Montes Taurus J. Pure Appl. Math. 2 (1), 1–35, 2020

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Non-Linear Effects in Incompressible Viscous Unidirectional Fluid Flows

Alexey Zhirkin[a]

[a]National Research Center Kurchatov Institute (NRC KI), Kurchatov Complex of Fusion Power and Plasma Technologies, Fusion Reactor Department, Reactor Problem Laboratory, Academician Kurchatov square 1, Moscow 123182, Russia

Abstract

More accurate nonlinear equations for the divergence free velocity field are obtained by considering small dissipation due to inelastic collisions in the three-dimensional Navier-Stokes equations. The approach of fluid incompressibility is not broken. The modified equations are used within the boundary layer. The nonlinear solutions obtained for the Couette and Poiseuille flow explain the paradoxes of symmetry and turbulence of viscous fluid flow. Laminar-turbulent transition of a steady and unsteady flow is analyzed. The process of space and time (blow-up) symmetry breaking of the Cauchy problem solution for the homogeneous and non-homogeneous Navier-Stokes equations is described. The dynamics of solitary waves in a dissipative medium without dispersion is investigated. Dissipative forms of the Korteweg-de Vries and Korteweg-de Vries-Burgers equation in one and two spatial dimensions as well as the Kadomtsev-Petviashvili equation is derived. Solitary wave solutions of these equations are presented as dissipative structures rather than wave packets. Solutions of the modified equations reveal the dynamo effect for a plane-parallel incompressible viscous hydro magnetic fluid flow. The results of the research show that the used approach is quite productive.

Keywords: Partial differential equations of mathematical physics, Navier-Stokes equations, Boltzmann equation in fluid mechanics, incompressible viscous fluid, boundary-layer theory, separation, transition to turbulence, waves for incompressible viscous fluids, instability of magneto-hydrodynamic flows

2010 MSC: P35Q30, 35Q35, 76D03, 76D05, 35C11, 35Q20, 76B25, 76D10, 76D33, 76E25, 76E30, 76F02, 76F06, 76F40, 76P99, 76W05 .05C10, 05C30, 68R10, 68R10, 68Uxx

1. Introduction

A mathematical model of incompressible viscous fluid motion is based on the Navier-Stokes equations. This model is still not successful. It can't adequately describe a real flow whose typical feature is space-time instability. This has led to emergence of the paradoxes of modeling the viscous fluid flows [1, 3]. An open problem in mathematical analysis is to decide whether smooth, physically reasonable solutions exist in three spatial dimensions [4]. Also, a key problem of fluid mechanics is to obtain solutions of the Navier-Stokes equations satisfying the conditions of fluid adhesion to a surface [5]. The accurate solutions satisfying this condition have been obtained extremely few. They don't reveal nonlinear effects [3]. This problem is closely related to understanding of the nature of viscosity.

Alone, a divergence-free velocity field does not possess properties sufficient to describe a motion of a viscous incompressible flow observed in experiments. The solutions are obtained only in the class of harmonic functions

†*Article ID:* MTJPAM-D-20-00001
Email address: aleksej-zhirkin@yandex.ru (Alexey Zhirkin)
Received:28 January 2020, Accepted:12 March 2020
⋆*Correspending Author:* Alexey Zhirkin

which are effective in space of two variables. We can show that there exist no strong viscous solutions of the Navier-Stokes equations in three dimensions [6].

The reason is inaccurate considering dissipation in the Navier-Stokes equations. Viscosity is a dissipative process. The mechanism of viscosity for an incompressible fluid consisting of one kind of molecules is based on the mixing of molecules involved in elastic collisions [7]. The elastic collisions conserve momentum and kinetic energy and dont cause dissipation.

To extend the class of solutions, it is necessary to modify the Navier-Stokes equations by use of viscosity related to intermolecular inelastic interactions. Due to inelastic collisions the fluid sticks to the body surface, and the equations for the divergence-free velocity acquires the terms which are responsible to nonlinear effects like instabilities and development of turbulence.

The Navier-Stokes equations are derived from the Boltzmann kinetic equation [8]. According to the Boltzmann kinetic equation, the existence of inelastic collisions in the collision integral leads to the appearance of additional terms in the Navier-Stokes equations. The velocity is presented as a superposition of a divergence-free and curl-free field. It performs the mass continuity equation in which the velocity divergence is not equal to zero. In turn, it results to the conservation of the term with the non-zero divergence and the second viscosity in the Cauchy momentum equation. Therefore, the viscosity associated with the inelastic collisions should be a kind of the second viscosity.

In spite of the velocity divergence is not equal to zero a fluid can be accepted as incompressible. An incompressible fluid is only approximately divergence-free [9]. We can introduce a small divergence of the curl-free velocity field in the mass continuity equation without breaking the incompressibility condition.

We accept that a fluid is incompressible when a fluid velocity is small relative to speed of sound, and the pressure variations are not great enough to cause considerable compression of the fluid, i.e. the interaction between elements of the fluid is instantaneous. Speed of the interaction is finite. The incompressibility condition is violated at distances and times larger some critical values. Thus, there is an external boundary (*S,T*) in space-time beyond which the compressibility and relativistic effect must be taken into account. We assume that the flow is steady on this boundary and is not equal to zero at all its points. It is reasonable to extrapolate a solution for the steady flow to an entire domain in which the solutions is obtained.

The flow is represented as a superposition of the steady and unsteady one. So, there exists a constant of integration $\vec{u}_0$ that does not depend on coordinates and time. The emergence of the constant is usual in the solution presented as a Fourier series. The eigenfunctions of the Fourier series are defined in infinite space-time. We can formally extrapolate the solution from a space-time domain inside the boundary (*S,T*) into an infinite domain outside (*S,T*) it in spite of the solution is the physically incorrect in it. Moreover, in infinite space we must use the Fourier integral instead of the Fourier series to represent the solution. In the extrapolated solution the constant $\vec{u}_0$ is connected to a non-zero constant fluid flow at infinity. The other functions comprising the solution can be equal zero at infinity. The validity of this construction of the solution is confirmed by defining the boundary value problems to study solvability of the stationary and non-stationary Navier-Stokes equations in the work [3].

In this research we apply the advanced model for solving the mathematical paradoxes of incompressible viscous fluid motion which have not been solved to this day. We also re-evaluate the significance of dissipation in the generation of stationary nonlinear waves and their turbulence.

The essence of the first paradox [1, 3] is that at any Reynolds number *R* the only possible solution of the Navier-Stokes equations obtaining the assumed symmetry (the stationary flow of viscous fluid in an infinitely long round pipe) is Poiseuille flow determined by the formula

$$v_x = a \cdot (c^2 - r^2), \quad v_r = v_\theta = 0, \quad a = const, \quad c = const,$$

where c is the pipe radius. The pipe axis is directed along the axis x. The fluid velocity is parallel to the pipe axis and is equal to zero on the boundary. However, the Poiseuille flows are observed for the Reynolds numbers not exceeding some critical value, if exceed they become turbulent.

The second paradox is that the symmetric solutions for Couette flows exist at all *R*, but the symmetric flows are observed only at small *R*. For large *R* they are replaced by laminar but asymmetric flows. This paradox contradicts the fact that symmetric reasons cause symmetric effects. For the large Reynolds numbers the flows do not behave as Couette and Poiseuille one at the pipe edges shifted to infinity, that is, they take other boundary value when the distance along the pipe is infinitely large [2, 3].

The general opinion is that a soliton appears as a result of the competition of dispersion and nonlinearity. It is a wave packet which is self-reinforcing during the movement due to nonlinear effect. The Korteweg-de Vries equation doesn't include dissipation [10]. And dissipation isn't a necessary condition of turbulence [11]. We examine this statement more carefully.

In the last section of this article we consider the loss of stability and magnetic dynamo model for an incompressible electrically conducting viscous fluid flow between two parallel non-conductive plates in the presence of a magnetic field.

The article is consisted of seven Sections. Section 1 is an introduction. The modification of the Navier-Stokes equations is considered in Section 2. In Section 3 the modified Navier-Stokes equations are transformed to the equations for boundary layer. In Section 4 these equations are used to explain the paradoxes of Couette and Poiseuille flows. Section 5 is devoted to the mechanism of the laminar-turbulent transition for a steady and unsteady flow. Models of solitary waves as kinds of dissipative structures are described in Section 6. These waves are solutions of dissipative nonlinear partial differential equations obtained from the Navier-Stokes equations. The application of the dynamo theory for the Couette magneto hydrodynamic flow is presented in Section 7.

2. Transform of Navier-Stokes equations

In this section we analyze the consideration of dissipation in the Boltzmann kinetic equation and modify the Navier-Stokes equations to take into account dissipation more accurately.

The equations of Newtonian fluid mechanics are obtained by the integration of the Boltzmann transport equation over the space domain [8]:

$$\frac{\partial \rho}{\partial t} + div(\rho \cdot \vec{u}) = \int \mathrm{St}\, f \cdot d\Gamma \tag{2.1}$$

$$\frac{\partial(\rho \cdot u_i)}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = \int \vec{p} \cdot \mathrm{St}\, f \cdot d\Gamma, \quad i,k = 1,2,3, \tag{2.2}$$

$$\frac{\partial(N \cdot \bar{\varepsilon})}{\partial t} + div\vec{q} = \int \varepsilon \cdot \mathrm{St}\, f \cdot d\Gamma, \tag{2.3}$$

where where ρ is the fluid density, $\vec{u} = (u_1, u_2, u_3)$ is the fluid velocity vector at time t and the point with the coordinates $\vec{r} = (x_1, x_2, x_3)$, St f is the collision integral, $f = f(t, \vec{r}, \Gamma)$ is the distribution function of the molecules, Γ is the set of all variables on which the distribution function depends except the coordinates and the time, Π_{ik} is the momentum flux tensor, $\vec{p}$ is the molecule momentum, N is the number of molecules per unit volume, ε is the molecule energy, $\vec{q}$ is the energy flux.

The form of the collision integral is established as

$$\mathrm{St}\, f = \int w' \cdot (f' \cdot f'_1 - f \cdot f_1) \cdot d\Gamma_1 \cdot d\Gamma' \cdot d\Gamma'_1. \tag{2.4}$$

The expression

$$w \cdot f \cdot f_1 \cdot d\Gamma \cdot d\Gamma_1 \cdot d\Gamma' \cdot d\Gamma'_1 \tag{2.5}$$

is the number of the collisions with the transition $\Gamma, \Gamma_1 \to \Gamma', \Gamma'_1$, w is the collision probability density.

$$w = w(\Gamma', \Gamma'_1, \Gamma, \Gamma_1), w' = w(\Gamma, \Gamma_1, \Gamma', \Gamma'_1). \tag{2.6}$$

$$f = f(t, \vec{r}, \Gamma), f' = f(t, \vec{r}, \Gamma'). \tag{2.7}$$

Γ is the variables of one from a pair colliding molecules, Γ' is the variables of the other one.

At the absence of energy dissipation the principal of detailed balancing is performed and the equilibrium statistical distribution $f_0 = f_0(\Gamma)$ satisfies the transport equation identically. The law of the conservation of energy in the collision of two molecules is performed: $\varepsilon + \varepsilon_1 = \varepsilon' + \varepsilon'_1$. Hence

$$f_0 \cdot f_{01} = f'_0 \cdot f'_{01}. \tag{2.8}$$

As a result, the collision integral is equal to zero [8]:

$$\mathrm{St}\, f = 0, \tag{2.9}$$

and the right part of the equations of (2.1)-(2.3) is equal to zero also.

Motion of a viscous fluid is a dissipative process [12]. Dissipation is an irreversible one. At dissipation the principal of detailed balancing is invalid. The part of kinetic energy of molecules is transformed to heat. Momentum is conserved but kinetic energy isn't in molecule collisions. These collisions are inelastic. The collision integral isn't equal to zero:

$$\mathrm{St}\, f \neq 0. \tag{2.10}$$

For this reason,

$$\int \mathrm{St}\, f \cdot d\Gamma \neq 0, \int \vec{p} \cdot \mathrm{St}\, f \cdot d\Gamma \neq 0, \int \varepsilon \cdot \mathrm{St}\, f \cdot d\Gamma \neq 0. \tag{2.11}$$

As follows from the proof of Liouville's theorem [13], dissipation breaks it. The Hamiltonian of the dissipative system isn't conserved along the trajectories in the phase-space due to the presence of the Rayleigh dissipation function [14]. As a result, the flow of the system points can't be described as a divergence free velocity field.

Let's consider an incompressible fluid. According to Batchelor [9], the condition of fluid incompressibility is

$$\left|\frac{1}{\rho} \cdot \frac{D\rho}{Dt}\right| << \frac{U}{L}$$

or

$$\left|div(\vec{v}(t, \vec{r}))\right| << \frac{U}{L} \tag{2.12}$$

where L is a length scale ($\vec{v}(t, \vec{r})$ varies slightly over distances small compared with the scale), U is a value of the variations of $\left|\vec{v}(t, \vec{r})\right|$ with respect to both position and time.

Even if this condition is performed, the velocity field distribution is only approximately divergence-free.

For the incompressible flow we have

$$\rho \cdot div(\vec{u}) = \int \mathrm{St}\, f \cdot d\Gamma \Rightarrow div(\vec{u}) \neq 0. \tag{2.13}$$

Our conclusion is that the equations of viscous fluid mechanics for compressible and incompressible flow have to include additional terms connected to the presence of a field which isn't divergence free. We suppose that these terms are fundamental in describing nonlinear effects.

The collision integral is a nonlinear functional, and the additional terms can be the nonlinear one.

The condition of incompressibility which takes into account the non-divergent-free velocity field is applicable for all problems of fluid or gas viscous motion in space-time. We accept that the fluid is incompressible if the propagation of interactions in it is instantaneous [12]. i.e. the time taken by an interaction signal to pass the distance equal to the length scale has to be small compared with the time scale during which the motion of fluid changes significantly.

In Newtonian mechanics there is no restriction on the speed of interaction propagation. Later the special theory of relativity was accepted as a more accurate representation of physical phenomena [14]. One of the consequences of special relativity postulates that the speed of light in vacuum is the limit speed of all processes and movements accompanied by a transfer of energy. It means that the length scale L and the limit of the velocity variations U in the incompressibility condition always takes the finite values, and we can accept the statement $div(\vec{v}(t, \vec{r})) \neq 0$ as generall in nonlinear viscous fluid mechanics.

We consider two groups of molecules with an arbitrary interaction potential [15]. One group of molecules of mass m and density n move with velocity ξ in a gas consisted of identical but stationary molecules of the other group. At collisions the molecules passing through a layer of unit area and thickness dx perpendicular to ξ lose a momentum

$$dP = P \cdot \frac{dx}{2 \cdot \lambda}, \tag{2.14}$$

where $P = m \cdot N \cdot \xi$ is the momentum of molecules passing through a unit area at section x, $N = n \cdot \xi$, λ is the mean free pass.

The momentum at section x is

$$P = P_0 \cdot \exp\left(-\frac{\lambda x}{2}\right), \tag{2.15}$$

where P_0 is the momentum at section $x = 0$.

According to this expression, we assume that the velocity $\vec{u} = \vec{u}(t, \vec{r})$ of viscous fluid moving with the inelastic collisions of molecules is presented by the linear vector function of the divergence-free velocity vector field $\vec{v} = \vec{v}(t, \vec{r})$

$$\begin{aligned} &\vec{u} = \textstyle\sum_{i=1}^{3} \exp(-\alpha_i \cdot x_i) \cdot v_i \cdot \vec{e}_i = \sum_{i=1}^{3} \vec{a}_i \cdot (\vec{e}_i \cdot \vec{v}) = A\vec{v}, \\ &div(\vec{v}) = 0, \end{aligned} \tag{2.16}$$

where $\vec{e}_i$ is the unit vector in the direction of the x_i axis;

$$\vec{a}_i = \exp(-\alpha_i \cdot x_i) \cdot \vec{e}_i, \quad i = 1, 2, 3;$$

α_i is the dissipation coefficient describing the dependence of the velocity directed along the x_i axis on dissipation due to the inelastic collisions;

$$A = \sum_{i=1}^{3} \vec{a}_i \otimes \vec{e}_i = \begin{pmatrix} \exp(-\alpha_1 \cdot x_1) & 0 & 0 \\ 0 & \exp(-\alpha_2 \cdot x_2) & 0 \\ 0 & 0 & \exp(-\alpha_3 \cdot x_3) \end{pmatrix} \tag{2.17}$$

is the symmetric tensor of rank 2.

