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ON CERTAIN SOLUTIONS OF A GENERALIZED PERL'S VECTOR EQUATION INVOLVING FRACTIONAL TIME DERIVATIVE

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Abstract. In this paper, we extricate a generalized Perl's vector equation of heat and matter distribution in body tissue by degenerating it into an equation involving several space-dimensional Sturm-Liouville operators along with fractional time derivative. Then we evaluate some of its solutions by imposing different initial and boundary conditions. The analytical and numerical studies of this problem has revealed interesting properties, which in some sense can be regarded as an extension of the properties of the special functions like Voigt functions, Lauricella functions and Sturve functions on using certain improper integral transformations. In this connection the relevance of Voigt functions, Lauricella's functions and Sturve functions and their multivariable extensions in mathematical physics has been emphasized.
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1. Introduction

In [4] Evans studied and analyzed solutions of the vector differential equations by transforming them into linear and non - linear partial differential equations in to problems in a multidimensional space by imposing some boundary conditions. Some remarkable problems of mathematical physics are also studied and solved by Churchill [2] in 1972. At the present time,the Sturm - Liouville theory ([5], [6], [19], [32]) with fractional calculus techniques ([3], [9], [20], [24], [26]) has been utilized by many researchers (for example [7], [8], [10], [12] - [17]) for solving of various diffusion and wave problems. In this connection, to explore new ideas for representing the development in the theory of Voigt functions, Sturve functions and Lauricella's functions, we obtain different special functions of mathematical physics by providing an unifying view to the theory of Sturm - Liouville operators, fractional calculus and special functions. It covers extremely wide domain of study by formulating the problems consisting of multidimensional Sturm - Liouville operators and fractional time derivative with different initial and

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boundary conditions, and their solutions produce continuous new applications and achievements.

The generalized form of Perl's vector equation [22] involving fractional time derivative, for study of heat and matter distribution in body tissue or determination of tissue blood flow by local clearance, is presented as

$$\rho\bar{c}\, {}^{C}_{t}D^{\alpha}_{0+} u = div[K\ grad(u)] + m_b c_b(u_A - u) + S, \alpha > 0. \tag{1.1}$$

Here, in Eqn. (1.1),α is the order of the derivative, u is the temperature distribution at a point in vivo tissue at time $t, t > 0, \rho$ is the density of the tissue, the thermal coefficients K and $\bar{c}$ are either constants or independent of time t, where, $\bar{c}$ the specific heat and K the thermal conductivity of that tissue; m_b and c_b are the blood mass flow rate and specific heat of blood, respectively. S is the rate of metabolic heat generation per unit volume and u_A the arterial blood temperature.

Clearly, putting $\alpha = 1$ in the generalized vector equation (1.1), it becomes Perl's vector equation [22]. Again, in Eqn. (1.1), setting $\alpha = 1$ and taking gradient in three dimensional space,we get Churchill's diffusion problem [2] of a solid which is solved by Duhamel's principle, given by

$$\frac{1}{\bar{c}}\left[\frac{\partial}{\partial x}\left(K\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial U}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial U}{\partial z}\right)\right] = \frac{\partial u}{\partial t},\ [\forall(x,y,z) \in \Omega, t > 0]. \tag{1.2}$$

Here, $U = U(x,y,z,t)$ is the temperature in the region Ω of that solid at time $t, t > 0$, where, the thermal coefficients K and $\bar{c}$ are either constants or independent of time $t, t > 0$, and the boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2,\ \partial\Omega_1 \cap \partial\Omega_2 = \phi$ (null set); with initial and boundary conditions
$U(x,y,z,0) = 0$, interior to Ω that is $[\forall(x,y,z) \in \Omega$, at $t = 0]$ and

$$U(x,y,z,t) = \begin{cases} F(t), [\forall(x,y,z) \in \partial\Omega_1 \text{ (some part of the boundary)}, t > 0], \\ 0, [\forall(x,y,z) \in \partial\Omega_2 \text{ (rest part of the boundary)}, \ t > 0]; \end{cases}$$

Again, on supposing that $U(x,y,z,t) = V(x,y,z,t)$, whenever, $V(x,y,z,0) = 0$, and

$$V(x,y,z,t) = \begin{cases} 1, [\forall(x,y,z) \in \partial\Omega_1, t > 0], \\ 0, [\forall(x,y,z) \in \partial\Omega_2, t > 0]; \end{cases}, \tag{1.3}$$

and thus obtained solution $[\forall(x,y,z) \in \Omega, t > 0]$, is given in Churchill [2] as.

$$U(x,y,z,t) = \int_0^t F(t-\tau)\frac{\partial}{\partial\tau}V(x,y,z,\tau)d\tau. \tag{1.4}$$

Now, we generalize the Eqns. (1.1) to (1.3) and present a solvable multivariable diffusion - wave problem involving fractional time derivative as

$$\left[\frac{\partial}{\partial x_1}\left(p_1(x_1)\frac{\partial U}{\partial x_1}\right)+...+\frac{\partial}{\partial x_k}\left(p_k(x_k)\frac{\partial U}{\partial x_k}\right)-\sum_{r=1}^{k}q_r(x_r)U\right] + F(x_1,...,x_k,t)= {}_t^{\mathsf{C}}D_{0+}^{\alpha}U \tag{1.5}$$

Here, in Eqn. (1.5), $0<\alpha\le 2, U=U(x_1,...,x_k,t)$ be the temperature distribution or wave propagation in any solid, filling a region $\Omega\ \forall(x_1,...,x_k)\in\Omega$ at any time $t, t>0$. Initially, in solid the temperature and its time derivative are zero throughout, that is $U(x_1,...,x_k,0)=0$ and

$\partial/\partial t U(x_1,...,x_k,t)|_{t=0}=0$. Also, at some points of tiny part of the boundary $\partial\Omega_1$ at time $t,\ t>0$;

$U(a_1,...,x_k,t)=0=U(l_1,...,x_k,t), U(x_1,a_2,...,x_k,t)=0$
$=U(x_1,l_2,...,x_k,t),...,U(x_1,...,a_{k-1},x_k,t)=0=U(x_1,...,l_{k-1},x_k,t),$
$U(x_1,...,a_k,t)=0=U(x_1,...,l_k,t);\forall a_1\le x_1\le l_1, a_2\le x_2\le l_2,...,$
$a_{k-1}\le x_{k-1}\le l_{k-1}, a_k\le x_k\le l_k;\forall r=1,2,...,k, l_r>a_r;$

Also, on rest part of the boundary $\partial\Omega_2$, the temperature obeys the ruling $U(x_1,...,x_k,t)=F(x_1,...,x_k,t)$, at time $t,t>0$. The boundary $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2, \partial\Omega_1\cap\partial\Omega_2=\phi$. The thermal coefficients $p_1(x_1),...,p_k(x_k)$ are independent of time t.

