\ x \geq -1,$ we have*

(3.1)
$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \left(H_{M+k}^{(r)}(x+1) - H_k^{(r)}(x+1)\right)$$
$$= \sum_{j=1}^{M} \left(\prod_{k=0}^{n} \frac{1}{k+j+x+1}\right) \sum_{m_1+2m_2+3m_3+\cdots=r-1} \frac{n!}{m_1!m_2!m_3!\cdots}$$
$$\times \left(\frac{H_{n+1}(x+j)}{1}\right)^{m_1} \left(\frac{H_{n+1}^{(2)}(x+j)}{2}\right)^{m_2} \left(\frac{H_{n+1}^{(3)}(x+j)}{3}\right)^{m_3} \cdots .$$

Proof. Letting $x \longmapsto x+1$ in (2.2). We have

(3.2)
$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{(x+k+1)^r} = \prod_{k=0}^{n} \frac{1}{x+k+1} \sum_{m_1+2m_2+3m_3+\cdots=r-1} \frac{n!}{m_1!m_2!m_3!\cdots}$$
$$\times \left(\frac{H_{n+1}(x)}{1}\right)^{m_1} \left(\frac{H_{n+1}^{(2)}(x)}{2}\right)^{m_2} \left(\frac{H_{n+1}^{(3)}(x)}{3}\right)^{m_3} \cdots .$$

Applying (3.2) $(x \longmapsto x+j)$, we get

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \left(H_{M+k}^{(r)}(x+1) - H_k^{(r)}(x+1)\right)$$
$$= \sum_{k=0}^{n}(-1)^k \binom{n}{k} \sum_{j=1}^{M} \frac{1}{(x+k+j+1)^r}$$
$$= \sum_{j=1}^{M} \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{(x+k+j+1)^r}$$
$$= \sum_{j=1}^{M} \left(\prod_{k=0}^{n} \frac{1}{x+k+j+1}\right) \sum_{m_1+2m_2+3m_3+\cdots=r-1} \frac{n!}{m_1!m_2!m_3!\cdots}$$
$$\times \left(\frac{H_{n+1}(x+j)}{1}\right)^{m_1} \left(\frac{H_{n+1}^{(2)}(x+j)}{2}\right)^{m_2} \left(\frac{H_{n+1}^{(3)}(x+j)}{3}\right)^{m_3} \cdots .$$

This proof is complete. □

Setting $x = -1$ in (3.1), we easily arrive at

Corollary 3.2. *For $n, r, M \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \left(H_{M+k}^{(r)} - H_k^{(r)}\right) \tag{3.3}$$
$$= \sum_{j=1}^{M} \left(\prod_{k=0}^{n} \frac{1}{k+j}\right) \sum_{m_1+2m_2+3m_3+\cdots=r-1} \frac{n!}{m_1!m_2!m_3!\cdots}$$
$$\times \left(\frac{H_{n+1}(j-1)}{1}\right)^{m_1} \left(\frac{H_{n+1}^{(2)}(j-1)}{2}\right)^{m_2} \left(\frac{H_{n+1}^{(3)}(j-1)}{3}\right)^{m_3} \cdots .$$

Next we give some special cases of the binomial identity above.

Case 1. Taking $r = 1$ in (3.3), we get the following binomial identities including the harmonic numbers

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} (H_{M+k} - H_k) = \sum_{j=1}^{M} \frac{1}{j\binom{n+j}{j}}. \tag{3.4}$$

Let $M \to \infty$ in (3.4). Applying the formula [4,p.19, (2.11)]

$$\sum_{k=1}^{\infty} \frac{1}{k\binom{n+k}{k}} = \frac{1}{n},$$

we obtain the following familiar formula

$$\sum_{k=0}^{n}(-1)^{k-1} \binom{n}{k} H_k = \frac{1}{n}.$$

Taking $M = 1$ in (3.4), we have

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{k+1} = \frac{1}{n+1}.$$

Taking $M = n$ in (3.4), we have

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} (H_{n+k} - H_k) = \sum_{j=1}^{n} \frac{1}{j\binom{n+j}{j}}.$$