The divergence and curl of the $\vec{u}$ field isn't equal to zero. According to Helmholtz's theorem [16], this field is decomposed into a curl-free component $\vec{u}_{POT} = \vec{u}_{POT}(t, \vec{r})$ and a divergence-free component $\vec{u}_{SOL} = \vec{u}_{SOL}(t, \vec{r})$

$$\vec{u} = \vec{u}_{POT} + \vec{u}_{SOL}. \tag{2.18}$$

Then

$$div(\vec{u}) = div(\vec{u}_{POT}) = \sum_{i=1}^{3} \left(\frac{\partial v_i}{\partial x_i} - \alpha_i \cdot v_i \right) \cdot \exp(-\alpha_i \cdot x_i). \tag{2.19}$$

To satisfy the incompressibility condition we accept

$$\left| div(\vec{u}) \right| \leq \left| \sum_{i=1}^{3} \left(\frac{\partial v_i}{\partial x_i} - \alpha_i \cdot v_i \right) \right| = \left| div(\vec{v}) - \vec{\alpha} \cdot \vec{v} \right| = \left| \vec{\alpha} \cdot \vec{v} \right| << \frac{U}{L}. \tag{2.20}$$

This condition is satisfied if dissipation is small

$$\alpha = \sqrt{{\alpha_1}^2 + {\alpha_2}^2 + {\alpha_3}^2} << 1. \tag{2.21}$$

We consider the Cauchy momentum equation for the incompressible fluid. If $div(\vec{u}) \neq 0$, we can't neglect the term with the dilatational viscosity in the right part of this equation. So, we use the expression [12]:

$$\begin{aligned} &\rho \cdot \frac{\partial \vec{u}}{\partial t} + \rho \cdot (\vec{u} \cdot \nabla) \cdot \overleftarrow{u} = -\nabla p + \eta \cdot \Delta \vec{u} + \eta_1 \cdot \nabla div(\vec{u}), \\ &\eta_1 = \zeta + \eta/3, \end{aligned} \tag{2.22}$$

where $p = p(t, \vec{r})$ is the pressure of fluid, η is the dynamic viscosity, ζ is the second viscosity coefficient related to the intermolecular interactions.

We consider the expression

$$\eta_1 \cdot \nabla div(\vec{u}) = \eta_1 \cdot \nabla \left[\sum_{i=1}^{3} \left(\frac{\partial v_i}{\partial x_i} - \alpha_i \cdot v_i \right) \cdot \exp(-\alpha_i \cdot x_i) \right]. \tag{2.23}$$

Omitting the terms with the coefficients of dissipation multiplied by the viscosity as a small one and accepting that $\exp(-\alpha_i \cdot x_i) \approx 1$, we obtain

$$\eta_1 \cdot \nabla div(\vec{u}) \approx \eta_1 \cdot \nabla div(\vec{v}) = 0. \tag{2.24}$$

We use the expression (2.16) implying that $\exp(-\alpha_i \cdot x_i) \approx 1$. As a result, we obtain the Cauchy momentum equation for the divergence-free velocity $\vec{v}$ as

$$\rho \cdot \frac{\partial v_i}{\partial t} - \rho \cdot \alpha_i \cdot {v_i}^2 + \rho \cdot (\vec{v} \cdot \nabla) \cdot v_i = -\nabla p + \eta \cdot \left({\alpha_i}^2 \cdot v_i - 2 \cdot \alpha_i \cdot \frac{\partial v_i}{\partial x_i} + \Delta v_i \right). \tag{2.25}$$

We omit the terms with the coefficients of dissipation multiplied by the viscosity and obtain

$$\rho \cdot \frac{\partial v_i}{\partial t} - \rho \cdot \alpha_i \cdot {v_i}^2 + \rho \cdot (\vec{v} \cdot \nabla) \cdot v_i = -\nabla p + \eta \cdot \Delta v_i, \quad i = 1, 2, 3, \tag{2.26}$$

or

$$\rho \cdot \frac{\partial \vec{v}}{\partial t} - \rho \cdot V \cdot \vec{v} + \rho \cdot (\vec{v} \cdot \nabla) \cdot \vec{v} = -\nabla p + \eta \cdot \Delta \vec{v} \tag{2.27}$$

where $V = \begin{pmatrix} \alpha_1 \cdot v_1 & 0 & 0 \\ 0 & \alpha_2 \cdot v_2 & 0 \\ 0 & 0 & \alpha_3 \cdot v_3 \end{pmatrix}$.

This equation together with the conditions

$$div(\vec{v}) = 0, \quad \alpha = \sqrt{\alpha_1{}^2 + \alpha_2{}^2 + \alpha_3{}^2} << 1 \tag{2.28}$$

is the Navier-Stokes equations considering the small dissipation due to molecule inelastic collisions.

We can introduce the constant vector $\vec{c} = (c_1, c_2, c_3)$ in the expression for the velocity [3, 12]

$$\vec{u}(t, \vec{r}) \to \vec{c} + \vec{u}(t, \vec{r}), \quad \vec{c} = const. \tag{2.29}$$

We obtain the equation

$$\rho \cdot \frac{\partial \vec{v}}{\partial t} - \rho \cdot C \cdot \vec{v} + \rho \cdot (\vec{c} \cdot \nabla) \cdot \vec{v} - \rho \cdot V \cdot \vec{v} + \rho \cdot (\vec{v} \cdot \nabla) \cdot \vec{v} = -\nabla p + \eta \cdot \Delta \vec{v}, \tag{2.30}$$

where $C = \begin{pmatrix} \alpha_1 \cdot c_1 & 0 & 0 \\ 0 & \alpha_2 \cdot c_2 & 0 \\ 0 & 0 & \alpha_3 \cdot c_3 \end{pmatrix}$.

The velocity form (2.16) isn't unique. Using the expression (2.18), we can construct other presentations satisfying the incompressibility condition (2.12).

3. Equations for boundary layer

To verify the capability of the modified Navier-Stokes equations to solve the actual problems of modern fluid mechanic, we have to compare the solutions of these equations with unexplained experimental results. The paradoxal behavior is observed for Couette and Poiseuille flows in boundary layer [1, 2, 3]. In this Section we derive the equations for boundary layer from the Navier-Stokes equations obtained in the previous section.

Sticking fluid molecules to a body surface is a kind of viscosity. The molecules which have stuck to the surface collide with other fluid molecules. These collisions are inelastic. After collisions some stuck molecules can lose their bounds with atoms and pass into fluid layers with rather faster or slower molecules. These layers slow down or accelerate. New molecules occupy places of the molecules that have left. As a result, the fluid boundary layer is formed. This layer moves overcoming resistance of the electromagnetic field created by the body due to loss of the kinetic energy of the directed movement to dissipation.

Let's consider viscosity due to the inelastic collisions of mixing molecules in the boundary layer.

We use the equation (2.26). We suppose that the velocity in the boundary layer is

$$\vec{u}(t, \vec{r}) = \exp(-\alpha \cdot x) \cdot v(t, y, z) \cdot \vec{e}_x. \tag{3.1}$$

We obtain the system of the equations for $v = v(t, y, z)$

$$\begin{aligned} &\rho \cdot \tfrac{\partial v}{\partial t} - \rho \cdot \alpha \cdot v^2 = -\tfrac{\partial p}{\partial x} + \eta \cdot \left(\tfrac{\partial^2 v}{\partial y^2} + \tfrac{\partial^2 v}{\partial z^2} \right), \\ &\tfrac{\partial p}{\partial y} = 0, \quad \tfrac{\partial p}{\partial z} = 0. \end{aligned} \tag{3.2}$$

The solution of the second and the third equation of this system is the function of pressure

$$p = P(t, x), \tag{3.3}$$

where $P(t, x)$ is an arbitrary function of the variables t and x.

We substitute this expression in the first equation and obtain

$$\rho \cdot \frac{\partial v}{\partial t} - \rho \cdot \alpha \cdot v^2 = -\frac{\partial P}{\partial x} + \eta \cdot \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \tag{3.4}$$

$$\vec{v} = v(t, y, z) \cdot \vec{e}_x, div(\vec{v}) = 0. \tag{3.5}$$

In this equation

$$\frac{\partial P}{\partial x} = const, \tag{3.6}$$

because $v(t, y, z)$ doesn't depend on x.

4. Flow in boundary layers

In this part we apply the obtained nonlinear equations for boundary layer to analyze Couette and Poiseuille flows which are well-studied in experiments.

4.1. Couette flow

This flow [9] is a movement of viscous incompressible fluid in a horizontal layer with the thickness d (Fig.1). The bottom boundary of the layer is motionless, and the top one moves with the constant velocity v_0 directed along the axis x. The planar horizontal flow is considered, that is $v_y \equiv 0$, $v_z \equiv 0$, $v_x \equiv v(z)$. The horizontal gradient of pressure is equal to zero.

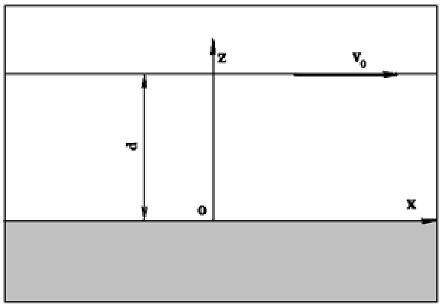

Figure 1. Couette flow

We consider the stationary problem and use the equation (3.4).

Couette flow is defined under the condition of

$$\frac{dP}{dx} = 0. \tag{4.1}$$

We use only one spatial coordinate:

$$v(y, z) \equiv v(z). \tag{4.2}$$

We define

$$\gamma = \frac{\alpha}{\kappa}, \tag{4.3}$$

where $\kappa = \frac{\eta}{\rho}$ is the coefficient of kinematical viscosity.

We obtain the nonlinear equation for Couette flow

$$\frac{d^2 v(z)}{dz^2} = -\gamma \cdot v(z)^2 \tag{4.4}$$

with the boundary conditions

$$v(0) = 0, v(d) = v_0. \tag{4.5}$$

We consider that

$$\vec{v} = v(r) \cdot \vec{e}_x.$$

Then

$$div(\vec{v}) = 0. \tag{4.6}$$

The solution of the equation (4.4) can be presented as Weierstrass's elliptic functions [17, 18]. But this solution is inconvenient to qualitative study of Couette flow.

Let's present the solution of the equations (4.4-4.6) as a power series

$$v(z) = \sum_{n=0}^{\infty} a_n \cdot \left(1 - \frac{z}{d}\right)^n, \tag{4.7}$$

where a_n is an unknown constant value, $n = 0, 1, 2, \ldots, \infty$.

$$v(z)^2 = \sum_{n=0}^{\infty} b_n \cdot \left(1 - \frac{z}{d}\right)^n, \tag{4.8}$$

$$b_n = \sum_{i=0}^{n} a_i \cdot a_{n-i}, \quad n = 0, 1, 2, \ldots, \infty. \tag{4.9}$$

We substitute these expressions in the equation (4.4) and obtain the formula

$$a_{n+2} = -\tfrac{\gamma \cdot d^2}{(n+1)\cdot(n+2)} \cdot \textstyle\sum_{i=0}^{n} a_i \cdot a_{n-i}, \qquad n = 0, 1, 2, \ldots, \infty. \tag{4.10}$$

We obtain from the boundary condition $v(d) = v_0$ that

$$a_0 = v_0. \tag{4.11}$$

We suggest that the coefficient a_1 in the series (4.7) is much larger than the others:

$$|a_1| >> |a_n|, \quad n = 2, 3, \ldots. \tag{4.12}$$

So, to satisfy the boundary condition $v(0) = 0$ we use the approximate expression

$$v(z) \approx a_0 + a_1 \cdot \left(1 - \frac{z}{d}\right). \tag{4.13}$$

We obtain

$$a_1 = -v_0. \tag{4.14}$$

We define the value

$$\beta = \gamma \cdot d^2 \cdot v_0. \tag{4.15}$$

As a result, we obtain the coefficient values

$$\begin{aligned}
a_0 &= v_0, \\
a_1 &= -v_0, \\
a_2 &= -\frac{\gamma \cdot d^2}{2} \cdot a_0{}^2 = -\frac{\gamma \cdot d^2 \cdot v_0{}^2}{2} = -\frac{\beta}{2} \cdot v_0, \\
a_3 &= -\frac{\gamma \cdot d^2}{2 \cdot 3} \cdot 2 \cdot a_0 \cdot a_1 = \frac{\gamma \cdot d^2 \cdot v_0{}^2}{3} = \frac{\beta}{3} \cdot v_0, \\
a_4 &= -\frac{\gamma \cdot d^2}{3 \cdot 4} \cdot (2a_0 \cdot a_2 + a_1{}^2) = \left(\frac{\beta^2}{12} - \frac{\beta}{12}\right) \cdot v_0, \\
a_5 &= -\frac{\gamma \cdot d^2}{4 \cdot 5} \cdot 2 \cdot (a_0 \cdot a_3 + a_1 \cdot a_2) = -\frac{\beta^2}{12} \cdot v_0, \\
a_6 &= -\frac{\gamma \cdot d^2}{5 \cdot 6} \cdot [2 \cdot (a_0 \cdot a_4 + a_1 \cdot a_3) + a_2{}^2] = \left(\frac{\beta^2}{36} - \frac{\beta^3}{45}\right) \cdot v_0, \\
a_7 &= -\frac{\gamma \cdot d^2}{6 \cdot 7} \cdot 2 \cdot (a_0 \cdot a_5 + a_1 \cdot a_4 + a_2 \cdot a_3) = \left(\frac{\beta^3}{63} - \frac{\beta^2}{252}\right) \cdot v_0, \\
a_8 &= -\frac{\gamma \cdot d^2}{7 \cdot 8} \cdot [2 \cdot (a_0 \cdot a_6 + a_1 \cdot a_5 + a_2 \cdot a_4) + a_3{}^2] = \left(\frac{23 \cdot \beta^4}{10080} - \frac{5 \cdot \beta^3}{672}\right) \cdot v_0,
\end{aligned} \tag{4.16}$$

$$a_9 = -\frac{\gamma \cdot d^2}{8 \cdot 9} \cdot 2 \cdot (a_0 \cdot a_7 + a_1 \cdot a_6 + a_2 \cdot a_5 + a_3 \cdot a_4) = \left(\frac{5 \cdot \beta^3}{3024} - \frac{7317 \cdot \beta^4}{2449440}\right) \cdot v_0.$$

If β<<1 the condition (4.12) is performed.