Also in Eqn. (1.5), the Caputo fractional derivative ${}_t^{\mathsf{C}}D_{0+}^{\alpha}$, $m-1<\alpha\le m$, of function $Y(t)$ is defined by (see Diethelm [3, p. 49])

$$\left({}_t^{\mathsf{C}}D_{0+}^{\alpha}Y\right)(t)=\left(I^{m-\alpha}Y^{(m)}\right)(t), \forall m\in\mathbb{N},$$

where, $Y^{(m)}(t)=\frac{d^mY}{dt^m}(t)$, $I^{m-\alpha}$ being the Riemann - Liouville fractional integral

$$\left(I^{m-\alpha}Y\right)(t)=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-\alpha-1}Y(\tau)\,d\tau, t>0, m-1<\alpha\le m,\\ Y(t), \alpha=m, \forall m\in\mathbb{N}.\end{cases} \tag{1.6}$$

In this work, we also use the Laplace transformation of the function defined in Eqn. (1.6), for $L[Y(t)]=\overline{Y}(s), s>0$, given by (Kilbas, Srivastava and Trujillo [9, p. 312])

$$L\left[\left({}_t^{\mathsf{C}}D_{0+}^{\alpha}Y\right)(t)\right]=s^{\alpha}\overline{Y}(s)-s^{\alpha-1}Y(0)-s^{\alpha-2}Y^{(1)}(0)-...-s^{\alpha-m}Y^{(m-1)}(0), \quad m-1<\alpha\le m \tag{1.7}$$

For the theory and analysis of the fractional differential equations, we refer the work of the researchers including authors (e. g. [3], [7], [8], [9], [18], [20], [24] and [26]).

It may be observed that for $\alpha=1$, and $k=1$ the equation (1.5) converts into a linear second order parabolic partial differential equation and a diffusion problem with initial and boundary conditions. Again, for $\alpha=2, k=1$, the equation (1.5) reduces to a linear second order elliptic partial differential equation of wave problem with given initial and boundary conditions (see Evans [4]). On the other hand, when $0 < \alpha \le 1, k = 1$ above problem (1.5) becomes identical to the initial - boundary value problem for the one dimensional time fractional diffusion equation because of the availability of the vast literature due to researchers (e. g. [7], [8], [10], [12], [15], [16], [17] and [18]). The computation of anomalous diffusion problems has been obtained in the form of integral equations ([13] and [14]). Consequently, reformulating a physical problem in terms of special functions, allows for a more elegant mathematical model. Then for easier handling of the relevant expected value of any function due to the potential function and fractional time derivative evolution diffusion and wave problems consisting of different initial and boundary conditions, we first separate above Eqn. (1.5) into various problems consisting of different space variable Sturm - Liouville operator with fractional time derivative by setting the initial and boundary conditions and then find out the series solution by using the Green function in the form of the Mercer formula [23]. The Sturm - Liouville problems are referred to (e. g. [5], [6], [12 - 17], [19], [32]).

2. **Solution of the Problem (1.5)**

In this section, we solve the problem (1.5) on degenerating it in separate space variables on supposing that $U(x_1, ..., x_k, t) = \prod_{i=1}^{k} Y(x_i, t)$, and $F(x_1, ..., x_k, t) = \sum_{r=1}^{k} Q_r(x_r, t) \prod_{i=1, i\neq r}^{k} Y(x_i, t)$. Then, we prove:

Theorem 2.1. *If $U(x_1, ..., x_k, t) = \prod_{i=1}^{k} Y(x_i, t)$ and $F(x_1, ..., x_k, t)$ $= \sum_{r=1}^{k} Q_r(x_r, t) \prod_{i=1, i\neq r}^{k} Y(x_i, t)$, then from the Eqn. (1.5), the difference of fractional time derivative and Sturm - Liouville operators operates $Y(x_r, t)$ to $Q_r(x_r, t)\, \forall r = 1, 2, ..., k, \alpha > 0, t > 0$.*

Proof. Introducing the assumptions of the Theorem 2.1 in the Eqn. (1.5), yields the form

$$\prod_{i=1, i\neq r}^{k} Y(x_i, t) \left\{ \left[\frac{\partial}{\partial x_r} \left(p_r(x_r) \frac{\partial}{\partial x_r} \right) - q_r(x_r) \right] Y(x_r, t) \right\} + Q_r(x_r, t) \prod_{i=1, i\neq r}^{k} Y(x_i, t) = \prod_{i=1, i\neq r}^{k} Y(x_i, t)\, {}_t^C D_{0+}^{\alpha} Y(x_r, t) \quad \forall r = 1, 2, ..., k, \alpha > 0. \tag{2.1}$$

Then, the Eqn. (2.1) gives us

$$\left\{ {}_t^C D_{0+}^{\alpha} - \left[\frac{\partial}{\partial x_r} \left(p_r(x_r) \frac{\partial}{\partial x_r} \right) - q_r(x_r) \right] \right\} Y(x_r, t) = Q_r(x_r, t)\, \forall r = 1, 2, ..., k, t > 0, \alpha > 0. \tag{2.2}$$

Hence, the theme of the Theorem has been proved. □

Theorem 2.2. *Let in Eqn. (1.5),* $U(x_1,...,x_k,t) = \prod_{i=1}^{k} Y(x_i,t)$ *and* $F(x_1,...,x_k,t) = \sum_{r=1}^{k} Q_r(x_r,t) \prod_{i=1,i\neq r}^{k} Y(x_i,t)$ *, with the initial and boundary conditions*

$$Y(x_i,0) = 0, \frac{\partial}{\partial t} Y(x_i,0) = 0 \forall i = 1,2,3,...,k;$$

$$Y(a_i,t) = 0 = Y(l_i,t), a_i \leq x_i \leq l_i \forall i = 1,2,...k, 0 < \alpha \leq 2, t > 0. \tag{2.3}$$

Then the solution of the problem (1.5) with boundary conditions (2.3), is given by

$$U(x_1,...,x_k,t) = \sum_{m=1}^{\infty} \prod_{r=1}^{k} \frac{Y(x_r, s_{m_r}^{\alpha})}{\left\{\int_{a_r}^{l_r} \left(Y(\theta_r, s_{m_r}^{\alpha})\right)^2 d\theta_r\right\}} \times \int_0^t \int_{a_r}^{l_r} e_{\alpha}^{s_{m_r}^{\alpha}(t-\tau)} Y(\varrho_r, s_{m_r}^{\alpha}) Q_r(x_r,\tau) d\varrho_r d\tau. \tag{2.4}$$