Case 2. Taking $r = 2$ in (3.3), we obtain the following binomial identities including the harmonic numbers of order 2

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \left(H_{M+k}^{(2)} - H_k^{(2)}\right) = \sum_{j=1}^{M} \frac{H_{n+1}(j-1)}{j\binom{n+j}{j}}. \tag{3.5}$$

Taking $M = 1$ in (3.5), we deduce that

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{(k+1)^2} = \frac{H_{n+1}}{n+1}.$$

Taking $M = n$ in (3.5), we get the following identity of harmonic number.

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \left(H_{n+k}^{(2)} - H_k^{(2)}\right) = \sum_{j=1}^{n} \frac{H_{n+1}(j-1)}{j\binom{n+j}{j}}.$$

Case 3. Taking $r = 3$ in (3.3), we get the following binomial identities including the harmonic numbers of order 3

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \left(H_{M+k}^{(3)} - H_k^{(3)}\right) = \sum_{j=1}^{M} \frac{H_{n+1}^2(j-1) + H_{n+1}^{(2)}(j-1)}{2j\binom{n+j}{j}}. \tag{3.6}$$

Taking $M = 1$ in (3.6), we get

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{(k+1)^3} = \frac{H_{n+1}^2 + H_{n+1}^{(2)}}{2(n+1)}.$$

Taking $M = n$ in (3.6), we have the following identity of harmonic number.

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \left(H_{n+k}^{(3)} - H_k^{(3)}\right) = \sum_{j=1}^{n} \frac{H_{n+1}^2(j-1) + H_{n+1}^{(2)}(j-1)}{2j\binom{n+j}{j}}.$$

Case 4. Taking $M = 1$ in (3.3), we deduce that

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{(k+1)^r} =$$

$$\frac{1}{n+1} \sum_{m_1+2m_2+3m_3+\cdots=r-1} \frac{1}{m_1!m_2!m_3!\cdots} \left(\frac{H_{n+1}}{1}\right)^{m_1} \left(\frac{H_{n+1}^{(2)}}{2}\right)^{m_2} \left(\frac{H_{n+1}^{(3)}}{3}\right)^{m_3} \cdots.$$

Acknowledgments

The present investigation was supported by *Natural Science Foundation General Project of Chongqing, China* under Grant cstc2019jcyj-msxmX0143.

References

[1] J. Choi, *Certain summation formulas involving harmonic numbers and generalized harmonic numbers*, Applied Mathematics and Computation **218**, 734-740, 2011.

[2] J. Choi, H. M. Srivastava, *Some summation formulas involving harmonic numbers and generalized harmonic numbers*, Mathematical and Computer Modelling **54**, 2220-223, 2011.

[3] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Springer, Reidel, Dordrecht and Boston, 1974.

[4] H.W. Gould, *Combinatorial Identities: Astandardized Set of Tables Listing 500 Binomial Coefficient Summations*, Morgantown, W. Va. 1972.

[5] J. Peterson, *A probabilistic proof of a binomial identity*, Amer. Math. Monthly **120**, 558-562, 2013.

[6] T. M. Rassias, H. M. Srivastava, *Some classes of infinite series associated with the Riemann Zeta and Polygamma functions and generalized harmonic numbers*, Applied Mathematics and Computation **131**, 593-605, 2002.

(Ting-Ting Bai) Department of Mathematics, Chongqing Normal University Chongqing Higher Education Mega Center, Huxi Campus Chongqing 401331, People's Republic of China

E-mail address: baitt@163.com

(Qiu-Ming Luo) Department of Mathematics, Chongqing Normal University Chongqing Higher Education Mega Center, Huxi Campus Chongqing 401331, People's Republic of China

E-mail address: luomath2007@163.com