The solution of the equations (4.4)–(4.6) is (4.17)

$$v(z) = v_0 \cdot \tag{4.17}$$

$$\left[\frac{z}{d} - \frac{\beta}{2} \cdot \left(1 - \frac{z}{d}\right)^2 + \frac{\beta}{3} \cdot \left(1 - \frac{z}{d}\right)^3 + \left(\frac{\beta^2}{12} - \frac{\beta}{12}\right) \cdot \left(1 - \frac{z}{d}\right)^4 - \frac{\beta^2}{12} \cdot \left(1 - \frac{z}{d}\right)^5 + \ldots\right].$$

We can present it in the form

$$v(z) = v_0 \cdot \tag{4.18}$$

$$\left[\frac{z}{d} - \frac{\beta}{2} \cdot \left(1 - \frac{z}{d}\right)^2 + \frac{\beta}{3} \cdot \left(1 - \frac{z}{d}\right)^3 + \left(\frac{\beta^2}{12} - \frac{\beta}{12}\right) \cdot \left(1 - \frac{z}{d}\right)^4 + r_1(z) + r_2(z)\right],$$

where $r_1(z)$and $r_2(z)$are

$$r_1(z) = -\frac{\beta^2}{12} \cdot \left(1 - \frac{z}{d}\right)^5 + \frac{\beta^2}{36} \cdot \left(1 - \frac{z}{d}\right)^6 - \frac{\beta^2}{252} \cdot \left(1 - \frac{z}{d}\right)^7 + \tag{4.19}$$

$$\frac{23 \cdot \beta^4}{10080} \cdot \left(1 - \frac{z}{d}\right)^8 - \frac{7317 \cdot \beta^4}{2449440} \cdot \left(1 - \frac{z}{d}\right)^9 + \ldots,$$

$$r_2(z) = -\frac{\beta^3}{45} \cdot \left(1 - \frac{z}{d}\right)^6 + \frac{\beta^3}{63} \cdot \left(1 - \frac{z}{d}\right)^7 - \frac{5 \cdot \beta^3}{672} \cdot \left(1 - \frac{z}{d}\right)^8 + \frac{5 \cdot \beta^3}{3024} \cdot \left(1 - \frac{z}{d}\right)^9 + \ldots, \tag{4.20}$$

The power series $r_1(z)$ and $r_2(z)$ converge at least for $\beta << 1$ and $z \in (0, d]$ according to the Leibniz test [19]. If we take $z = d$ we obtain the alternating series which terms are monotonically decreased and converge to zero:

$$|a_{n+2}| = \frac{\gamma \cdot d^2}{(n+1) \cdot (n+2)} \cdot \left|\sum_{i=1}^{n-1} a_i \cdot a_{n-i}\right| < \frac{\beta \cdot v_0 \cdot n}{(n+1) \cdot (n+2)} = c_n \to 0 \text{ if } n \to \infty. \tag{4.21}$$

So, the power series $v(z)$ converges at least for $\beta << 1$ and $z \in (0, d]$.

Let's reveal the physical essence of the parameter β. The Reynolds number for Couette flow is

$$Re = \frac{v_0 \cdot d}{\kappa}. \tag{4.22}$$

The fluid flow loses its energy in the dissipation. According to the formula (2.13), we define the value

$$L = \frac{1}{\alpha} \tag{4.23}$$

as the fluid flow relaxation length. It is the average distance which the fluid molecules pass to the complete loss of their energy in dissipation.

We also define the critical Reynolds number which determines the loss of stability of Couette flow

$$Re_c = \frac{L}{d}. \tag{4.24}$$

We obtain

$$\beta = \gamma \cdot d^2 \cdot v_0 = \frac{d}{L} \cdot \frac{v_0 \cdot d}{\kappa} = \frac{Re}{Re_c}. \tag{4.25}$$

Let's show that the parameter β governs the separation of the boundary layer from the surface.

The boundary layer is separated from the motionless surface located at z=0 when the flow reverses in the backward direction. In this case the distribution of the velocity has negative values near this surface. These values correspond to the fluid backflow [20].

In the field of negative values there is a minimum of the function $v(z)$. The point of location of this minimum z_0 is defined by the formula

$$\frac{dv(z)}{dz} = 0. \tag{4.26}$$

To simplify the calculations we use the approximate expression for *v(z)*

$$v(z) \approx v_0 \cdot \left[\frac{z}{d} - \frac{\beta}{2} \cdot \left(1 - \frac{z}{d}\right)^2 \right]. \tag{4.27}$$

This expression satisfies the boundary condition $v(0)$=0 with a sufficient accuracy if β<<1. We use $\beta < 1$, $\beta = 1$, $\beta > 1$ only for the qualitative analysis. We obtain

$$z_0 = d \cdot \left(1 - \frac{1}{\beta}\right) = d \cdot \frac{Re - Re_c}{Re}. \tag{4.28}$$

If $Re < Re_c$ we have $z_0 < 0$ and $z_0 \notin (0, d]$. The backflow isn't revealed. At $Re = Re_c$ the backflow is appeared at the point $z_0 = 0$. If $Re > Re_c$ the backflow occupies more space and the location of the minimum of $v(z)$ is at the point $z_0 \in (0, d]$ (Fig. 2).

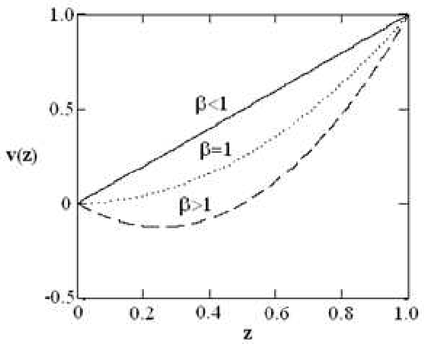

Figure 2. The space velocity distribution of Couette flow

We note that as follows from the boundary condition *v(z)*=0, generally the coefficient a_1 is dependent on coefficients $a_0, a_2, a_3, \ldots$ and, therefore, on the coefficient β.

4.2. **Plane Poiseuille flow**

We consider a flow of viscous incompressible fluid in a layer with firm motionless boundaries [9] under the action of the pressure difference at the edges (Fig. 3). The thickness of the layer is equal to $2 \cdot d$, the length is L_0. The pressure P_1 and P_2 is defined at the ends of the layer correspondingly. The flow is a planar horizontal one, that is $v_y \equiv 0$, $v_z \equiv 0$, $v_x \equiv v(z)$. The fluid velocity is equal to the constant value v_0 at the middle of the layer.

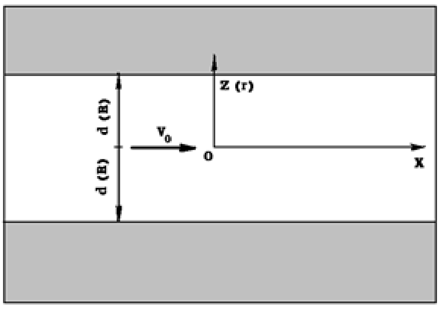

Figure 3. Poiseuille flow

Poiseuille flow is defined under the condition of

$$P = P_1 - \frac{P_1 - P_2}{L_0} \cdot x. \tag{4.29}$$

We use the designation

$$K_0 = -\frac{dP}{dx} = \frac{P_1 - P_2}{L_0}. \tag{4.30}$$

We substitute this value in the stationary equation (3.4) in which we use only one spatial variable

$$v(y, z) \equiv v(z). \tag{4.31}$$

We define

$$\gamma = \frac{\alpha}{\kappa}, \kappa = \frac{\eta}{\rho}, K = \frac{K_0}{\rho \cdot \kappa} = \frac{K_0}{\eta}, \tag{4.32}$$

and obtain the equation

$$\frac{d^2 v(z)}{dz^2} = -K - \gamma \cdot v(z)^2 \tag{4.33}$$

with the boundary condition

$$v(\pm d) = 0. \tag{4.34}$$

We consider that

$$\vec{v} = v(z) \cdot \vec{e}_x.$$

Then

$$div(\vec{v}) = 0. \tag{4.35}$$

The solution of the equation (4.33) can be represented as Weierstrass's elliptic functions as well as the solution of the equation (4.4).

We present the solution of the equations (4.33-4.35) in the form which is more convenient to study Poiseuille flow. This is a power series

$$v(z) = \sum_{n=0}^{\infty} a_n \cdot \left(\frac{z}{d}\right)^{2n}, \tag{4.36}$$

where a_n is an unknown constant value, $n = 0, 1, 2, \ldots, \infty$.

$$v(z)^2 = \sum_{n=0}^{\infty} b_n \cdot \left(\frac{z}{d}\right)^{2n}, \tag{4.37}$$

$$b_n = \sum_{i=0}^{n} a_i \cdot a_{n-i}, \quad n = 0, 1, 2, \ldots, \infty. \tag{4.38}$$

We substitute these expressions in the equation (4.33) and obtain the formula

$$a_1 = \frac{-K \cdot d^2 - \gamma \cdot d^2 \cdot {a_0}^2}{2}. \tag{4.39}$$

$$a_{n+1} = -\frac{\gamma \cdot d^2}{2 \cdot n \cdot (2n+1)} \cdot \sum_{i=0}^{n} a_i \cdot a_{n-i}, \quad n = 1, 2, 3, \ldots, \infty. \tag{4.40}$$

We obtain from the boundary condition $v(\pm d) = 0$ that

$$a_0 = -\sum_{n=1}^{\infty} a_n \,. \tag{4.41}$$

We suggest that the coefficient a_1 in the series (4.36) is much larger than the others:

$$|a_1| >> |a_n|, \quad n = 2, 3, \ldots. \tag{4.42}$$

So, to satisfy the boundary condition we use the approximate expression

$$v(z) \approx a_0 + a_1 \cdot \left(\frac{z}{d}\right)^2. \tag{4.43}$$

We obtain

$$a_1 = -a_0. \tag{4.44}$$

We define the values

$$a_1 = -v_0, \quad Re = \frac{v_0 \cdot d}{\kappa}, \quad L = \frac{1}{\alpha}, \ Re_c = \frac{L}{d}, \ \beta = \gamma \cdot d^2 \cdot v_0 = \frac{d}{L} \cdot \frac{v_0 \cdot d}{\kappa} = \frac{Re}{Re_c}, \tag{4.45}$$

as well as for Couette flow.

We assume that

$$\beta << 1 \tag{4.46}$$

and obtain

$$a_1 = \frac{-K \cdot d^2 - \gamma \cdot d^2 \cdot {a_0}^2}{2} = \frac{-K \cdot d^2 - \beta \cdot v_0}{2} \approx -\frac{K \cdot d^2}{2}. \tag{4.47}$$

As a result, we obtain the coefficient values

$$\begin{gathered} a_0 = v_0 = \frac{K \cdot d^2}{2}, \\ a_1 = -v_0 = -\frac{K \cdot d^2}{2}, \\ a_2 = -\frac{\gamma \cdot d^2}{3} \cdot a_0 \cdot a_1 = \frac{\gamma \cdot d^2 \cdot {v_0}^2}{3} = \frac{\beta}{3} \cdot v_0, \\ a_3 = -\frac{\gamma \cdot d^2}{20} \cdot (2 \cdot a_0 \cdot a_2 + {a_1}^2) = -\left(\frac{\beta^2}{30} + \frac{\beta}{20}\right) \cdot v_0, \\ a_4 = -\frac{\gamma \cdot d^2}{42} \cdot 2 \cdot (a_0 \cdot a_3 + a_1 \cdot a_2) = \left(\frac{\beta^3}{630} + \frac{23 \cdot \beta^2}{1260}\right) \cdot v_0, \\ a_5 = -\frac{\gamma \cdot d^2}{72} \cdot [2 \cdot (a_0 \cdot a_4 + a_1 \cdot a_3) + {a_2}^2] = -\left(\frac{\beta^4}{22680} + \frac{3 \cdot \beta^3}{1008} + \frac{\beta^2}{720}\right) \cdot v_0. \end{gathered} \tag{4.48}$$

If β<<1 the condition (4.42) is performed.

The solution of the equations (4.33)-(4.35)

is

$$v(z) = v_0 \cdot \tag{4.49}$$

$$\left[1 - \left(\frac{z}{d}\right)^2 + \frac{\beta}{3} \cdot \left(\frac{z}{d}\right)^4 - \left(\frac{\beta^2}{30} + \frac{\beta}{20}\right) \cdot \left(\frac{z}{d}\right)^6 + \left(\frac{\beta^3}{630} + \frac{23 \cdot \beta^2}{1260}\right) \cdot \left(\frac{z}{d}\right)^8 - \ldots\right],$$

where

$$v_0 = \frac{P_1 - P_2}{2L_0 \cdot \eta} \cdot d^2. \tag{4.50}$$

This power series converge at least for $\beta << 1$ and $z \in [-d, d]$ according to the Leibniz test. If $z = \pm d$ we obtain the alternating series which terms are monotonically decreased and converge to zero:

$$|a_{n+1}| = \frac{\gamma \cdot d^2}{2 \cdot n \cdot (2n+1)} \cdot \left|\sum_{i=0}^{n} a_i \cdot a_{n-i}\right| < \frac{\beta \cdot v_0 \cdot (n+1)}{2 \cdot n \cdot (2n+1)} = c_n \to 0 \ \text{ if } n \to \infty. \tag{4.51}$$

Let's show that the parameter β governs the separation of the boundary layer from the surfaces with the coordinates $z = \pm d$ in Fig. 3.

We determine the presence of the fluid backflow by defining the position of the minimum of the function $v(z)$ in the field of negative values as well as for Couette flow. The point of location of this minimum z_0 is defined by the formula

$$\frac{dv(z)}{dz} = 0. \tag{4.52}$$

To simplify the calculations we use the approximate expression for $v(z)$

$$v(z) \approx v_0 \cdot \left[1 - \left(\frac{z}{d}\right)^2 + \frac{\beta}{3} \cdot \left(\frac{z}{d}\right)^4\right]. \tag{4.53}$$

This expression satisfies the boundary condition $v(\pm d) = 0$ with a sufficient accuracy if $\beta << 1$. For the qualitative analysis we use $\beta < 1, \ \beta = 1, \ \beta > 1$. We obtain

$$z_0 = \pm d \cdot \sqrt{\frac{3}{2 \cdot \beta}} = \pm d \cdot \sqrt{\frac{3 \cdot Re_c}{2 \cdot Re}} \approx \pm d \cdot \sqrt{\frac{Re_c}{Re}}. \tag{4.54}$$

If $Re < Re_c$ we have $z_0 \notin [-d, d]$. The backflow isn't revealed. At $Re = Re_c$ the backflow is appeared at the points $z_0 = \pm d$. If $Re > Re_c$ the backflow occupies more space and the location of the minimum of $v(z)$ is at the point $z_0 \in (-d, d)$ (Fig. 4).

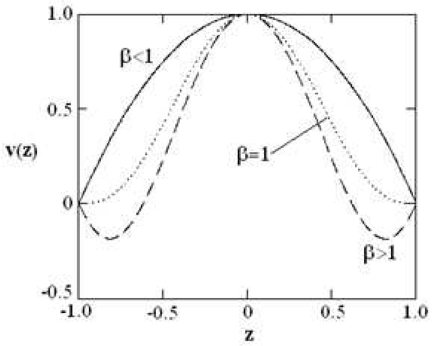

Figure 4. The space velocity distribution of Poiseuille flow

4.3. Annular Poiseuille flow

Let's consider Poiseuille flow in a cylindrical pipe with the length L_0 and radius $R = d$. We use the cylindrical coordinates (x, r, φ). The axis x is the one of the cylinder (Fig. 3).

We transform the Cartesian coordinates (y, z) to the polar one (r, φ) in the equation (3.4). We imply the cylindrically symmetric problem, that is

$$\varphi \equiv 0.$$

Then

$$v(y, z) \equiv v(r)$$

and the equation (3.4) takes the form

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial v(r)}{\partial r}\right) = -K - \gamma \cdot v(r)^2, \tag{4.55}$$

where $\gamma = \frac{\alpha}{\kappa}$, $K = \frac{K_0}{\rho \cdot \kappa} = \frac{K_0}{\eta}$.