Moreover, another solution is also given by

$$U(x_1,...,x_k,t) = \sum_{m=1}^{\infty} \prod_{r=1}^{k} \frac{Y(x_r, s_{m_r}^{\alpha})}{\left\{\int_{a_r}^{l_r} \left(Y(\theta_r, s_{m_r}^{\alpha})\right)^2 d\theta_r\right\}} \times \int_0^t \int_{a_r}^{l_r} e_{\alpha}^{s_{m_r}^{\alpha}\tau} Y(\varrho_r, s_{m_r}^{\alpha}) Q_r(x_r,(t-\tau)) d\varrho_r d\tau \tag{2.5}$$

where, $Y(x_r, s_{m_r}^{\alpha})$ *satisfies the Eqn. (2.7) and* α*-exponential functions* $e_{\alpha}^{\lambda z}$*, defined by* $e_{\alpha}^{\lambda z} = z^{\alpha-1}E_{\alpha,\alpha}(\lambda z^{\alpha})$*,where* $E_{\alpha,\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\alpha)}$ *is the Mittag-Leffler function, (see [9, p. 50])* $\forall r = 1,2,...,k, t > 0, s_{m_r}^{\alpha} > 0 \forall m = 1,2,3,....$

Proof. Take Laplace transformation of both sides of the Eqn. (2.2) $\forall r = 1,2,...,k, t > 0$ of the Theorem 2.1 under the boundary values (2.3) and on applying the formula (1.7),we find that

$$\mathcal{L}\bar{Y}(x_r,s) - s^{\alpha}\bar{Y}(x_r,s) = -\bar{Q}_r(x_r,s), \mathcal{L} \equiv \left[\frac{\partial}{\partial x_r}(p_r(x_r)\frac{\partial}{\partial x_r}) - q_r(x_r)\right],$$
$$L\{Y(x_r,t)\} = \bar{Y}(x_r,s),$$

with the transform prescribed conditions

$$\bar{Y}(x_r,0) = 0, \bar{Y}(a_r,s) = 0 = \bar{Y}(l_r,s), a_r \leq x_r \leq l_r \forall r = 1,2,...k, s > 0. \tag{2.6}$$

Now for any eigenvalue $s_{n_r} > 0$, let $Y(x_r, s_{n_r}^{\alpha})$ denote the real - valued eigenfunctions of the homogeneous problem

$\mathcal{L}Y(x_r, s_{n_r}^{\alpha}) - s_{n_r}^{\alpha} Y(x_r, s_{n_r}^{\alpha}) = 0$, with the prescribed conditions

$$Y(a_r, s_{n_r}^{\alpha}) = 0 = Y(l_r, s_{n_r}^{\alpha}) \forall n = 1,2,3,...,r = 1,2,3,...,k, \tag{2.7}$$

then one of the solution of problem (2.7) is $\psi^{\alpha}_{n_r}(x_r) = \frac{Y(x_r, s^{\alpha}_{n_r})}{\int_{a_r}^{l_r}\{Y(\theta_r, s^{\alpha}_{n_r})\}^2 d\theta_r} \forall a_r \leq x_r \leq l_r, r = 1, 2, 3, ..., k$ such that for $0 < \alpha \leq 2$ and $, \forall r = 1, 2, 3, ..., k$;

$$\int_{a_r}^{l_r} \psi^{\alpha}_{n_r}(\varrho_r)\psi^{\alpha}_{m_r}(\varrho_r)d\varrho_r = \begin{cases} 1, m = n; \\ 0, m \neq n. \end{cases} \tag{2.8}$$

Again, the Laplace transformation of $Y(x_r, s^{\alpha}_{n_r})$ has the equality $Y(x_r, s^{\alpha}_{n_r}) = \bar{Y}(x_r, s^{\alpha}_{n_r})$.

Then, by Eqn. (2.7), we write $\mathcal{L}\bar{Y}(x_r, s^{\alpha}_{n_r}) - s^{\alpha}_{n_r}\bar{Y}(x_r, s^{\alpha}_{n_r}) = 0$, with the prescribed conditions

$$\bar{Y}(a_r, s^{\alpha}_{n_r}) = 0 = \bar{Y}(l_r, s^{\alpha}_{n_r}) \forall n = 1, 2, 3, ..., r = 1, 2, 3, ..., k. \tag{2.9}$$

Further, $\forall r = 1, 2, 3, ..., k, s > 0$, to find series solution, we suppose the functions

$$\bar{Q_r}(x_r, s) = \sum_{n=1}^{\infty} A_{n_r}(s)\psi^{\alpha}_{n_r}(x_r), A_{n_r} \neq 0 \text{ and } \bar{Y}(x_r, s) = \sum_{n=1}^{\infty} C_{n_r}(s)\psi^{\alpha}_{n_r}(x_r). \tag{2.10}$$

Again then, introduce these functions of Eqn. (2.10) in the Eqns. (2.6) and (2.9) to get

$$\sum_{n=1}^{\infty} C_{n_r}(s)\{s^{\alpha} - s^{\alpha}_{n_r}\}\psi^{\alpha}_{n_r}(x_r)\psi^{\alpha}_{m_r}(x_r) = \sum_{n=1}^{\infty} A_{n_r}(s)\psi^{\alpha}_{n_r}(x_r)\psi^{\alpha}_{m_r}(x_r) \tag{2.11}$$

Then, to reach our goal, for $0 < \alpha \leq 2$ and $, \forall n = 1, 2, 3, ..., r = 1, 2, 3, ..., k$; we define

$$C_{n_r}(s) = \frac{A_{n_r}(s)}{[\alpha[_{s^{\alpha}_{n_r}, s}}, A_{n_r}(s) \neq 0, \text{ where, } [\alpha[_{s^{\alpha}_{n_r}, s} = \begin{cases} (s - s_{n_r})|\alpha|_{s^{\alpha}_{n_r}, s}, s > s_{n_r} \\ (s - s_{n_r})^{-2}, s = s_{n_r} \end{cases},$$

and

$$|\alpha|_{s^{\alpha}_{n_r}, s} = \frac{\{s^{\alpha} - s^{\alpha}_{n_r}\}}{(s - s_{n_r})} \tag{2.12}$$

Therefore, on integrating both sides of Eqn. (2.11) and on application of the orthogonal property (2.8) and the relations (2.12), $\forall n = 1, 2, 3, ..., r = 1, 2, 3, ..., k$, we obtain

$$C_{n_r}(s) = \begin{cases} \frac{A_{n_r}(s)}{[\alpha[_{s^{\alpha}_{n_r}, s}}, A_{n_r}(s) \neq 0, s > s_{n_r}; \\ 0, s = s_{n_r}. \end{cases} \tag{2.13}$$

Thus, with the help of Eqns. (2.10) and (2.13), for $s \geq s_{n_r}, s_{n_r} > 0$, $\forall r = 1, 2, 3, ..., k$, we obtain the solution of Eqn. (2.6) in the form

$$\bar{Y}(x_r, s) = \sum_{n=1}^{\infty} \frac{A_{n_r}(s)}{\{s^{\alpha} - s^{\alpha}_{n_r}\}}\psi^{\alpha}_{n_r}(x_r) \tag{2.14}$$

and the relation

(2.15)

$$\sum_{m=1}^{\infty}\frac{Q_r(x_r,s)}{\{s^\alpha - s^\alpha_{m_r}\}}\psi^\alpha_{m_r}(x_r)\psi^\alpha_{m_r}(\varrho_r)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_{n_r}(s)}{\{s^\alpha - s^\alpha_{n_r}\}}\psi^\alpha_{n_r}(x_r)\psi^\alpha_{m_r}(x_r)\psi^\alpha_{m_r}(\varrho_r)$$

Further, in Eqn. (2.15) interchanging m with n and then integrating both sides with respect to x_r from a_r to l_r, for $s \geq s_{n_r}, s_{n_r} > 0$, $\forall r = 1, 2, 3, ..., k$, and thus with the aid of the orthogonal property (2.8), we find

$$\begin{aligned}\bar{Y}(\varrho_r, s) &= \sum_{n=1}^{\infty}\frac{\psi^\alpha_{n_r}(\varrho_r)}{(s^\alpha - s^\alpha_{n_r})}\int_{a_r}^{l_r}\psi^\alpha_{n_r}(x_r)Q_r(x_r,s)dx_r\\ &= \int_{a_r}^{l_r} G(s, s_{n_r}; x_r, \varrho_r)Q_r(x_r,s)dx_r\end{aligned}$$

where, for all $s \geq s_{n_r}, s_{n_r} > 0$ and $\forall r = 1, 2, 3, ..., k, 0 < \alpha \leq 2$, the series,

$$\sum_{n=1}^{\infty}\frac{\psi^\alpha_{n_r}(x_r)\psi^\alpha_{n_r}(\varrho_r)}{(s^\alpha - s^\alpha_{n_r})} = G(s, s_{n_r}; x_r, \varrho_r) \text{ (let)} \tag{2.16}$$

is identical to the Mercer type Green function [23].

Now, in Eqn. (2.16) on interchanging x_r with ϱ_r and making an appeal to the result due to Kilbas, Srivastava and Trujillo [9, Eqn. 1.10.26, p. 52] and by convolution theorem, for $s_{m_r} > 0$ $\forall r = 1, 2, 3, ..., k, \alpha > 0, t > 0$, we find that

$$Y(x_r,t) = \sum_{m=1}^{\infty}\psi^\alpha_{m_r}(x_r)\int_0^t\int_{a_r}^{l_r} e_\alpha^{s^\alpha_{m_r}(t-\tau)}\psi^\alpha_{m_r}(\varrho_r)Q_r(\varrho_r,\tau)d\varrho_r d\tau \tag{2.17}$$

So that by theme of the Theorem 2.2 and with the aid of Eqn. (2.17), we obtain

(2.18)

$$U(x_1, ..., x_k, t) = \prod_{r=1}^{k}\sum_{m=1}^{\infty}\psi^\alpha_{m_r}(x_r)\int_0^t\int_{a_r}^{l_r} e_\alpha^{s^\alpha_{m_r}(t-\tau)}\psi^\alpha_{m_r}(\varrho_r)Q_r(\varrho_r,\tau)d\varrho_r d\tau$$

Another solution is given by

(2.19)

$$U(x_1, ..., x_k, t) = \prod_{r=1}^{k}\sum_{m=1}^{\infty}\psi^\alpha_{m_r}(x_r)\int_0^t\int_{a_r}^{l_r} e_\alpha^{s^\alpha_{m_r}(\tau)}\psi^\alpha_{m_r}(\varrho_r)Q_r(\varrho_r,(t-\tau))d\varrho_r d\tau$$

Finally, by the Eqns. (2.18) and (2.19), we obtain the results (2.4) and (2.5) respectively. □

Theorem 2.3. *For $0 < \alpha \leq 2$, if the multidimensional partial differential equation with fractional time derivative is given in the form*

$$\left[\frac{\partial}{\partial x_1}\left(p_1(x_1)\frac{\partial U}{\partial x_1}\right) + ... + \frac{\partial}{\partial x_k}(p_k(x_k)\frac{\partial U}{\partial x_1}) - \sum_{r=1}^{k}q_r(x_r)U\right] = {}_t{}^C D^\alpha_{0+}U,$$

where, $U = U(x_1, ..., x_k, t)$ *is the temperature distribution in the region* $\Omega \forall (x_1, ..., x_k) \in \Omega$ *at any time* $t, t > 0$.

Along with the conditions that in the region Ω,

$$U(x_1, ..., x_k, 0) = 0, \frac{\partial}{\partial t} U(x_1, ..., x_k, t)|_{t=0} = 0;$$

and on tiny part of the boundary surface $\partial\Omega_1$,

$U(a_1, ..., x_k, t) = 0 = U(l_1, ..., x_k, t), U(x_1, a_2, ..., x_k, t) = 0 = U(x_1, l_2, ..., x_k, t),$ $..., U(x_1, ..., a_{k-1}, x_k, t) = 0 = U(x_1, ..., l_{k-1}, x_k, t), U(x_1, ..., a_k, t) = 0$ $= U(x_1, ..., l_k, t); \forall a_1 \leq x_1 \leq l_1, a_2 \leq x_2 \leq l_2, ..., a_{k-1} \leq x_{k-1} \leq l_{k-1},$ $a_k \leq x_k \leq l_k$; *while in other part of the boundary surface on rest part of the boundary* $\partial\Omega_2$, *the temperature obeys the ruling*

$$U(x_1, ..., x_k, t) = 0, t > 0. \tag{2.20}$$

Then, by setting a degeneration formula $U(x_1, ..., x_k, t) = \prod_{i=1}^{k} Y(x_i, t)$, *in the problem (2.20), it terminates in k homogeneous Sturm - Liouville fractional time derivative equations*

$$\left\{ {}_t^C D_{0+}^{\alpha} - \left[\frac{\partial}{\partial x_i}(p_i(x_i)\frac{\partial}{\partial x_i}) - q_i(x_i) \right] \right\} Y(x_i, t) = 0 \forall i = 1, 2, 3, ..., k.$$

Along with the initial and boundary conditions

$$Y(x_i, 0) = 0, \frac{\partial}{\partial t} Y(x_i, t)|_{t=0} = 0; Y(a_i, t) = 0 = Y(l_i, t); \forall i = 1, 2, 3, ..., k. \tag{2.21}$$