The boundary condition is

$$v(R) = 0. \tag{4.56}$$

We consider that

$$\vec{v} = v(r) \cdot \vec{e}_x.$$

Then

$$div(\vec{v}) = 0. \tag{4.57}$$

We present the solution of the equations (4.55)-(4.57) as a power series

$$v(r) = \sum_{n=0}^{\infty} a_n \cdot \left(\frac{r}{R}\right)^{2n}, \tag{4.58}$$

where a_n is an unknown constant value, $n = 0, 1, 2, \dots, \infty$.

$$v(r)^2 = \sum_{n=0}^{\infty} b_n \cdot \left(\frac{r}{R}\right)^{2n}, \tag{4.59}$$

$$b_n = \sum_{i=0}^{n} a_i \cdot a_{n-i}, \quad n = 0, 1, 2, \dots, \infty. \tag{4.60}$$

We substitute these expressions in the equation (4.55) and obtain the formula

$$a_1 = \frac{-K \cdot R^2 - \gamma \cdot R^2 \cdot {a_0}^2}{4}. \tag{4.61}$$

$$a_{n+1} = -\tfrac{\gamma \cdot R^2}{4 \cdot (n+1)^2} \cdot \textstyle\sum_{i=0}^{n} a_i \cdot a_{n-i}, \qquad n = 1, 2, 3, \dots, \infty. \tag{4.62}$$

We obtain from the boundary condition $v(R) = 0$ that

$$a_0 = -\sum_{n=1}^{\infty} a_n\ . \tag{4.63}$$

We suggest that the coefficient a_1 in the series (4.58) is much larger than the others:

$$|a_1| >> |a_n|, \quad n = 2, 3, \dots. \tag{4.64}$$

To satisfy the boundary condition we use the approximate expression

$$v(z) \approx a_0 + a_1 \cdot \left(\frac{r}{R}\right)^2. \tag{4.65}$$

We obtain

$$a_1 = -a_0. \tag{4.66}$$

We define the values

$$a_1 = -v_0, \ \ Re = \tfrac{v_0 \cdot R}{\kappa}, \ \ L = \tfrac{1}{\alpha}, \ \ Re_c = \tfrac{L}{R}, \ \beta = \gamma \cdot R^2 \cdot v_0 = \tfrac{R}{L} \cdot \tfrac{v_0 \cdot R}{\kappa} = \tfrac{Re}{Re_c}. \tag{4.67}$$

We assume that

$$\beta << 1 \tag{4.68}$$

and obtain

$$a_1 = \frac{-K \cdot R^2 - \gamma \cdot R^2 \cdot {a_0}^2}{4} = \frac{-K \cdot R^2 - \beta \cdot v_0}{4} \approx -\frac{K \cdot R^2}{4}. \tag{4.69}$$

As a result, we obtain the coefficient values

$$\begin{aligned} a_0 &= v_0 = \frac{K \cdot R^2}{4}, \\ a_1 &= -v_0 = -\frac{K \cdot R^2}{4}, \\ a_2 &= -\frac{\gamma \cdot R^2}{8} \cdot a_0 \cdot a_1 = \frac{\gamma \cdot R^2 \cdot {v_0}^2}{8} = \frac{\beta}{8} \cdot v_0, \\ a_3 &= -\frac{\gamma \cdot R^2}{36} \cdot (2 \cdot a_0 \cdot a_2 + {a_1}^2) = -\left(\frac{\beta^2}{144} + \frac{\beta}{36}\right) \cdot v_0, \\ a_4 &= -\frac{\gamma \cdot R^2}{64} \cdot 2 \cdot (a_0 \cdot a_3 + a_1 \cdot a_2) = \left(\frac{\beta^3}{4608} + \frac{145 \cdot \beta^2}{1152}\right) \cdot v_0. \end{aligned} \tag{4.70}$$

If β<<1 the condition (4.64) is performed.

The solution of the equations (4.55)-(4.57) is

$$v(r) = v_0 \cdot \tag{4.71}$$

$$\left[1-\left(\frac{r}{R}\right)^2+\frac{\beta}{8}\cdot\left(\frac{r}{R}\right)^4-\left(\frac{\beta^2}{144}+\frac{\beta}{36}\right)\cdot\left(\frac{r}{R}\right)^6+\left(\frac{\beta^3}{4608}+\frac{145\cdot\beta^2}{1152}\right)\cdot\left(\frac{r}{R}\right)^8-\ldots\right],$$

where

$$v_0=\frac{P_1-P_2}{4L_0\cdot\eta}\cdot R^2. \tag{4.72}$$

This power series converge at least for $\beta << 1$ and $r \in (0, R]$ according to the Leibniz test. If $r = R$ we obtain the alternating series which terms are monotonically decreased and converge to zero:

$$|a_{n+1}|=\frac{\gamma\cdot R^2}{2\cdot n\cdot(2n+1)}\cdot\left|\sum_{i=0}^{n}a_i\cdot a_{n-i}\right|<\frac{\beta\cdot v_0\cdot(n+1)}{2\cdot n\cdot(2n+1)}=c_n\to 0\ \text{ if } n\to\infty. \tag{4.73}$$

4.4. Poiseuille's law

Let's derive the Poiseuille's law using the nonlinear solution. We define the value Q as the fluid discharge that is the fluid amount passing through the pipe cross-section per one second [12].

We use the formula (4.72):

$$\begin{aligned}
&Q=\int_0^R v(r)\cdot 2\pi\cdot r\cdot dr=\pi\cdot R^2\cdot\textstyle\sum_{n=0}^{\infty}\frac{a_n}{n+1}=\\
&=\pi\cdot R^2\cdot v_0\cdot\left[1-\frac{1}{2}+\frac{\beta}{24}-\frac{1}{4}\cdot\left(\frac{\beta^2}{144}+\frac{\beta}{36}\right)+\frac{1}{5}\cdot\left(\frac{\beta^3}{4608}+\frac{145\cdot\beta^2}{1152}\right)-\ldots\right]=\\
&=\pi\cdot R^2\cdot v_0\cdot\left[1-\frac{1}{2}+\frac{\beta}{24}-\left(\frac{\beta^2}{576}+\frac{\beta}{144}\right)+\left(\frac{\beta^3}{23040}+\frac{29\cdot\beta^2}{1152}\right)-\ldots\right]=\\
&=\pi\cdot R^2\cdot v_0\cdot\left[\frac{1}{2}+\frac{5\cdot\beta}{144}+\frac{27\cdot\beta^2}{1152}+\ldots\right],
\end{aligned} \tag{4.74}$$

where

$$v_0=\frac{P_1-P_2}{4\cdot L_0\cdot\eta}\cdot R^2,\ \ \beta << 1. \tag{4.75}$$

This series can be presented as an alternating one which terms are monotonically decreased and converge to zero. Therefore, it is converged at least for $\beta << 1$ according to the Leibniz test. We can obtain the same result applying d'Alembert's ratio test [19].

If $\beta = 0$ we obtain the ordinary Poiseuille formula [12]

$$Q=\frac{\pi\cdot R^4\cdot(P_1-P_2)}{8\cdot L_0\cdot\eta}. \tag{4.76}$$

5. Symmetry breaking of steady and unsteady flow

In this Section we study symmetry breaking of a steady and unsteady flow. We derive the nonlinear equations for these flows and apply their solutions to describe a spontaneous laminar-turbulence transition both at absence and at presence of an external force.

5.1. Equation for steady and unsteady flow

We consider the equation (2.22)

$$\rho\cdot\frac{\partial\vec{u}}{\partial t}+\rho\cdot(\vec{u}\cdot\nabla)\cdot\overleftarrow{u}=-\nabla p+\eta\cdot\Delta\vec{u}+\eta_1\cdot\nabla div(\vec{u}).$$

We define the velocity $\vec{u}=\vec{u}(t,\vec{r})$ as the sum of the solution $\vec{v}=v(\vec{r})$ for a steady and $\vec{w}=\vec{w}(t,\vec{r})$ for an unsteady flow. We accept that the velocity solution includes the constant $\vec{c}$. We define the pressure $p_v = p_v(\vec{r})$ for the steady and $p_w = p_w(t,\vec{r})$ for the unsteady flow. We have the expressions

$$\begin{aligned}&\vec{u}=\vec{c}+\vec{v}+\vec{w},\quad \vec{c}=const,\\ &p=p_v+p_w.\end{aligned} \tag{5.1}$$

We obtain the stationary equation

$$\rho \cdot (\vec{c} \cdot \nabla) \cdot \vec{v} + \rho \cdot (\vec{v} \cdot \nabla) \cdot \vec{v} = -\nabla p_v + \eta \cdot \Delta \vec{v} + \eta_1 \cdot \nabla div(\vec{v}), \tag{5.2}$$

and the non-stationary one

$$\rho \cdot \frac{\partial \vec{w}}{\partial t} + \rho \cdot (\vec{c} \cdot \nabla) \cdot \vec{w} + \rho \cdot (\vec{w} \cdot \nabla) \cdot \vec{w} + \rho \cdot (\vec{v} \cdot \nabla) \cdot \vec{w} + \rho \cdot (\vec{w} \cdot \nabla) \cdot \vec{v} = -\nabla p_w + \eta \cdot \Delta \vec{w} + \eta_1 \cdot \nabla div(\vec{w}). \tag{5.3}$$

5.2. **Steady flow**

We solve the stationary equation (5.2) for the velocity

$$v(\vec{r}) = \exp(-\alpha \cdot x) \cdot v(z) \cdot \vec{e}_x, \text{ and } \vec{c} = c \cdot \vec{e}_x. \tag{5.4}$$

From the stationary form of the equation (2.30), we obtain the equation for *v=v(z)* on the condition that $\frac{dP}{dx} = 0$:

$$\frac{d^2 v}{dz^2} + \omega_0{}^2 \cdot v = -\gamma \cdot v^2, \tag{5.5}$$

where $\kappa = \frac{\eta}{\rho}$, $\quad \gamma = \frac{\alpha}{\kappa}$, $\quad \omega_0{}^2 = \frac{c \cdot \alpha}{\kappa} = c \cdot \gamma$.

We define the solution in the form of the uniformly converging series

$$v = \sum_{j=1}^{\infty} v_j(z), j = 1, 2, \ldots, \infty. \tag{5.6}$$

We obtain the linear system of the equations for $v_j = v_j(z)$

$$\begin{aligned}
&\frac{d^2 v_1}{dz^2} + \omega_0{}^2 \cdot v_1 = 0, \\
&\frac{d^2 v_2}{dz^2} + \omega_0{}^2 \cdot v_2 = -\gamma \cdot v_1{}^2, \\
&\frac{d^2 v_3}{dz^2} + \omega_0{}^2 \cdot v_3 = -2 \cdot \gamma \cdot v_1 \cdot v_2, \\
&\ldots \\
&\frac{d^2 v_j}{dz^2} + \omega_0{}^2 \cdot v_j = -\gamma \cdot \sum_{n=1}^{j-1} v_n \cdot v_{j-n}. \\
&\ldots.
\end{aligned} \tag{5.7}$$

We define the solution of the first equation like

$$v_1 = u_0 \cdot \cos(kz + \varphi_0), \tag{5.8}$$

where *k* is an arbitrary real value.

We have to prevent the appearance of secular terms with $\cos(kz + \varphi_0)$ in the solutions of the rest equations of the system (5.7). We use the method of Poincare for this [21, 22].

We suppose that

$$k = k^{(0)} + k^{(1)} + k^{(2)} + \ldots, k^{(0)} = \omega_0. \tag{5.9}$$

We modify the first equation of the system:

$$\begin{aligned}
&\frac{d^2 v_1}{dz^2} + \omega_0^2 \cdot v_1 = 0, \\
&\frac{\omega_0^2}{k^2} \cdot \frac{d^2 v_1}{dz^2} + \omega_0^2 \cdot v_1 = -\left(1 - \frac{\omega_0^2}{k^2}\right) \cdot \frac{d^2 v_1}{dz^2}, \\
&\tfrac{\omega_0^2}{k^2} \cdot \tfrac{d^2 v_1}{dz^2} + \omega_0^2 \cdot v_1 = -\tfrac{(k^{(0)} + k^{(1)} + k^{(2)} + \ldots)^2 - \omega_0^2}{k^2} \cdot \tfrac{d^2 v_1}{dz^2} = \\
&= -\tfrac{(k^{(1)})^2 + (k^{(2)})^2 + \ldots + 2 \cdot \omega_0 \cdot k^{(1)} + 2 \cdot \omega_0 \cdot k^{(2)} + 2 \cdot k^{(1)} \cdot k^{(2)} + \ldots}{k^2} \cdot \tfrac{d^2 v_1}{dz^2}.
\end{aligned} \tag{5.10}$$

We modify the system (5.7) collecting the terms of the same order of smallness:

$$\frac{\omega_0^2}{k^2} \cdot \frac{d^2 v_1}{dz^2} + \omega_0^2 \cdot v_1 = 0,$$

$$\frac{d^2v_2}{dz^2} + \omega_0^2 \cdot v_2 = -\gamma \cdot {v_1}^2 - \frac{2 \cdot \omega_0 \cdot k^{(1)}}{k^2} \cdot \frac{d^2v_1}{dz^2},$$
$$\frac{d^3v_1}{dz^2} + \omega_0^2 \cdot v_3 = -2 \cdot \gamma \cdot v_1 \cdot v_2 - \frac{(k^{(1)})^2 + 2 \cdot \omega_0 \cdot k^{(2)}}{k^2} \cdot \frac{d^2v_1}{dz^2}, \tag{5.11}$$
$$\dots$$
$$\frac{d^2v_j}{dz^2} + \omega_0^2 \cdot v_j = -\gamma \cdot \sum_{n=1}^{j-1} v_n \cdot v_{j-n} - \frac{(k^{(1)})^{j-1} + 2 \cdot \omega_0 \cdot k^{(j-1)} + \dots}{k^2} \cdot \frac{d^2v_1}{dz^2}.$$
$$\dots.$$

The solutions of the initial three equations are

$$v_1 = u_0 \cdot \cos(kz + \varphi_0),$$
$$v_2 = -\frac{\gamma \cdot {u_0}^2}{2{\omega_0}^2} + \frac{\gamma \cdot {u_0}^2}{6{\omega_0}^2} \cdot \cos(2kz + 2\varphi_0), k^{(1)} = 0, \tag{5.12}$$
$$v_3 = \frac{\gamma^2 \cdot {u_0}^3}{48{\omega_0}^4} \cdot \cos(3kz + 3\varphi_0), k^{(2)} = -\frac{5\gamma^2 \cdot {u_0}^2}{12{\omega_0}^3}.$$

The solution of the stationary the equation (5.5) is

$$v(z) = u_0 \cdot \cos(kz + \varphi_0) - \frac{\gamma \cdot {u_0}^2}{2{\omega_0}^2} + \frac{\gamma \cdot {u_0}^2}{6{\omega_0}^2} \cdot \cos(2kz + 2\varphi_0) + \frac{\gamma^2 \cdot {u_0}^3}{48{\omega_0}^4} \cdot \cos(3kz + 3\varphi_0), \tag{5.13}$$

$$k = \omega_0 - \frac{5\gamma^2 \cdot {u_0}^2}{12{\omega_0}^3} + \dots. \tag{5.14}$$

Applying this solution to the boundary layer of the width d, we define

$$\beta = \gamma \cdot d^2 \cdot u_0 = \frac{d}{L} \cdot \frac{u_0 \cdot d}{\kappa} = \frac{Re}{Re_c}, \tag{5.15}$$

where $Re = \frac{u_0 \cdot d}{\kappa}$ is the Reynolds number, $Re_c = \alpha \cdot d = \frac{L}{d}$ is the critical Reynolds number.

We obtain

$$v(z) = u_0 \cdot \tag{5.16}$$
$$\left[\cos(kz + \varphi_0) - \frac{\beta}{2{\omega_0}^2 \cdot d^2} + \frac{\beta}{6{\omega_0}^2 \cdot d^2} \cdot \cos(2kz + 2\varphi_0) + \frac{\beta^2}{48{\omega_0}^4 \cdot d^4} \cdot \cos(3kz + 3\varphi_0)\right].$$

If

$$c \to 0 \ \ (\Rightarrow {\omega_0}^2 \to 0), \tag{5.17}$$

the equation (5.5) transforms to the equation (4.4) for Couette flow, and the solution (5.16) transforms to the one for Couette flow.

5.3. **Unsteady flow**

We solve the equation (5.3). To simplify the expressions we imply that

$$\vec{v} = \vec{v}(\vec{r}) \equiv 0. \tag{5.18}$$

We present the velocity for the unsteady flow as

$$\vec{w}(t, \vec{r}) = \exp(-\alpha \cdot x) \cdot v(t, z) \cdot \vec{e}_x, \text{ and } \vec{c} = c \cdot \vec{e}_x. \tag{5.19}$$

From the equation (2.30) we obtain the equation for $v = v(t, z)$

$$-\frac{1}{\kappa} \cdot \frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial z^2} + \omega_0^2 \cdot v = -\gamma \cdot v^2, \tag{5.20}$$

where $\kappa = \frac{\eta}{\rho}$, $\gamma = \frac{\alpha}{\kappa}$, ${\omega_0}^2 = \frac{c \cdot \alpha}{\kappa} = c \cdot \gamma$.

Let's define the solution as

$$v = \sum_{j=1}^{\infty} v_j(t, z) = \sum_{j=1}^{\infty} \exp[j \cdot \kappa \cdot (\omega_0^2 - \omega^2) \cdot t] \cdot v_j(z), j = 1, 2, \dots, \infty, \tag{5.21}$$

where ω^2 is a real value.

We obtain the linear system of equations for $v_j = v_j(z)$

$$\begin{gathered}
\frac{d^2v_1}{dz^2} + \omega^2 \cdot v_1 = 0, \\
\frac{d^2v_2}{dz^2} + (2\cdot\omega^2 - \omega_0)\cdot v_2 = -\gamma\cdot {v_1}^2, \\
\frac{d^2v_3}{dz^2} + (3\cdot\omega^2 - 2\cdot{\omega_0}^2)\cdot v_3 = -2\cdot\gamma\cdot v_1(z)\cdot v_2, \\
\cdots \\
\frac{d^2v_j}{dz^2} + [j\cdot\omega^2 - (j-1)\cdot{\omega_0}^2]\cdot v_j = -\gamma\cdot\sum_{n=1}^{j-1} v_n\cdot v_{j-n}. \\
\cdots
\end{gathered} \tag{5.22}$$

We have to prevent the appearance of secular terms with $\cos(kz+\varphi_0)$ in the solutions of the rest equations of the system (5.22). We use the method of Poincare for this.

The solutions of the initial three equations are

$$\begin{gathered}
v_1 = u_0\cdot\cos(kz+\varphi_0), \\
v_2 = \frac{\gamma\cdot u_0^2}{4\omega^2 + 2{\omega_0}^2}\cdot\cos(2kz+2\varphi_0) - \frac{\gamma\cdot u_0^2}{4\omega^2 - 2{\omega_0}^2}, \\
v_3 = \frac{\gamma^2\cdot u_0^3\cdot\cos(3kz+3\varphi_0)}{(4\omega^2+2{\omega_0}^2)\cdot(6\omega^2+2{\omega_0}^2)} + \frac{\gamma^2\cdot u_0^3\cdot(2\omega^2+3{\omega_0}^2)\cdot\cos(kz+\varphi_0)}{2\cdot(8\omega^4-2{\omega_0}^4)\cdot(\omega^2-{\omega_0}^2)},
\end{gathered} \tag{5.23}$$

where u_0, k and φ_0 is an arbitrary real value.

The solution of the equation (5.20) is

$$\begin{aligned}
&v(t,z) = u_0\cdot\exp[\kappa\cdot(\omega_0^2-\omega^2)\cdot t]\cdot\cos(kz+\varphi_0)+ \\
&+\gamma\cdot u_0^2\cdot\exp[2\cdot\kappa\cdot(\omega_0^2-\omega^2)\cdot t]\cdot\left[\frac{\cos(2kz+2\varphi_0)}{4\omega^2+2{\omega_0}^2} - \frac{1}{4\omega^2-2{\omega_0}^2}\right] + \gamma^2\cdot u_0^3\times \\
&\times\exp[3\cdot\kappa\cdot(\omega_0^2-\omega^2)\cdot t]\cdot\left[\frac{\cos(3kz+3\varphi_0)}{(4\omega^2+2{\omega_0}^2)\cdot(6\omega^2+2{\omega_0}^2)} + \frac{(2\omega^2+3{\omega_0}^2)\cdot\cos(kz+\varphi_0)}{2\cdot(8\omega^4-2{\omega_0}^4)\cdot(\omega^2-{\omega_0}^2)}\right] + \cdots .
\end{aligned} \tag{5.24}$$

The Krylov-Bogoliubov averaging method [21] can be used to prevent the appearance of secular terms in the solution. In this method the value k is dependent on time and the amplitude u_0.

The fluid flow blows up and the turbulence develops if

$$\omega_0^2 \geq \omega^2 \quad \text{or} \quad \frac{\alpha\cdot c}{\kappa} = \frac{c}{\kappa\cdot L} \geq \omega^2. \tag{5.25}$$

For the boundary layer of the width d this condition is

$$\frac{c}{\kappa\cdot L} = \frac{1}{d^2}\cdot\frac{c\cdot d}{\kappa\cdot}\cdot\frac{d}{L} = \frac{1}{d^2}\cdot\frac{Re}{Re_c} = \frac{1}{d^2}\cdot\beta \geq \omega^2 \Rightarrow \beta \geq \omega^2\cdot d^2, \tag{5.26}$$

where $Re = \frac{c\cdot d}{\kappa}$ is the Reynolds number, $Re_c = \frac{L}{d}$ is the critical Reynolds number.

The value of ω^2 can be obtained from the boundary conditions.

If $\omega_0^2 = \omega^2$, the nonlinear solution is stationary but unstable. It is obtained in the previous section.

Let's consider the flow if one of its rigid boundary moved suddenly and one held stationary [9] as an example.

5.4. **Flow which one of its rigid boundary moved suddenly and one held stationary**

We consider the fluid is bounded by two rigid plane boundaries at *z=0* and *z=d* [9]. The fluid is at rest at *t=0*. The lower boundary is stationary. The upper one is moved suddenly to the steady velocity *U*.

We define the velocity of fluid $v(t,z)$ as the sum of the solution $v(z)$ for the steady and $w(t,z)$ for the unsteady flow
$$v(t,z) = c + v(z) + w(t,z), \quad c = const. \tag{5.27}$$

These solutions are obtained from the linearized equation (5.5) for $v = v(z)$ and (5.20) for $w = w(t,z)$:
$$\frac{d^2v}{dz^2} + \omega_0^2 \cdot v = 0, \tag{5.28}$$
$$-\frac{1}{\kappa} \cdot \frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial z^2} + \omega_0^2 \cdot w = 0, \tag{5.29}$$
$$\omega_0^2 = \frac{c \cdot \alpha}{\kappa} = \frac{c}{L \cdot \kappa}. \tag{5.30}$$

The initial condition is
$$v(0,z) = 0, \quad 0 < z \le d. \tag{5.31}$$

The boundary conditions are
$$v(t,0) = 0, v(t,d) = U, t > 0. \tag{5.32}$$

We define the stationary solution as
$$v(z) = u_0 \cdot \sin(\omega_0(z-d)). \tag{5.33}$$

We suppose that
$$\omega_0 << 1, U = u_0 \cdot \omega_0 \cdot d, c = U. \tag{5.34}$$

We obtain
$$c + v(z) = U \cdot \frac{z}{d}. \tag{5.35}$$

We define the non-stationary solution as [9]
$$w(t,z) = -\frac{2U}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{n} \cdot \exp[(\alpha \cdot U - n^2\pi^2\kappa/d^2) \cdot t] \cdot \sin\left[n\pi\left(1 - \frac{z}{d}\right)\right]. \tag{5.36}$$

The solution of the problem satisfying the initial and boundary conditions is
$$v(t,z) = U \cdot \frac{z}{d} - \frac{2U}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{n} \cdot \exp[(\alpha \cdot U - n^2\pi^2\kappa/d^2) \cdot t] \cdot \sin\left[n\pi\left(1 - \frac{z}{d}\right)\right]. \tag{5.37}$$

We use the Reynolds number
$$Re = \frac{U \cdot d}{\kappa}. \tag{5.38}$$

We obtain
$$v(t,z) = U \cdot \frac{z}{d} - \frac{2U}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{n} \cdot \exp\left[\left(Re - \frac{n^2\pi^2}{\alpha \cdot d}\right) \cdot \frac{\alpha \cdot \kappa \cdot t}{d}\right] \cdot \sin\left[n\pi\left(1 - \frac{z}{d}\right)\right]. \tag{5.39}$$

This solution blows up if
$$Re \ge \frac{\pi^2}{\alpha \cdot d} = \pi^2 \frac{L}{d}. \tag{5.40}$$

The value of
$$Re_c = \pi^2 \frac{L}{d} \tag{5.41}$$
is the critical Reynolds number.

Using the Reynolds number and the critical Reynolds number, we have the expression
$$v(t,z) = U \cdot \frac{z}{d} - \frac{2U}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{n} \cdot \exp\left[\frac{(Re - n^2 \cdot Re_c)}{L} \cdot \frac{\kappa \cdot t}{d}\right] \cdot \sin\left[n\pi\left(1 - \frac{z}{d}\right)\right]. \tag{5.42}$$

According this formula, the flow turns from laminar to turbulent if the hydraulic diameter of the pipe is increased. If

$$Re = Re_c = \pi^2 \frac{L}{d}, \tag{5.43}$$

$$v(t,z) = U\cdot \tag{5.44}$$

$$\left\{\frac{z}{d} - \frac{2}{\pi}\cdot \sin\left[\pi\left(1-\frac{z}{d}\right)\right]\right\} - \frac{2U}{\pi}\cdot\sum_{n=2}^{\infty}\frac{1}{n}\cdot\exp\left[-\frac{(n^2-1)\cdot\pi^2}{d^2}\cdot\kappa\cdot t\right]\cdot\sin\left[n\pi\left(1-\frac{z}{d}\right)\right].$$

The value of L can be defined from the experimental data. According to [9], Reynolds's estimation of the critical Reynolds number for Poiseuille flow in water was about 13000 at 20 ^{o}C for $U{\cdot}d$ less than about130 cm^2·s. Let suppose that d=10 cm. Then L is ~13000 cm.

According the formulas (5.42) and (5.44), the scenario of the turbulence development is the following. If the value of the Reynolds number is subcritical and small enough: $Re << Re_c$, the steady laminar flow is established asymptotically relatively quickly. Close to the critical value, the flow is unsteady for a long time. At the critical value the first term of the series in the solution is stationary. The fluid flow obtains the steady component which is unstable. Small fluctuations cause the turbulence.

We can obtain the similar results for Poiseuille flow.

5.5. **Turbulence under action of external force**

We consider the stationary inhomogeneous Cauchy momentum equation for $\vec{v} = \vec{v}(\vec{r})$ and $p = p(\vec{r})$ as

$$\rho\cdot(\vec{c}\cdot\nabla)\cdot\vec{v} + \rho\cdot(\vec{v}\cdot\nabla)\cdot\vec{v} = -\nabla p + \vec{F} + \eta\cdot\Delta\vec{v} + \eta_1\cdot\nabla div(\vec{v}),$$

where $\vec{c} = const$, $\vec{F} = \vec{F}(\vec{r})$ is the external force.

We find the solution in the form

$$v = \exp(-\alpha\cdot x)\cdot v(z)\cdot\vec{e}_x \tag{5.45}$$

for the external force

$$\vec{F} = f_0\cdot\cos(\omega_f\cdot z)\cdot\vec{e}_x,\quad f_0 = const,\quad \omega_f = const, \tag{5.46}$$

and

$$\vec{c} = c_x\cdot\vec{e}_x + c_z\cdot\vec{e}_z. \tag{5.47}$$

From the equation (2.30) we obtain the equation for $v = v(z)$

$$\frac{\partial^2 v}{\partial z^2} + 2\lambda\cdot\frac{\partial v}{\partial z} + \omega_0^2\cdot v = -\gamma\cdot v^2 + f\cdot\cos(\omega_f\cdot z),$$

where $\lambda = \frac{c_z}{2}$, $\gamma = \frac{\alpha}{\kappa}$, $\omega_0^2 = \frac{c_x\cdot\alpha}{\kappa} = c_x\cdot\gamma$, $f = \frac{f_0}{\eta}$.

We perform the qualitative analysis of the non-linear resonance as in [22].

We consider the linearized equation

$$\frac{\partial^2 v}{\partial z^2} + 2\lambda\cdot\frac{\partial v}{\partial z} + \omega_0^2\cdot v = f\cdot\cos(\omega_f\cdot z). \tag{5.48}$$

The private solution of this equation for $f\cdot\cos(\omega_f\cdot z)$ is

$$v = u_0\cdot\cos(\omega_f\cdot z + \delta), \tag{5.49}$$

$$u_0 = \frac{f}{\sqrt{(\omega_0^2 - {\omega_f}^2)^2 + 4\cdot(\lambda\cdot\omega_f)^2}},\ tg\delta = \frac{2\lambda\cdot\omega_f}{{\omega_f}^2 - \omega_0^2}. \tag{5.50}$$

We take into account the dependence of ω_0 on the amplitude u_0 due to the non-linear effect and obtain the equation

$$\varepsilon = \chi\cdot u_0^2 \pm \sqrt{\left(\frac{f}{2\omega_0\cdot u_0}\right)^2 - \lambda^2},\ tg\delta \approx \frac{\lambda}{\varepsilon - \chi\cdot u_0^2}. \tag{5.51}$$

where $\varepsilon = \omega_f - \omega_0, \chi$ is the defined function of anharmonic coefficients.

If

$$f > f_k = \sqrt{\frac{32\omega_0^2 \cdot \lambda^3}{3\sqrt{3}|\chi|}}, \tag{5.52}$$

the solution (5.49) shows bifurcations and flow instability.

We can obtain the analogous results for the non-stationary equation $v = v(t, z)$

$$\frac{1}{\kappa} \cdot \frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial z^2} + 2\lambda \cdot \frac{\partial v}{\partial z} + \omega_0^2 \cdot v = -\gamma \cdot v^2 + f \cdot \exp(-\tau \cdot t) \cdot \cos(\omega_f \cdot z), \tag{5.53}$$

where τ is a constant value.

Let's show that the solution of the equation (5.48) can be reduced to the solution of the linear inhomogeneous equation for Couette flow obtained from (4.4).

The general solution of the equation (5.48) is [22]

$$v(z) = a \cdot \exp(-\lambda \cdot z) \cdot \cos(\omega \cdot z + \varphi_0) + u_0 \cdot \cos(\omega_f \cdot z + \delta), \tag{5.54}$$

where a, $\omega = \sqrt{\omega_0^2 - \lambda^2}$, φ_0 are constant values.

If $\lambda << 1$, $\omega << 1$, $\varphi_0 = \frac{3\pi}{2}$,

$$a \cdot \exp(-\lambda \cdot z) \cdot \cos(\omega \cdot z + \varphi_0) \approx a \cdot \sin(\omega \cdot z) \approx a \cdot \omega \cdot z.$$

$$u_0 \cdot \cos(\omega_f \cdot z + \delta) = u_0 \cdot [\cos(\omega_f \cdot z) \cdot \cos\delta - \sin(\omega_f \cdot z) \cdot \sin\delta] = A \cdot \cos(\omega_f \cdot z) + B \sin(\omega_f \cdot z),$$

where $A = u_0 \cdot \cos\delta$, $B = -u_0 \cdot \sin\delta$.

If $\omega_f << 1$

$$\cos(\omega_f \cdot z) \approx 1 - \frac{(\omega_f \cdot z)^2}{2}, \quad \sin(\omega_f \cdot z) \approx \omega_f \cdot z.$$

$$v(z) = a \cdot \omega \cdot z + A - \frac{A \cdot {\omega_f}^2}{2} \cdot z^2 + B \cdot \omega_f \cdot z. \tag{5.55}$$

We consider the presence of the constant value c_0 in the velocity

$$\hat{v}(z) = c_0 + v(z).$$

$$\hat{v}(z) = c_0 + a \cdot \omega \cdot z + A - \frac{A \cdot {\omega_f}^2}{2} \cdot z^2 + B \cdot \omega_f \cdot z. \tag{5.56}$$

According to the expression (4.4), the linearized inhomogeneous equation for Couette flow is

$$\frac{\partial^2 \hat{v}(z)}{\partial z^2} = -\frac{K}{\eta}. \tag{5.57}$$

where K is a constant value, $K = -\frac{\partial P}{\partial x}$.