Then, for $s_{n_i}, \forall n = 1, 2, 3, ...;\ i = 1, 2, 3, ..., k$, *and by (2.7), the solution of Eqn. (2.21) is obtained by*

$$U(x_1, ..., x_k, t) = \sum_{n=1}^{\infty} \prod_{i=1}^{k} \frac{Y(x_i, -s_{n_i}^{\alpha})}{\int_{a_i}^{l_i} \{Y(\theta_i, -s_{n_i}^{\alpha})\}^2 d\theta_i} E_{\alpha}(-s_{n_i}^{\alpha} t^{\alpha}), 0 < \alpha \leq 2. \tag{2.22}$$

Again, in the problem (2.20), at the multiple points $(b_1, ..., b_k, t)$ *of the boundary* $\partial\Omega_2$, *the temperature is filling by the ruling* $U(b_1, ..., b_k, t) = Q(t)$, *where,* $b_1 = l_1 + \delta_1, ..., b_k = l_k + \delta_k, 0 < \delta_1 < 1, ..., 0 < \delta_k < 1$.

Further setting that, when $Q(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $U(x_1, ..., x_k, t) = V(x_1, ..., x_k, t)$, *and satisfying the conditions*

$$V(x_1, ..., x_k, 0) = 0, \frac{\partial}{\partial t} V(x_1, ..., x_k, t)|_{t=0} = 0. \tag{2.23}$$

Then, the solution of the problem (2.20)-(2.23) is obtained by

$$U(x_1, ..., x_k, t) = {}_t^C D_{0+}^{\alpha} \int_0^t V(x_1, ..., x_k, \tau) Q(t - \tau) d\tau. \tag{2.24}$$

Proof. Introducing $U(x_1, ..., x_k, t) = \prod_{i=1}^{k} Y(x_i, t)$ in Eqn. (2.20) and using Theorem 2.1, it becomes the problem (2.21). Now, in the problem (2.21), for $s_{n_i} > 0, \forall n = 1, 2, 3, ...$ and for $i = 1, 2, 3, ..., k$, consider that $Y(x_i, t) = Y(x_i) E_{\alpha}(-s_{n_i}^{\alpha} t^{\alpha})$, then, it becomes equivalent to the problem (2.7) of the

Theorem 2.2 and hence, for $s_{n_i} > 0, \forall n = 1,2,3,...;\ i = 1,2,3,...,k\ 0 < \alpha \leq 2$, the solution of Eqn. (2.20) is obtained as Eqn. (2.22).

Further for $0 < \alpha \leq 2$,the Laplace transformation of the problem (2.20) gives us

(2.25)
$$\frac{1}{s^\alpha}\left[\frac{\partial}{\partial x_1}\left(p_1(x_1)\frac{\partial \bar{U}}{\partial x_1}\right) + ... + \frac{\partial}{\partial x_k}\left(p_k(x_k)\frac{\partial \bar{U}}{\partial x_k}\right) - \sum_{r=1}^{k} q_r(x_r)\bar{U}\right] = \bar{U},$$

where, $L\{U(x_1,...,x_k,t)\} = \bar{U}(x_1,...,x_k,s) = \bar{U}, L\{Q(t)\} = \bar{Q}(s), s > 0$, satisfyng the condition that

$\bar{U}(b_1,...,b_k,s) = \bar{Q}(s)$. Again, by particular case given in Eqn. (2.23), we get $s^\alpha\ \bar{V}(b_1,...,b_k,s)\bar{Q}(s) = \bar{Q}(s)$.

Then, in general, we may write $\bar{U}(x_1,...,x_k,s) = s^\alpha\ \bar{V}(x_1,...,x_k,s)\bar{Q}(s)$ and thus putting it in Eqn. (2.25), we find

(2.26)
$$\left[\frac{\partial}{\partial x_1}\left(p_1(x_1)\frac{\partial \bar{V}}{\partial x_k}\right) + ... + \frac{\partial}{\partial x_k}\left(p_k(x_k)\frac{\partial \bar{V}}{\partial x_k}\right) - \sum_{r=1}^{k} q_r(x_r)\bar{V}\right] = s^\alpha \bar{V}$$

Since the equations (2.25) and (2.26) are equivalent, therefore in the relation $\bar{U}(x_1,...,x_k,s) = s^\alpha\ \bar{V}(x_1,...,x_k,s)\bar{Q}(s)$,we use convolution theorem and evaluate the result (2.24). □

Theorem 2.4. *For $0 < \alpha \leq 2$, if the multidimensional partial differential equation with fractional time derivative is given in the form*

$$\left[\frac{\partial}{\partial x_1}\left(p_1(x_1)\frac{\partial U}{\partial x_1}\right) + ... + \frac{\partial}{\partial x_k}\left(p_k(x_k)\frac{\partial U}{\partial x_k}\right) - \sum_{r=1}^{k} q_r(x_r)U\right] = {}_t{}^C D_{0^+}^\alpha U,$$

where, $U = U(x_1,...,x_k,t)$ is the temperature distribution in the region $\Omega\ \forall(x_1,...,x_k) \in \Omega$ at any time $t, t > 0$.

Along with the conditions that in the region $\Omega, U(x_1,...,x_k,0) = \prod_{i=1}^{k} G(x_i)$, and on tiny part of the boundary surface $\partial\Omega_1$ there exist, $U(a_1,...,x_k,t) = 0 = U(l_1,...,x_k,t), U(x_1,a_2,...,x_k,t) = 0 = U(x_1,l_2,...,x_k,t), ..., U(x_1,...,a_{k-1},x_k,t) = 0 = U(x_1,...,l_{k-1},x_k,t), U(x_1,...,a_k,t) = 0 = U$ $(x_1,...,l_k,t); \forall a_1 \leq x_1 \leq l_1, a_2 \leq x_2 \leq l_2,...,a_{k-1} \leq x_{k-1} \leq l_{k-1}, a_k \leq x_k \leq l_k$; while in other part of the boundary surface $\partial\Omega_2$, the temperature obeys the ruling

$$U(x_1,...,x_k,t) = 0, t > 0. \tag{2.27}$$

Then, by setting a degeneration formula $U(x_1,...,x_k,t) = \prod_{i=1}^{k} Y(x_i,t)$, in the problem (2.27), it terminates in k homogeneous Sturm - Liouville fractional time derivative equations

$$\left\{{}_t{}^C D_{0^+}^\alpha - \left[\frac{\partial}{\partial x_i}(p_i(x_i)\frac{\partial}{\partial x_i}) - q_i(x_i)\right]\right\} Y(x_i,t) = 0 \forall i = 1,2,3,...,k.$$

Along with the initial and boundary conditions

$$Y(x_i, 0) = G(x_i), Y(a_i, t) = 0 = Y(l_i, t); \forall i = 1, 2, 3, ..., k. \tag{2.28}$$

Then, for $0 < \alpha \leq 2$

$$U(x_1, ..., x_k, t) = \sum_{n=1}^{\infty} \prod_{i=1}^{k} \frac{Y(x_i, -s_{n_i}^{\alpha})}{\int_{a_i}^{l_i} \{Y(\theta_i, -s_{n_i}^{\alpha})\}^2 d\theta_i} E_\alpha(-s_{n_i}^{\alpha} t^\alpha) \int_{a_i}^{l_i} Y(\theta_i, -s_{n_i}^{\alpha}) G(\theta_i) d\theta_i. \tag{2.29}$$

Proof. In Eqn. (2.28), set $Y(x_i, t) = D_i \frac{Y(x_i, -s_{n_i}^{\alpha})}{\int_{a_i}^{l_i} \{Y(\theta_i, -s_{n_i}^{\alpha})\}^2 d\theta_i} E_\alpha(-s_{n_i}^{\alpha} t^\alpha)$, where $D_i \forall i = 1, 2, 3, ..., k$, is a constant. Then, by Eqn. (2.8), it satisfies the equality both the sides. Again, by the initial conditions given in Eqn. (2.28) and the orthogonal conditions (2.9), we find $D_i = \int_{a_i}^{l_i} Y(\theta_i, -s_{n_i}^{\alpha}) G(\theta_i) d\theta_i$. Finally, we obtain the result (2.28). □

3. **Special Cases**

In this section, we consider some particular values of the functions, parameters and boundary conditions given in our theorems of previous sections.

Corollary 3.1. *In Theorem 2.3, set* $k = 3$, $\alpha = 1$, *and* $\forall i = 1, 2, 3$, $a_i = 0$ $q_i(x_i) = 0$, $p_i(x_i) = \frac{K}{\bar{c}}$, $x_1 = x, x_2 = y, x_3 = z$, *then problem (2.20) - (2.23) become equivalent to the problem (1.2) - (1.3). Again, as* $Q(t) = 1, U(x, y, z, t) = V(x, y, z, t)$, *Then the solution by the Eqn. (2.24) is equal to the solution given in Eqn. (1.4), (see also, Churchill [2, Sec. 85, p. 247 - 249]) and is in the form*

$$U(x, y, z, t) = \frac{\partial}{\partial t} \int_0^t V(x, y, z, \tau) Q(t - \tau) d\tau,$$

where,

$$V(x, y, z, t) = \sum_{n=1}^{\infty} \prod_{i=1}^{3} \frac{Y(x_i, -s_{n_i})}{\int_0^{l_i} \{Y(\theta_i, -s_{n_i})\}^2 d\theta_i} exp(-s_{n_i} t),$$

where, $x_1 = x, x_2 = y, x_3 = z$, *and* $Y(x_i, -s_{n_i})$ *are the eigen functions corresponding to the eigen values*

$$s_{n_i} \forall i = 1, 2, 3; n = 1, 2, 3, ... \tag{3.1}$$

Corollary 3.2. *In Theorem 2.3, set* $k = 1$, $\alpha = 2$, *and* $q_1(x_1) = 1$, $p_1(x_1) = A^2, x_1 = x$, *then the problem (2.20) - (2.23) become equivalent to the wave problem due to Churchill [2, Sec. 93, p. 270 - 271 (for b = 0, h = 1)] when* $Q(t) = t, U(x, t) = V(x, t)$ *and its solution is found in the form* $U(x) = \frac{\partial^2}{\partial t^2} \int_0^t V(x, \tau) Q(t - \tau) d\tau$, *where,*

$$V(x, t) = \sum_{n=1}^{\infty} \frac{Y(x, -s_n^2)}{\int_0^l \{Y(\theta, -s_n^2)\}^2 d\theta} E_2(-s_n^2 t^2), Y(x, -s_n)$$

are the eigen functions corresponding to the eigen values

$$s_n \forall n = 1, 2, 3, ... \tag{3.2}$$

Corollary 3.3. *In Theorem 2.4, for* $0 < \alpha \leq 2$ *suppose* $J_{(n_r)^\alpha}(\beta_r^n) = 0, \forall r = 1, 2, 3, ..., k, n = 1, 2, 3, ...L, L < \infty$, *and then set* $q_r(x_r) = \frac{(n_r)^{2\alpha}}{x_r} + (n_r)^{2\alpha} - (\frac{\beta_r^n}{l_r})^2 x_r, p_r(x_r) = x_r, a_r = 0, s_{n_r}^\alpha = (n_r)^{2\alpha}, Y(x_r, t) = y(x_r) E_\alpha(-(n_r)^{2\alpha} t^\alpha) \forall r = 1, 2, 3, ..., k; n = 1, 2, 3, ...L, L < \infty$, *then,* $\forall r = 1, 2, 3, ..., k; n = 1, 2, 3, ...L, L < \infty$,

$$Y(x_r, t) = \frac{2J_{(n_r)^\alpha}(\frac{\beta_r^n}{l_r} x_r)}{(l_r)^2 (J_{(n_r)^\alpha+1}(\beta_r^n))^2} E_\alpha(-(n_r)^{2\alpha} t^\alpha) \int_0^{l_r} \theta_r J_{(n_r)^\alpha}(\frac{\beta_r^n}{l_r}\theta_r) G(\theta_r) d\theta_r \tag{3.3}$$

Therefore, on using the theory given in Eqns. (2.27)-(2.29) in the Eqn. (3.3), we get the solution of this problem equivalent to Kumar and Rai [15], in the form

$$U(x_1, ..., x_k, t) = \sum_{n=1}^{L} \prod_{r=1}^{k} \frac{2J_{(n_r)^\alpha}(\frac{\beta_r^n}{l_r} x_r)}{(l_r)^2 (J_{(n_r)^\alpha+1}(\beta_r^n))^2} E_\alpha(-(n_r)^{2\alpha} t^\alpha) \int_0^{l_r} \theta_r J_{(n_r)^\alpha}(\frac{\beta_r^n}{l_r}\theta_r) G(\theta_r) d\theta_r \tag{3.4}$$