To satisfy this equation we define

$$A \cdot \omega_f^2 = \frac{K}{\eta} \Rightarrow A = \frac{K}{\eta \cdot {\omega_f}^2}. \tag{5.58}$$

We have the boundary conditions

$$\hat{v}(0) = 0, \quad \hat{v}(d) = U. \tag{5.59}$$

To satisfy these conditions we define

$$a \cdot \omega = \frac{U}{d}, B = \frac{A \cdot \omega_f \cdot d}{2} = \frac{K \cdot d}{2\eta \cdot \omega_f}, c_0 + A = 0. \tag{5.60}$$

As a result, we obtain the solution of the equation (5.57) with the boundary conditions (5.59)

$$\hat{v}(z) = U \cdot \frac{z}{d} + \frac{K}{2\eta} \cdot z \cdot (d - z). \tag{5.61}$$

6. Navier-Stokes equations and solitary waves as dissipative structures

In this section we obtain the soliton solution of the Navier-Stokes equations and derive the Kortewegde Vries, Kortewegde VriesBurgers and KadomtsevPetviashvili equation from the Cauchy momentum equation for a dissipative medium. Solitary wave solutions of these equations are dissipative structures.

6.1. Soliton solutions of Navier-Stokes equations

We consider the stationary form of the equation (2.30) for the velocity

$$\vec{v}(\vec{r}) = \exp(-\alpha \cdot z) \cdot v(x) \cdot \vec{e}_z, \text{ and } \vec{c} = -c \cdot \vec{e}_z, \quad c = const, \tag{6.1}$$

on the condition that $\frac{dP}{dz} = 0$.

We obtain for $v = v(x)$

$$\frac{d^2 v}{dx^2} - {\omega_0}^2 \cdot v = -\gamma \cdot v^2, \tag{6.2}$$

where $\kappa = \frac{\eta}{\rho}, \quad \gamma = \frac{\alpha}{\kappa}, \quad {\omega_0}^2 = \frac{c \cdot \alpha}{\kappa} = c \cdot \gamma$.

Let's substitute the characteristic shape of the soliton [23] at the fixed time moment

$$v = \frac{a}{ch^2(b \cdot x + \varphi_0)} \tag{6.3}$$

in this equation, where a, b, φ_0 are constant values.

We obtain

$$\frac{4a \cdot b^2}{ch^2(b \cdot x + \varphi_0)} - \frac{6a \cdot b^2}{ch^4(b \cdot x + \varphi_0)} - {\omega_0}^2 \cdot \frac{a}{ch^2(b \cdot x + \varphi_0)} = -\gamma \cdot \frac{a^2}{ch^4(b \cdot x + \varphi_0)}.$$

$$a = \frac{6b^2}{\gamma}, \quad b = \pm\frac{\omega_0}{2},$$

$$a = \frac{3{\omega_0}^2}{2\gamma} = \frac{3c}{2}, \quad b = \pm\frac{\omega_0}{2}.$$

$$v = \frac{3 \cdot c}{2 \cdot ch^2\left(\pm\frac{\omega_0}{2} \cdot x + \varphi_0\right)} = \frac{3 \cdot c}{2 \cdot ch^2\left(\pm\sqrt{\frac{Re}{Re_c}} \cdot \frac{x}{2 \cdot d} + \varphi_0\right)}, \tag{6.4}$$

where $Re = \frac{c \cdot d}{\kappa}$ is the Reynolds number, $Re_c = \alpha \cdot d = \frac{L}{d}$ is the critical Reynolds number, d is the characteristic width of the soliton base.

The soliton is an accurate solution of the equation (6.2). In space of one dimension the shape of the plane stationary nonlinear waves at a characteristic velocity c_0 [24] is

$$v(t, x) = v(x \pm c_0 \cdot t + \varphi_0), \quad c_0 = const, \quad \varphi_0 = const, \tag{6.5}$$

Let's show that the Cauchy momentum equation (2.30) is reduced to the known equations obtaining the solutions (6.5).

6.2. Korteweg-de Vries equation

We define

$$x = \xi. \tag{6.6}$$

in (6.2) and differentiate this equation with respect to ξ

$$-{\omega_0}^2 \cdot \frac{dv}{d\xi} + 2\gamma \cdot v \cdot \frac{dv}{d\xi} + \frac{d^3 v}{d\xi^3} = 0. \tag{6.7}$$

We define

$$\begin{gathered} \xi = x - c_0 \cdot t + \varphi_0, \\ v = v(\xi) = v(t, x) = v(x - c_0 \cdot t + \varphi_0). \end{gathered} \tag{6.8}$$

where $c_0 = const, \quad \varphi_0 = const$,.

We have

$$\frac{\partial v}{\partial t} = -c_0 \cdot \frac{dv}{d\xi}, \qquad \frac{\partial v}{\partial x} = \frac{dv}{d\xi}. \tag{6.9}$$

We obtain the equation

$$\frac{{\omega_0}^2}{c_0} \cdot \frac{\partial v}{\partial t} + 2\gamma \cdot v \cdot \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = 0. \tag{6.10}$$

We change the variables and the solution as

$$c_0 \cdot \omega_0 \cdot t \to t, \qquad \omega_0 \cdot x \to x, \qquad \frac{-\gamma}{3 \cdot {\omega_0}^2} \cdot v \to v. \tag{6.11}$$

We obtain the Korteweg-de Vries equation (KdV) [23, 24]

$$\frac{\partial v}{\partial t} - 6v \cdot \frac{\partial v}{\partial x} + \frac{d^3 v}{dx^3} = 0. \tag{6.12}$$

6.3. **Korteweg- de Vries-Burgers equation**

We solve the equation (2.30) for the velocity

$$\vec{v}(\vec{r}) = \exp(-\alpha \cdot z) \cdot v(x) \cdot \vec{e}_z, \text{ and } \vec{c} = c_x \cdot \vec{e}_x - c \cdot \vec{e}_z, \tag{6.13}$$

on the condition that $\frac{dP}{dz} = 0$.

We obtain the equation for $v = v(x)$

$$-\frac{c_x}{\kappa} \cdot \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} - \omega_0^2 \cdot v = -\gamma \cdot v^2. \tag{6.14}$$

We define

$$v(x) = v(\xi).$$

We obtain

$$-\frac{c_x}{\kappa} \cdot \frac{dv}{d\xi} + \frac{d^2 v}{d\xi^2} - \omega_0^2 \cdot v = -\gamma \cdot v^2, \tag{6.15}$$

We differentiate this equation with respect to ξ

$$-{\omega_0}^2 \cdot \frac{dv}{d\xi} + 2\gamma \cdot v \cdot \frac{dv}{d\xi} + \frac{d^3 v}{d\xi^3} = \frac{c_x}{\kappa} \cdot \frac{d^2 v}{d\xi^2}. \tag{6.16}$$

We define

$$v = v(\xi) = v(t, x) = v(x - c_0 \cdot t + \varphi_0).$$

and

$$\frac{\partial v}{\partial t} = -c_0 \cdot \frac{dv}{d\xi}, \qquad \frac{\partial v}{\partial x} = \frac{dv}{d\xi}.$$

We obtain the equation

$$\frac{{\omega_0}^2}{c_0} \cdot \frac{\partial v}{\partial t} + 2\gamma \cdot v \cdot \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = \frac{c_x}{\kappa} \cdot \frac{\partial^2 v}{\partial x^2}. \tag{6.17}$$

We change the variables and the solution as

$$c_0 \cdot \omega_0 \cdot t \to t, \qquad \omega_0 \cdot x \to x, \qquad \frac{-\gamma}{3 \cdot {\omega_0}^2} \cdot v \to v.$$

We obtain the Korteweg- de Vries-Burgers (KdVB) equation [25]

$$\frac{\partial v}{\partial t} - 6\gamma \cdot v \cdot \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = \delta \cdot \frac{\partial^2 v}{\partial x^2}, \tag{6.18}$$

where $\delta = \frac{c_x}{\kappa \cdot \omega_0}$.

If $c_x = 0 \Rightarrow \delta = 0$ we have the KdV equation (6.12).

6.4. **Korteweg- de Vries and Korteweg- de Vries-Burgers equation in two spatial dimensions**

We solve the equation (2.30) for the velocity

$$\vec{v}(\vec{r}) = \exp(-\alpha \cdot z) \cdot v(x,y) \cdot \vec{e}_z, \text{ and } \vec{c} = c_x \cdot \vec{e}_x + c_y \cdot \vec{e}_y - c \cdot \vec{e}_z. \tag{6.19}$$

We obtain the equation for $v = v(x,y)$

$$-\frac{c_x}{\kappa} \cdot \frac{\partial v}{\partial x} - \frac{c_y}{\kappa} \cdot \frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \omega_0^2 \cdot v = -\gamma \cdot v^2, \tag{6.20}$$

We define the expression

$$v = v(\xi) = v\left(\frac{x}{c_{0x}} + \frac{y}{c_{0y}} - t + \varphi_0\right), \quad c_{0x} = const, \quad c_{0y} = const, \quad \varphi_0 = const. \tag{6.21}$$

From (6.20) we obtain the equation for $v = v(\xi)$

$$-\left(\frac{c_x}{\kappa \cdot c_{0x}} + \frac{c_y}{\kappa \cdot c_{0y}}\right) \cdot \frac{dv}{d\xi} + \left(\frac{1}{{c_{0x}}^2} + \frac{1}{{c_{0y}}^2}\right) \cdot \frac{d^2 v}{d\xi^2} - {\omega_0}^2 \cdot v = -\gamma \cdot v^2. \tag{6.22}$$

We differentiate this equation with respect to ξ

$$-{\omega_0}^2 \cdot \frac{dv}{d\xi} + 2\gamma \cdot v \cdot \frac{dv}{d\xi} - \left(\frac{c_x}{\kappa \cdot c_{0x}} + \frac{c_y}{\kappa \cdot c_{0y}}\right) \cdot \frac{d^2 v}{d\xi^2} + \left(\frac{1}{{c_{0x}}^2} + \frac{1}{{c_{0y}}^2}\right) \cdot \frac{d^3 v}{d\xi^3} = 0. \tag{6.23}$$

We use the expressions

$$\frac{\partial v}{\partial t} = -\frac{dv}{d\xi}, \qquad \frac{\partial v}{\partial x} = \frac{1}{c_{0x}} \cdot \frac{dv}{d\xi} \qquad \frac{\partial v}{\partial y} = \frac{1}{c_{0y}} \cdot \frac{dv}{d\xi}. \tag{6.24}$$

We obtain

$${\omega_0}^2 \cdot \frac{\partial v}{\partial t} + c_{0x} \cdot \left(\gamma \cdot v \cdot \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} - \frac{c_x}{\kappa} \cdot \frac{\partial^2 v}{\partial x^2}\right) + c_{0y} \cdot \left(\gamma \cdot v \cdot \frac{\partial v}{\partial y} + \frac{\partial^3 v}{\partial y^3} - \frac{c_y}{\kappa} \cdot \frac{\partial^2 v}{\partial y^2}\right) = 0 \tag{6.25}$$

We change the variables and the solution coefficient

$$\omega_0 \cdot t \to t, \qquad \omega_0 \cdot x \to x, \qquad \omega_0 \cdot y \to y, \qquad \frac{\gamma}{6 \cdot {\omega_0}^2} \cdot v \to v. \tag{6.26}$$

We obtain the KdVB equation in two spatial dimensions

$$\frac{\partial v}{\partial t} + c_{0x} \cdot \left(6v \cdot \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} - \delta_x \cdot \frac{\partial^2 v}{\partial x^2}\right) + c_{0y} \cdot \left(6v \cdot \frac{\partial v}{\partial y} + \frac{\partial^3 v}{\partial y^3} - \delta_y \cdot \frac{\partial^2 v}{\partial y^2}\right) = 0, \tag{6.27}$$

or

$$\frac{\partial v}{\partial t} + c_{0x} \cdot \frac{\partial}{\partial x}\left(3 \cdot v^2 + \frac{\partial^2 v}{\partial x^2} - \delta_x \cdot \frac{\partial v}{\partial x}\right) + c_{0y} \cdot \frac{\partial}{\partial y}\left(3 \cdot v^2 + \frac{\partial^2 v}{\partial y^2} - \delta_y \cdot \frac{\partial v}{\partial y}\right) = 0, \tag{6.28}$$

where $\delta_x = \frac{c_x}{\kappa \cdot \omega_0}, \quad \delta_y = \frac{c_y}{\kappa \cdot \omega_0}$.

If

$$c_x = 0, \quad c_y = 0 \Rightarrow \delta_x = 0, \quad \delta_y = 0, \tag{6.29}$$

we obtain the KdV equation in two spatial dimensions

$$\frac{\partial v}{\partial t} + c_{0x} \cdot \left(6v \cdot \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3}\right) + c_{0y} \cdot \left(6v \cdot \frac{\partial v}{\partial y} + \frac{\partial^3 v}{\partial y^3}\right) = 0, \tag{6.30}$$

or

$$\frac{\partial v}{\partial t} + c_{0x} \cdot \frac{\partial}{\partial x}\left(3 \cdot v^2 + \frac{\partial^2 v}{\partial x^2}\right) + c_{0y} \cdot \frac{\partial}{\partial y}\left(3 \cdot v^2 + \frac{\partial^2 v}{\partial y^2}\right) = 0. \tag{6.31}$$

If

$$c_{0y} = 0, \ \ c_y = 0 \text{ and } c_{0x} \cdot t \to t, \tag{6.32}$$

we obtain the KdV equation in one spatial dimension

$$\frac{\partial v}{\partial t} + 6v \cdot \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = 0. \tag{6.33}$$

6.5. **Kadomtsev-Petviashvili equation**

We consider the equation (6.23) on the condition that $c_x = 0, \; c_y = 0$. We suppose that the solution change is small in the direction parallel to the y axis [26]:

$$c_{0y} >> \; c_{0x} \Rightarrow \frac{1}{{c_{0x}}^2} + \frac{1}{{c_{0y}}^2} \approx \frac{1}{{c_{0x}}^2}. \quad (6.34)$$

We use the variable $\xi = \frac{x}{c_{0x}}$ and obtain

$$-{\omega_0}^2 \cdot \frac{\partial v}{\partial x} + 2\gamma \cdot v \cdot \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = 0. \quad (6.35)$$

We obtain the expressions from (6.21)

$$\frac{\partial v}{\partial t} + c_{0x} \cdot \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial t} + c_{0y} \cdot \frac{\partial v}{\partial y} = 0, \qquad c_{0x} \frac{\partial v}{\partial x} - c_{0y} \cdot \frac{\partial v}{\partial y} = 0. \quad (6.36)$$

We obtain from the first and the second expression

$$\frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y} - \left(\frac{1}{c_{0x}} + \frac{1}{c_{0y}}\right) \cdot \frac{\partial v}{\partial t}. \quad (6.37)$$

We take into account that $c_{0y} >> c_{0x}$:

$$\frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y} - \frac{1}{c_{0x}} \cdot \frac{\partial v}{\partial t}. \quad (6.38)$$

We substitute this expression in (6.35)

$${\omega_0}^2 \cdot \frac{\partial v}{\partial y} + {\omega_0}^2 \cdot \frac{1}{c_{0x}} \cdot \frac{\partial v}{\partial t} + 2\gamma \cdot v \cdot \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = 0. \quad (6.39)$$

We differentiate this equation with respect to x. We use the third formula from (6.36) in the expression

$$\frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \frac{c_{0y}}{c_{0x}} \cdot \frac{\partial^2 v}{\partial y^2}. \quad (6.40)$$

We obtain

$${\omega_0}^2 \cdot \frac{c_{0y}}{c_{0x}} \cdot \frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial x}\left(\frac{{\omega_0}^2}{c_{0x}} \cdot \frac{\partial v}{\partial t} + 2\gamma \cdot v \cdot \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3}\right) = 0, \quad (6.41)$$

and

$$c_{0y} \cdot \frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + \frac{2\gamma \cdot c_{0x}}{{\omega_0}^2} v \cdot \frac{\partial v}{\partial x} + \frac{c_{0x}}{{\omega_0}^2} \cdot \frac{\partial^3 v}{\partial x^3}\right) = 0. \quad (6.42)$$

We change the variables and the solution coefficient

$$c_{0x} \cdot \omega_0 \cdot t \to t, \qquad \omega_0 \cdot x \to x, \qquad \omega_0 \cdot y \to y, \qquad \frac{\gamma}{3 \cdot {\omega_0}^2} \cdot v \to v. \quad (6.43)$$

We obtain the Kadomtsev-Petviashvili equation

$$\delta \cdot \frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + 6 \cdot v \cdot \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3}\right) = 0, \quad (6.44)$$

where $\delta = \frac{c_{0y}}{c_{0x}}$.