Corollary 3.4. *In Theorem 2.4, for* $0 < \alpha \leq 2$, *set* $q_r(x_r) = \frac{\mu_r}{1-x_r^2}$, $p_r(x_r) = 1 - x_r^2, a_r = -1, l_r = 1, s_{n_r}^\alpha = (n_r)(n_r + 1)$, $Y(x_r, t) = y(x_r) E_\alpha(-(n_r)(n_r + 1)t^\alpha)$ $\forall r = 1, 2, 3, ..., k, n = 1, 2, 3, ...L, L < \infty$, *there exists a formula*

$$Y(x_r, t) = (-1)^{\mu_r} (\frac{2\mu_r + 1}{2}) P_{n_r}^{\mu_r}(x_r) E_\alpha(-(n_r)(n_r+1)t^\alpha) \int_{-1}^{1} P_{n_r}^{\mu_r}(x_r) G(\theta_r) d\theta_r \tag{3.5}$$

Then, by Eqn. (3.5), the solution of this problem, equivalent to Kumar [12], is obtained in the form

$$U(x_1, ..., x_k, t) = \sum_{n=1}^{l} \prod_{r=1}^{k} (-1)^{\mu_r} (\frac{2\mu_r + 1}{2}) P_{n_r}^{\mu_r}(x_r) E_\alpha(-(n_r)(n_r + 1)t^\alpha) \times \int_{-1}^{1} P_{n_r}^{\mu_r}(x_r) G(\theta_r) d\theta_r \tag{3.6}$$

where, $P_n^m(x) = \frac{\Gamma(n+m+1)}{(2^m m! \Gamma(n-m+1))} (x^2-1)^{(m/2)} \times {}_2F_1 \begin{pmatrix} m-n, n+m+1; \frac{1-x}{2} \\ m+1; \end{pmatrix}$, is a Legendres associated function.

4. **Application to Voigt functions, Lauricella's functions and Sturve functions**

In this section, we connect Voigt functions, Lauricella functions and Sturve functions with the Perl's vector equation (1.1) under certain initial and boundary conditions:

4.1. **Application I**

To obtain the representation of Voigt functions by solving the generalized Perl's vector equation (1.5), we consider
$\forall r = 1,2,3,...,k, n = 1,2,3,...L, L < \infty$,and for $0 < \alpha \leq 2$,consider $\xi_r, \zeta_r \in \mathbb{R}^+, \Re(\rho_r + (n_r)^\alpha) > -1$ to get the tiny area in form of the determinant $A^{(n_r)^\alpha}_{r,\rho_r}(\xi_r, \zeta_r, t)$ along the x_r - axis with weight function $x_r^{\rho_r} e^{-\xi_r x_r - \zeta_r x_r^2}$ when $x_r = 0$ to $x_r \to \infty$ at any fixed time $t, t > 0$, as

$$(4.1) \qquad A^{(n_r)^\alpha}_{r,\rho_r}(\xi_r, \zeta_r, t) = \int_0^\infty x_r^{\rho_r} e^{-\xi_r x_r - \zeta_r x_r^2} Y(x_r, t) dx_r$$

Again, $\forall r = 1,2,3,...,k, n = 1,2,3,...L, L < \infty$,and for $0 < \alpha \leq 2$, and at any fixed time $t, t > 0$ suppose that there exists a positive real number
(4.2)

$$B(l_r, \beta_r^n, t) = \frac{2}{(l_r)^2 (J_{(n_r)^\alpha + 1}(\beta_r^n))^2} E_\alpha(-(n_r)^{2\alpha} t^\alpha) \int_0^{l_r} \theta_r J_{(n_r)^\alpha}\left(\frac{\beta_r^n}{l_r}\theta_r\right) G(\theta_r) d\theta_r$$

where E_α is the Mittag-Leffeler function.

Then, on using the Eqns. (3.3), (4.1) and (4.2), $\forall r = 1,2,3,...,k, n = 1,2,3,...L, L < \infty$, and for $0 < \alpha \leq 2$, and at any fixed time $t, t > 0$, we find the relation

$$(4.3) \quad A^{(n_r)^\alpha}_{r,\rho_r}(\xi_r, \zeta_r, t) = B(l_r, \beta_r^n, t) \int_0^\infty x_r^{\rho_r} e^{-\xi_r x_r - \zeta_r x_r^2} J_{(n_r)^\alpha}\left(\frac{\beta_r^n}{l_r} x_r\right) dx_r$$

Now, use the theory of Voigt functions $V_{\rho,\nu}(x, y, z)$, defined by
$V_{\rho,\nu}(x, y, z) = \sqrt{\frac{x}{2}} \int_0^\infty t^\rho e^{-yt - zt^2} J_\nu(xt) dt$, $(\Re(\rho + \nu) > -1, \forall x, y, z \in \mathbb{R}^+)$, ([10], [19], [25] and [26]) for the classical Bessel functions $J_\nu(x), \Re(\nu) > -1$, ([1], [22] and [24]), from Eqns. (4.2) and (4.3), $\forall r = 1,2,3,...,k, n = 1,2,3,...L, L < \infty$, and for $0 < \alpha \leq 2$, at any fixed time $t, t > 0$, we find a determinant

$$(4.4) \qquad A^{(n_r)^\alpha}_{r,\rho_r}(\xi_r, \zeta_r, t) = B(l_r, \beta_r^n, t) \sqrt{\frac{2l_r}{\beta_r^n}} V_{\rho_r,(n_r)^\alpha}\left(\frac{\beta_r^n}{l_r}, \xi_r, \zeta_r\right)$$

Further, in Eqn. (3.4) replace x_1 by $x_1 u_1 ... x_k$ by $x_k u_k$ and multiply both the sides by $u_1^{\rho_1} e^{-\xi_1 u_1 - \zeta_1 u_1^2} ... u_k^{\rho_k} e^{-\xi_k u_k - \zeta_k u_k^2}$ and thus integrating with respect to u_1 from $u_1 = 0$ to $u_1 \to \infty$ ... u_k from $u_k = 0$ to $u_k \to \infty$, and then using the Eqns. (4.1) - (4.4), we obtain the relations of Voigt functions

$$\begin{aligned}(4.5)\quad &\int_0^\infty ... \int_0^\infty u_1^{\rho_1} e^{-\xi_1 u_1-\zeta_1 u_1^2} ... u_k^{\rho_k} e^{-\xi_k u_k-\zeta_k u_k^2} U(x_1u_1,...,x_ku_k,t)du_1...du_k \\ &= \sum_{n=1}^{L}\left[A_{1,\rho_1}^{(n_1)^\alpha}(\xi_1,\zeta_1,t)...(A_{k,\rho_k}^{(n_k)^\alpha}(\xi_k,\zeta_k,t))\right] \\ &= \sum_{n=1}^{L}\left[\sqrt{\frac{2l_1}{x_1\beta_1^n}} B(l_1,\beta_1^n,t) V_{\rho_1,(n_1)^\alpha}\left(\frac{\beta_1^n}{l_1},\xi_1,\zeta_1\right)... \right. \\ &\left. \left(\sqrt{\frac{2l_k}{x_k\beta_k^n}} B(l_k,\beta_k^n,t) V_{\rho_k,(n_k)^\alpha}\left(\frac{\beta_k^n}{l_k},\xi_k,\zeta_k\right)\right)\right]\end{aligned}$$