If $c_{0y} = 0 \Rightarrow \delta = 0$, we obtain the KdV equation.

6.6. KdV and KdVB equations in one and two spatial dimensions obtained from non-stationary Navier-Stokes equations

We can also derive the KdV and KdBV equations in one and two spatial dimensions from the non-stationary equation (2.30). We solve this equation for the velocity

$$\vec{v}(t,\vec{r}) = \exp(-\alpha\cdot z)\cdot v(t,x,y)\cdot\vec{e}_z, \text{ and } \vec{c} = \frac{c_x}{2}\cdot\vec{e}_x + \frac{c_y}{2}\cdot\vec{e}_y - c\cdot\vec{e}_z. \tag{6.45}$$

We obtain the equation for $v = v(t,x,y)$

$$-\frac{1}{\kappa}\cdot\frac{\partial v}{\partial t} - \frac{c_x}{2\kappa}\cdot\frac{\partial v}{\partial x} - \frac{c_y}{2\kappa}\cdot\frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \omega_0^2\cdot v = -\gamma\cdot v^2, \tag{6.46}$$

We define the expression

$$\begin{gathered} v = v(\xi) = v\left(\tfrac{x}{c_{0x}} + \tfrac{y}{c_{0y}} - t + \varphi_0\right), \\ c_{0x} = const, \quad c_{0y} = const, \quad c_{0x} \neq c_x, \quad c_{0y} \neq c_y, \quad \varphi_0 = const, \end{gathered} \tag{6.47}$$

and the expressions

$$\frac{\partial v}{\partial t} = -\frac{dv}{d\xi}, \qquad \frac{\partial v}{\partial x} = \frac{1}{c_{0x}}\cdot\frac{dv}{d\xi}, \qquad \frac{\partial v}{\partial y} = \frac{1}{c_{0y}}\cdot\frac{dv}{d\xi}. \tag{6.48}$$

We obtain the equation

$$\chi\cdot\frac{dv}{d\xi} + \left(\frac{1}{{c_{0x}}^2} + \frac{1}{{c_{0y}}^2}\right)\cdot\frac{d^2 v}{d\xi^2} - \omega_0^2\cdot v = -\gamma\cdot v^2, \tag{6.49}$$

where

$$\chi = \frac{1}{\kappa}\cdot\left(1 - \frac{c_x}{2c_{0x}} - \frac{c_y}{2c_{0y}}\right) = \frac{1}{\kappa}\cdot\left(\frac{c_{0x}-c_x}{2c_{0x}} + \frac{c_{0y}-c_y}{2c_{0y}}\right). \tag{6.50}$$

We differentiate this equation with respect to ξ

$$-{\omega_0}^2\cdot\frac{dv}{d\xi} + 2\gamma\cdot v\cdot\frac{dv}{d\xi} + \chi\cdot\frac{d^2 v}{d\xi^2} + \left(\frac{1}{{c_{0x}}^2} + \frac{1}{{c_{0y}}^2}\right)\cdot\frac{d^3 v}{d\xi^3} = 0.$$

We use the expressions (6.48)

$$\begin{gathered} {\omega_0}^2\cdot\frac{\partial v}{\partial t} + c_{0x}\cdot\left(\gamma\cdot v\cdot\frac{\partial v}{\partial x} + \frac{c_{0x}-c_x}{2\kappa}\cdot\frac{\partial^2 v}{\partial x^2} + \frac{\partial^3 v}{\partial x^3}\right) + \\ c_{0y}\cdot\left(\gamma\cdot v\cdot\frac{\partial v}{\partial y} + \frac{c_{0y}-c_y}{2\kappa}\cdot\frac{\partial^2 v}{\partial y^2} + \frac{\partial^3 v}{\partial y^3}\right) = 0. \end{gathered} \tag{6.51}$$

We change the variables and the solution as

$$\omega_0\cdot t \to t, \qquad \omega_0\cdot x \to x, \qquad \omega_0\cdot y \to y, \qquad \frac{\gamma}{6\cdot{\omega_0}^2}\cdot v \to v. \tag{6.52}$$

We obtain the KdVB equation in two spatial dimensions

$$\frac{\partial v}{\partial t} + c_{0x}\cdot\left(6\cdot v\cdot\frac{\partial v}{\partial x} + \delta_x\cdot\frac{\partial^2 v}{\partial x^2} + \frac{\partial^3 v}{\partial x^3}\right) + c_{0y}\cdot\left(6\cdot v\cdot\frac{\partial v}{\partial y} + \delta_y\cdot\frac{\partial^2 v}{\partial y^2} + \frac{\partial^3 v}{\partial y^3}\right) = 0, \tag{6.53}$$

where $\delta_x = \frac{c_{0x}-c_x}{2\kappa\cdot\omega_0}$, $\delta_y = \frac{c_{0y}-c_y}{2\kappa\cdot\omega_0}$.

If

$$c_{0x} = c_x, \quad c_{0y} = c_y \Rightarrow \delta_x = 0, \quad \delta_y = 0, \tag{6.54}$$

we obtain the KdV equation in two spatial dimensions

$$\frac{\partial v}{\partial t} + c_{0x}\cdot\left(6v\cdot\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3}\right) + c_{0y}\cdot\left(6v\cdot\frac{\partial v}{\partial y} + \frac{\partial^3 v}{\partial y^3}\right) = 0. \tag{6.55}$$

If

$$c_{0y} = 0 \text{ and } c_{0x}\cdot t \to t, \tag{6.56}$$

we obtain the KdVB equation in one spatial dimension

$$\frac{\partial v}{\partial t} + 6 \cdot v \cdot \frac{\partial v}{\partial x} + \delta_x \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial^3 v}{\partial x^3} = 0. \tag{6.57}$$

If

$$c_{0x} = c_x \Rightarrow \delta_x = 0, \tag{6.58}$$

and (6.56) is performed, we obtain the KdV equation in one spatial dimension

$$\frac{\partial v}{\partial t} + 6v \cdot \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = 0. \tag{6.59}$$

6.7. Instability of nonlinear waves

The monochromatic nonlinear wave can be expanded into the Fourier series

$$v(t, x) = \sum_{n=0}^{\infty} v_n(t) \cdot \exp(i \cdot nkx),$$

or

$$v(t, x) = \sum_{n=0}^{\infty} a_n \cdot \exp(i \cdot nk(x - ct)),$$

where a_n, c and $k$are constant values.

As shown in [11] that the nonlinear waves presented by this way can be involved in the resonance interaction with the development of turbulence. We can describe this development as the nonlinear resonance under the influence of external perturbation or force.

7. Instability of magnetohydrodynamic Couette flow

In this section we consider a mechanism of selfgenerating a magnetic field through movement of an electrically conducting fluid. We modify the equations of magnetohydrodynamics (MHD) and obtain the solutions which reveal the dynamo effect for the Couette magneto hydrodynamic flow.

7.1. Transform of magnetohydrodynamic equations for Couette flow

We take the equations of magnetohydrodynamics (MHD) for an incompressible fluid [28] and modify them. The modified MHD equations look as follows

$$\begin{gathered} \rho \cdot \partial \vec{v}(t, \vec{r})/\partial t + \rho \cdot (\vec{v}(t, \vec{r}) \cdot \nabla) \cdot \vec{v}(t, \vec{r}) = -\nabla \left(p(t, \vec{r}) + \tfrac{H^2}{8\pi} \right) + \eta \cdot \Delta \vec{v}(t, \vec{r}) + \\ + \tfrac{1}{4\pi} (\vec{H}(t, \vec{r}) \cdot \nabla) \cdot \vec{H}(t, \vec{r}) + (\eta/3 + \zeta) \cdot \nabla div(\vec{v}(t, \vec{r})), \end{gathered} \tag{7.1}$$

$$\partial \vec{H}(t, \vec{r})/\partial t = \frac{c^2}{4\pi\sigma} \Delta \vec{H}(t, \vec{r}) + (\vec{H}(t, \vec{r}) \cdot \nabla) \vec{v}(t, \vec{r}) - (\vec{v}(t, \vec{r}) \cdot \nabla) \vec{H}(t, \vec{r}) - \vec{H} \cdot div(\vec{v}(t, \vec{r})), \tag{7.2}$$

$$div(\vec{H}(t, \vec{r})) = 0, \tag{7.3}$$

$$div(\vec{v}(t, \vec{r})) \neq 0, \tag{7.4}$$

where$\vec{H}(t, \vec{r})$is the magnetic field, c is the speed of light, and $\sigma$is electrical conductivity.

The Couette flow of an unsteady incompressible electrically conducting viscous fluid which thickness is $2d$ is assumed to be flowing between two parallel non-conductive plates located at z=–d and z=d in the presence of a transverse magnetic field which is applied perpendicular to the walls as shown in Fig. 5.

The Cartesian x-axis is directed along the flow. The z-axis is perpendicular to the plates. The value of the uniform magnetic field G_0, which is directed along the z-axis, is assumed to be unaltered in time. The flow velocity v is dependent only on time t and coordinate z. The same applies to the longitudinal magnetic field H arising from the motion of the fluid. A pressure gradient in the x-direction is equal to zero.

We accept the initial condition at t=0 as the absence of the fluid flow. The plates move in the opposite directions with increase of the velocity from zero to a constant value U. The slip condition implies that the plates move with the velocity of the fluid on the plates. The magnetic field doesn't penetrate inside the walls. So H is equal to zero at *z=–d* and *z=d* because the tangential component of the magnetic field is continouse on the plates.

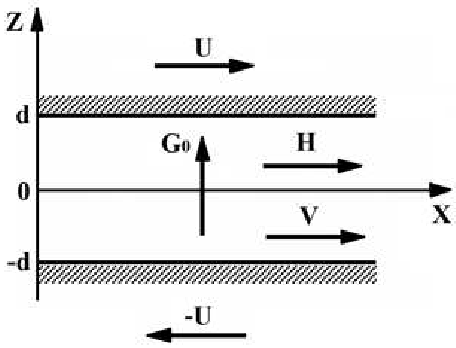

Figure 5. The Couette magneto hydrodynamic flow

So, we have

$$\vec{v} = v(t,z) \cdot \vec{e}_x, div(\vec{v}) = 0,$$

$$\vec{G_0} = G_0 \cdot \vec{e}_z, \quad G_0 = const,$$

$$\vec{H} = H(t,z) \cdot \vec{e}_x, div(\vec{H}) = 0, \frac{\partial}{\partial x}\left(p + \frac{H^2}{8\pi}\right) = 0.$$

From the equations (7.1), (7.2) and (5.20) we obtain the system

$$\begin{aligned} &-\tfrac{1}{\kappa} \cdot \tfrac{\partial v}{\partial t} + \tfrac{\partial^2 v}{\partial z^2} + \omega_0^2 \cdot v = -\gamma \cdot v^2 - \tfrac{G_0}{4\pi\eta} \cdot \tfrac{\partial H}{\partial z} \\ &-\tfrac{\partial H}{\partial t} + \kappa_m \cdot \tfrac{\partial^2 H}{\partial z^2} = -G_0 \cdot \tfrac{\partial v}{\partial z}, \end{aligned} \tag{7.5}$$

where $\kappa = \frac{\eta}{\rho}$, $\gamma = \frac{\alpha}{\kappa}$, ${\omega_0}^2 = \frac{v_0 \cdot \alpha}{\kappa} = v_0 \cdot \gamma$, $\kappa_m = \frac{c^2}{4\pi\sigma}$, $v = v(t,z)$, $H = H(t,z)$,

$$v = v(t,z).$$

7.2. **Stationary problem**

The stationary form of equations (7.5) is

$$\begin{aligned} &\tfrac{\partial^2 v}{\partial z^2} + v_0 \cdot \gamma \cdot v = -\gamma \cdot v^2 - \tfrac{G_0}{4\pi\eta} \cdot \tfrac{\partial H}{\partial z} \\ &\kappa_m \cdot \tfrac{\partial^2 H}{\partial z^2} = -G_0 \cdot \tfrac{\partial v}{\partial z}. \end{aligned} \tag{7.6}$$

$$v = v(z), H = H(z).$$

The boundary conditions for $v(z)$ are

$$v(-d) = -U - v_0, \qquad v(d) = U - v_0. \tag{7.7}$$

From the second equation we obtain

$$\frac{\partial H}{\partial z} = -\frac{G_0}{\kappa_m} \cdot v - C_0, C_0 = const. \tag{7.8}$$

We have the first equation as

$$\frac{\partial^2 v}{\partial z^2} - \left(\frac{{G_0}^2}{4\pi\rho \cdot \kappa_m \cdot \kappa} - v_0 \cdot \gamma\right) \cdot v = -\gamma \cdot v^2 + C, C = \frac{G_0}{4\pi\eta} \cdot C_0. \tag{7.9}$$

We define the function

$$v' = v + v_0 \Rightarrow v = v' - v_0. \tag{7.10}$$

We obtain the equation

$$\frac{\partial^2 v'}{\partial z^2} - \left(\frac{{G_0}^2}{4\pi\rho \cdot \kappa_m \cdot \kappa} + v_0 \cdot \gamma\right) \cdot v' + \left(\frac{{G_0}^2}{4\pi\rho \cdot \kappa_m \cdot \kappa} - v_0 \cdot \gamma\right) \cdot v_0 = -\gamma \cdot v'^2 - \gamma \cdot {v_0}^2 + C. \tag{7.11}$$

We accept that

$$C = \frac{v_0 \cdot {G_0}^2}{4\pi\rho \cdot \kappa_m \cdot \kappa}, C_0 = \frac{v_0 \cdot G_0}{\kappa_m}. \tag{7.12}$$

and obtain the equation

$$\frac{\partial^2 v'}{\partial z^2} - \left(\frac{{G_0}^2}{4\pi\rho \cdot \kappa_m \cdot \kappa} + v_0 \cdot \gamma\right) \cdot v' = -\gamma \cdot v'^2. \tag{7.13}$$

We define

$$\frac{{G_0}^2}{4\pi\rho \cdot \kappa_m \cdot \kappa} = \frac{4\pi\sigma}{c^2} \cdot \frac{{G_0}^2}{4\pi\eta} = \frac{{G_0}^2 \cdot d^2}{c^2} \cdot \frac{\sigma}{\eta} \cdot \frac{1}{d^2} = \frac{Ha^2}{d^2}, \tag{7.14}$$

where $Ha = \frac{G_0 \cdot d}{c} \cdot \sqrt{\frac{\sigma}{\eta}}$ is Hartmann number.