4.2. **Application II**

This subsection consists of a transformation formula of the solution of generalized Perl's vector equation (1.5) into Lauricella's functions. So that on making an appeal to the Eqns. (3.3) and (4.2) and at any fixed time $t, t > 0, 0 < \alpha \leq 2$, $\forall n = 1,2,3,...,L, L < \infty$, we define a sequence of functions

$$(4.6)\qquad R_n^\alpha(x_1,...,x_k,t;v) = \prod_{r=1}^{k}\frac{Y(x_rv,t)}{B(l_r,\beta_r^n,t)} = \prod_{r=1}^{k} J_{(n_r)^\alpha}\left(\frac{\beta_r^n}{l_r}x_rv\right)$$

Now, in Eqn. (4.6), consider that $\frac{\beta_r^n}{l_r}x_r \in \mathbb{R}^+, \forall r = 1,2,3,...,k$, and $n = 1,2,3,...,L, L < \infty$, such that $\frac{\beta_k^n}{l_k}x_k = \frac{\beta_1^n}{l_1}x_1 + ... + \frac{\beta_{k-1}^n}{l_{k-1}}x_{k-1}, k \geq 2$; for $0 < \alpha \leq 2$, choose $M = \mu+(n_1)^\alpha+...+(n_k)^\alpha$, where, $((n_1)^\alpha+...+(n_k)^\alpha+1) > \Re(1-\mu) > (-\frac{1}{2}k)$, and then use the formula of Srivastava and Exton [28, p. 4, Eqn. (2.8)] (see also Srivastava and Karlsson [29, p. 50, Eqn. (12)]), we obtain a relation

$$\begin{aligned}(4.7)\quad &\int_0^\infty v^{\mu-1}R_n^\alpha(x_1,...,x_k,t;v)dv = \int_0^\infty v^{\mu-1}\prod_{r=1}^{k}\frac{Y(x_rv,t)}{B(l_r,\beta_r^n,t)}dv \\ &= \frac{2^{\mu-1}(\frac{\beta_1^n}{l_1}x_1)^{(n_1)^\alpha}...(\frac{\beta_{k-1}^n}{l_{k-1}}x_{k-1})^{(n_{k-1})^\alpha}(\frac{\beta_k^n}{l_k}x_k)^{(n_k)^\alpha-M}\Gamma(\frac{M}{2})}{\Gamma((n_1)^\alpha+1)...\Gamma((n_{k-1})^\alpha+1)\Gamma((n_k)^\alpha+\frac{M}{2}+1)} \\ &\times F_C^{(k-1)}\left[\frac{M}{2},\frac{M}{2}-(n_k)^\alpha;(n_1)^\alpha+1,...,(n_{k-1})^\alpha+1;\right. \\ &\left. \frac{(\frac{\beta_1^n}{l_1}x_1)^2}{(\frac{\beta_k^n}{l_k}x_k)^2},...,\frac{(\frac{\beta_{k-1}^n}{l_{k-1}}x_{k-1})^2}{(\frac{\beta_k^n}{l_k}x_k)^2}\right]\end{aligned}$$

Here, in Eqn. (4.7) the k- variables Lauricella's function, $F_C^{(k)}[.,...,.]$, is defined in the Literature of Srivastava and Karlsson [29, p.33, Eqn. (3)].

4.3. **Application III**

In this subsection, to obtain the solution of generalized Perl's vector equation (1.5) in terms of Sturve functions at any fixed time $t, t > 0, 0 < \alpha \leq 2$, $\forall n = 1, 2, 3, ..., L, L < \infty$, we substitute

$$U(x_1, ..., x_k, t) = \prod_{i=1}^{k} X(x_i) E_\alpha(-\eta_i^n t^\alpha), \forall \eta_i^n \in \mathbb{R}^+, i = 1, 2, 3, ..., k$$

and

$$F(x_1, ..., x_k, t) = -\sum_{r=1}^{k} \frac{2(\frac{x_r}{2})^{\eta_r^n}}{\sqrt{\pi}\Gamma(\eta_r^n + \frac{1}{2})} E_\alpha(-\eta_r^n t^\alpha) \prod_{i=1, i\neq r}^{k} X(x_i) E_\alpha(-\eta_i^n t^\alpha)$$

and then find that

$$\left[\frac{\partial}{\partial x_r}(p_r(x_r))\frac{\partial}{\partial x_r} - q_r(x_r) + \eta_r^n\right] X(x_r) = \frac{2(\frac{x_r}{2})^{\eta_r^n}}{\sqrt{\pi}\Gamma(\eta_r^n + \frac{1}{2})} \tag{4.8}$$

Again, in Eqn. (4.8), set $p_r(x_r) = x_r, q_r(x_r) = \frac{\eta_r^{2n}}{x_r} - x_r + \eta_r^n, \forall n = 1, 2, 3, ..., L, L < \infty$, $r = 1, 2, 3, ..., k$, it becomes

$$x_r^2 \frac{d^2}{dx_r^2} X(x_r) + x_r \frac{d}{dx_r} X(x_r) + \{x_r^2 - \eta_r^{2n}\} X(x_r) = \frac{4(\frac{x_r}{2})^{\eta_r^n + 1}}{\sqrt{\pi}\Gamma(\eta_r^n + \frac{1}{2})} \tag{4.9}$$

In Eqn. (4.9), we claim the theory of Hermann Sturve (1882) in form of Sturve functions $H_{\eta_r^n}(x_r) \forall n = 1, 2, 3, ..., L, L < \infty$, $r = 1, 2, 3, ..., k$, (see also [1]) and obtain its solution in the form

$$X(x_r) = H_{\eta_r^n}(x_r) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\Gamma(\nu + \frac{3}{2})\Gamma(\nu + \eta_r^n + \frac{3}{2})} (\frac{x_r}{2})^{2\nu + \eta_r^n + 1} \tag{4.10}$$

Finally, at any fixed time $t, t > 0, 0 < \alpha \leq 2$, we obtain the solution of generalized Perl's vector equation (1.5) in terms of Surve functions $H_{\eta_r^n}(x_r)$, as

(4.11)

$$U(x_1, ..., x_k, t) = \prod_{r=1}^{k} H_{\eta_r^n}(x_r) E_\alpha(-\eta_r^n t^\alpha), \forall \eta_r^n \in \mathbb{R}^+, n = 1, 2, 3, ..., L, L < \infty.$$

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