We define

$$\alpha = \frac{1}{L}, \quad Re' = \frac{v_0 \cdot d}{\kappa}, \quad Re_c = \frac{L}{d}, \quad v_0 \cdot \gamma = \frac{v_0 \cdot \alpha}{\kappa} = \frac{1}{d^2} \cdot \frac{v_0 \cdot d}{\kappa} \cdot \frac{d}{L} = \frac{1}{d^2} \cdot \frac{Re'}{Re_c}. \tag{7.15}$$

$$\frac{\partial^2 v'}{\partial z^2} - \frac{1}{d^2}\left(Ha^2 + \frac{Re'}{Re_c}\right) \cdot v' = -\gamma \cdot v'^2. \tag{7.16}$$

We define

$$z' = \frac{z}{d}, \quad M^2 = Ha^2 + \frac{Re'}{Re_c}, \tag{7.17}$$

and obtain the nonlinear equation

$$\frac{\partial^2 v'}{\partial z'^2} - M^2 \cdot v' = -\gamma \cdot d^2 \cdot v'^2, \tag{7.18}$$

with the boundary conditions

$$v'(-1) = -U, \quad v'(1) = U. \tag{7.19}$$

We define the solution in the form of the uniformly converging series

$$v' = \sum_{j=1}^{\infty} v_j\left(\frac{z}{d}\right), j = 1, 2, \ldots, \infty. \tag{7.20}$$

We obtain the linear system of the equations for $v_j = v_j\left(\frac{z}{d}\right)$

$$\frac{d^2 v_1}{dz^2} - M^2 \cdot v_1 = 0,$$

$$\frac{d^2 v_2}{dz^2} - M^2 \cdot v_2 = -\gamma \cdot d^2 \cdot {v_1}^2,$$

$$\frac{d^2 v_3}{dz^2} - M^2 \cdot v_3 = -2 \cdot \gamma \cdot d^2 \cdot v_1 \cdot v_2, \tag{7.21}$$

$$\ldots$$

$$\frac{d^2 v_j}{dz^2} - M^2 \cdot v_j = -\gamma \cdot d^2 \cdot \sum_{n=1}^{j-1} v_n \cdot v_{j-n}.$$

$$\cdots.$$

We define the solution of the first equation like

$$v_1 = u_0 \cdot sh\left(k \cdot z' + \varphi_0\right), \tag{7.22}$$

where k is an arbitrary real value, $u_0 = const$, $\varphi_0 = const$.

We have to prevent the appearance of secular terms with $sh\left(k \cdot z' + \varphi_0\right)$ in the solutions of the rest equations of the system (7.21). We use the method of Poincare for this [21, 22].

We suppose that

$$k = k^{(0)} + k^{(1)} + k^{(2)} + \ldots, k^{(0)} = M. \tag{7.23}$$

We modify the first equation of the system:

$$\begin{gathered} \frac{d^2 v_1}{dz'^2} - M^2 \cdot v_1 = 0, \\ \frac{M^2}{k^2} \cdot \frac{d^2 v_1}{dz'^2} - M^2 \cdot v_1 = -\left(1 - \frac{M^2}{k^2}\right) \cdot \frac{d^2 v_1}{dz'^2}, \\ \frac{M^2}{k^2} \cdot \frac{d^2 v_1}{dz'^2} - M^2 \cdot v_1 = -\frac{(k^{(0)}+k^{(1)}+k^{(2)}+\ldots)^2 - M^2}{k^2} \cdot \frac{d^2 v_1}{dz'^2} = \\ = -\frac{(k^{(1)})^2+(k^{(2)})^2+\ldots+2 \cdot M \cdot k^{(1)}+2 \cdot M \cdot k^{(2)}+2 \cdot k^{(1)} \cdot k^{(2)}+\ldots}{k^2} \cdot \frac{d^2 v_1}{dz'^2}. \end{gathered} \tag{7.24}$$

We modify the system (7.21) collecting the terms of the same order of smallness:

$$\begin{gathered} \frac{M^2}{k^2} \cdot \frac{d^2 v_1}{dz'^2} - M^2 \cdot v_1 = 0, \\ \frac{d^2 v_2}{dz'^2} - M^2 \cdot v_2 = -\gamma \cdot d^2 \cdot {v_1}^2 - \frac{2 \cdot M \cdot k^{(1)}}{k^2} \cdot \frac{d^2 v_1}{dz'^2}, \\ \frac{d^2 v_3}{dz'^2} - M^2 \cdot v_3 = -2 \cdot \gamma \cdot d^2 \cdot v_1 \cdot v_2 - \frac{(k^{(1)})^2 + 2 \cdot M \cdot k^{(2)}}{k^2} \cdot \frac{d^2 v_1}{dz'^2}, \\ \cdots \\ \frac{d^2 v_j}{dz'^2} - M^2 \cdot v_j = -\gamma \cdot d^2 \cdot \sum_{n=1}^{j-1} v_n \cdot v_{j-n} - \frac{(k^{(1)})^{j-1} + 2 \cdot M \cdot k^{(j-1)} + \ldots}{k^2} \cdot \frac{d^2 v_1}{dz'^2}. \\ \cdots. \end{gathered} \tag{7.25}$$

The solutions of the initial three equations are

$$\begin{gathered} v_1 = u_0 \cdot sh\left(\frac{kz}{d} + \varphi_0\right), \\ v_2 = \frac{\gamma \cdot d^2 \cdot {u_0}^2}{2M^2} - \frac{\gamma \cdot d^2 \cdot {u_0}^2}{6M^2} \cdot ch\left(\frac{2kz}{d} + 2\varphi_0\right), k^{(1)} = 0, \\ v_3 = \frac{\gamma^2 \cdot d^4 \cdot {u_0}^3}{48M^4} \cdot sh\left(\frac{3kz}{d} + 3\varphi_0\right), k^{(2)} = -\frac{7\gamma^2 \cdot d^4 \cdot {u_0}^2}{12M^3}. \end{gathered} \tag{7.26}$$

The solution for the stationary velocity is

$$\begin{gathered} v'\left(\frac{z}{d}\right) = u_0 \cdot sh\left(\frac{kz}{d} + \varphi_0\right) + \frac{\gamma \cdot d^2 \cdot {u_0}^2}{2M^2} - \frac{\gamma \cdot d^2 \cdot {u_0}^2}{6M^2} \cdot ch\left(\frac{2kz}{d} + 2\varphi_0\right) + \\ + \frac{\gamma^2 \cdot d^4 \cdot {u_0}^3}{48M^4} \cdot sh\left(\frac{3kz}{d} + 3\varphi_0\right) + \ldots, \end{gathered} \tag{7.27}$$

$$k = M - \frac{7\gamma^2 \cdot d^4 \cdot {u_0}^2}{12M^3} + \ldots. \tag{7.28}$$

We obtain the solution for the magnetic field H from the expression

$$\frac{\partial H}{\partial z} = -\frac{G_0}{\kappa_m} \cdot v - C_0 = -\frac{G_0}{\kappa_m} \cdot v' + \frac{v_0 \cdot G_0}{\kappa_m} - \frac{v_0 \cdot G_0}{\kappa_m} = -\frac{G_0}{\kappa_m} \cdot v'. \tag{7.29}$$

$$H = -\tfrac{G_0 \cdot d \cdot u_0}{k \cdot \kappa_m} \times$$

$$\left[ch\left(\tfrac{kz}{d} + \varphi_0\right) + \tfrac{\gamma \cdot d^2 \cdot u_0}{2M^2} \cdot \tfrac{kz}{d} - \tfrac{\gamma \cdot d^2 \cdot u_0}{12M^2} \cdot sh\left(\tfrac{2kz}{d} + 2\varphi_0\right) + \tfrac{\gamma^2 \cdot d^4 \cdot u_0{}^2}{144M^4} \cdot ch\left(\tfrac{3kz}{d} + 3\varphi_0\right) + \ldots\right] - \tag{7.30}$$

$$-C_1,$$

$$C_1 = const.$$

We satisfy the boundary conditions

$$v'(-1) = -U, \qquad v'(1) = U, \qquad H(-d) = H(d) = 0. \tag{7.31}$$

We have the system of four equations

$$v'(-1) = u_0 \cdot sh[k - \varphi_0] - \tfrac{\gamma \cdot d^2 \cdot u_0{}^2}{2M^2} + \tfrac{\gamma \cdot d^2 \cdot u_0{}^2}{6M^2} \cdot ch[2 \cdot (k - \varphi_0)] +$$

$$+ \tfrac{\gamma^2 \cdot d^4 \cdot u_0{}^3}{48M^4} \cdot sh[3 \cdot (k - \varphi_0)] + \ldots = U.$$

$$v'(1) = u_0 \cdot sh[k + \varphi_0] + \tfrac{\gamma \cdot d^2 \cdot u_0{}^2}{2M^2} - \tfrac{\gamma \cdot d^2 \cdot u_0{}^2}{6M^2} \cdot ch[2 \cdot (k + \varphi_0)] + \tag{7.32}$$

$$+ \tfrac{\gamma^2 \cdot d^4 \cdot u_0{}^3}{48M^4} \cdot sh[3 \cdot (k + \varphi_0)] + \ldots = U.$$

$$ch[k - \varphi_0] - \frac{\gamma \cdot d^2 \cdot u_0}{2M^2} \cdot k + \frac{\gamma \cdot d^2 \cdot u_0}{12M^2} \cdot sh[2 \cdot (k - \varphi_0)] + \frac{\gamma^2 \cdot d^4 \cdot u_0{}^2}{144M^4} \cdot ch[3 \cdot (k - \varphi_0)] + \ldots = C_1$$

$$ch[k + \varphi_0] + \frac{\gamma \cdot d^2 \cdot u_0}{2M^2} \cdot k - \frac{\gamma \cdot d^2 \cdot u_0}{12M^2} \cdot sh[2 \cdot (k + \varphi_0)] + \frac{\gamma^2 \cdot d^4 \cdot u_0{}^2}{144M^4} \cdot ch[3 \cdot (k + \varphi_0)] + \ldots = C_1$$

Having solved this system, we obtain the values of four coefficients: $u_0, \ \varphi_0, \ v_0, \ \ C_1$. The coefficient v_0 is obtained from the expressions

$$k = M - \tfrac{7\gamma^2 \cdot d^4 \cdot u_0{}^2}{12M^3} + \ldots \tag{7.33}$$

$$M^2 = Ha^2 + \gamma \cdot d^2 \cdot v_0.$$

We define

$$\beta = \gamma \cdot d^2 \cdot u_0 = \frac{d}{L} \cdot \frac{u_0 \cdot d}{\kappa} = \frac{Re}{Re_c}, \tag{7.34}$$

where $Re = \frac{u_0 \cdot d}{\kappa}$ is the Reynolds number, $Re_c = \alpha \cdot d = \frac{L}{d}$ is the critical Reynolds number.

We obtain the expressions

$$v'\left(\frac{z}{d}\right) = u_0 \cdot \tag{7.35}$$

$$\left[sh\left(\frac{kz}{d} + \varphi_0\right) + \frac{\beta}{2M^2} - \frac{\beta}{6M^2} \cdot ch\left(\frac{2kz}{d} + 2\varphi_0\right) + \frac{\beta^2}{48M^4} \cdot sh\left(\frac{3kz}{d} + 3\varphi_0\right) + \ldots\right]$$

$$H = -\frac{G_0 \cdot d \cdot u_0}{k \cdot \kappa_m} \times$$

$$\left[ch\left(\frac{kz}{d} + \varphi_0\right) + \frac{\beta}{2M^2} \cdot \frac{kz}{d} - \frac{\beta}{12M^2} \cdot sh\left(\frac{2kz}{d} + 2\varphi_0\right) + \frac{\beta^2}{144M^4} \cdot ch\left(\frac{3kz}{d} + 3\varphi_0\right) + \ldots\right]$$

$$-C_1.$$

The convergence of the series is dependent on the value $\frac{\beta}{M^2}$. If $\frac{\beta}{M^2} > 1$, the series aren't converged, and we have blow up of magnetic field.

7.3. **Non-stationary problem**

We solve the non-stationary equations

$$-\frac{\partial u}{\partial t} + v_0 \cdot \alpha \cdot u + \kappa \cdot \frac{\partial^2 u}{\partial z^2} = -\frac{G_0}{4\pi\rho} \cdot \frac{\partial H}{\partial z}$$
$$-\frac{\partial H}{\partial t} + \kappa_m \cdot \frac{\partial^2 H}{\partial z^2} = -G_0 \cdot \frac{\partial u}{\partial z}. \tag{7.36}$$

We find the solution of this homogeneous system as

$$u = \sum_{n=0}^{\infty}[u_{1,n} \cdot ch(a(k_n) \cdot t) + u_{2,n} \cdot sh(a(k_n) \cdot t)] \cdot \exp(-\kappa_m \cdot {k_n}^2 \cdot t) \cdot \sin(k_n z + \varphi_n),$$
$$H = \sum_{n=0}^{\infty}[H_{1,n} \cdot ch(a(k_n) \cdot t) + H_{2,n} \cdot sh(a(k_n) \cdot t)] \cdot \exp(-\kappa_m \cdot {k_n}^2 \cdot t) \cdot \cos(k_n z + \varphi_n), \tag{7.37}$$
$$k_n = const, \quad \varphi_n = const.$$

Note: We can use the functions $sh(k_n z + \varphi_n)$ and $ch(k_n z + \varphi_n)$ instead of $\sin(k_n z + \varphi_n)$ and $\cos(k_n z + \varphi_n)$.

We substitute this solution in the equations and obtain

$$-a(k_n) \cdot u_{1,n} + r_n \cdot u_{2,n} = \frac{G_0}{4\pi\rho} \cdot k_n \cdot H_{2,n}$$
$$r_n \cdot u_{1,n} - a(k_n) \cdot u_{1,n} - a(k_n) \cdot u_{2,n} = \frac{G_0}{4\pi\rho} \cdot k_n \cdot H_{1,n}$$
$$H_{1,n} = -\frac{G_0 \cdot k_n}{a(k_n)} \cdot u_{2,n} \tag{7.38}$$
$$H_{2,n} = -\frac{G_0 \cdot k_n}{a(k_n)} \cdot u_{1,n}$$

where

$$r_n = (\kappa_m - \kappa) \cdot {k_n}^2 + v_0 \cdot \alpha. \tag{7.39}$$

We simplify the equations

$$\left[a(k_n)^2 - \frac{{G_0}^2}{4\pi\rho} \cdot {k_n}^2\right] \cdot u_{1,n} - a(k_n) \cdot r_n \cdot u_{2,n} = 0$$
$$a(k_n) \cdot r_n \cdot u_{1,n} - \left[a(k_n)^2 - \frac{{G_0}^2}{4\pi\rho} \cdot {k_n}^2\right] \cdot u_{2,n} = 0$$
$$H_{1,n} = -\frac{G_0 \cdot k_n}{a(k_n)} \cdot u_{2,n} \tag{7.40}$$
$$H_{2,n} = -\frac{G_0 \cdot k_n}{a(k_n)} \cdot u_{1,n}$$
$$a(k_n) \neq 0.$$

The square matrix consisted of the first and second equation has an infinite number of solutions if its determinant is equal to zero:

$$\det = \left[a(k_n)^2 - \frac{{G_0}^2}{4\pi\rho} \cdot {k_n}^2\right]^2 - [a(k_n) \cdot r_n]^2 = 0. \tag{7.41}$$

As a result, we obtain

$$a(k_n) = \pm\left(\frac{r_n}{2} + \sqrt{\frac{{G_0}^2}{4\pi\rho} \cdot {k_n}^2 + \frac{{r_n}^2}{4}}\right),$$
$$a(k_n) = \pm\left(\frac{r_n}{2} - \sqrt{\frac{{G_0}^2}{4\pi\rho} \cdot {k_n}^2 + \frac{{r_n}^2}{4}}\right). \tag{7.42}$$

We have also the equations for the coefficients

$$u_{2,n} = \frac{a(k_n)^2 - \frac{{G_0}^2}{4\pi\rho} \cdot {k_n}^2}{a(k_n) \cdot r_n} \cdot u_{1,n}, \tag{7.43}$$

$$H_{1,n} = -\frac{G_0 \cdot k_n}{a(k_n)} \cdot u_{2,n} = \frac{G_0 \cdot k_n \cdot \left(\frac{{G_0}^2}{4\pi\rho} \cdot {k_n}^2 - a(k_n)^2\right)}{a(k_n)^2 \cdot r_n} \cdot u_{1,n},$$

$$H_{2,n} = -\frac{G_0 \cdot k_n}{a(k_n)} \cdot u_{1,n}, a(k_n) \neq 0, r_n \neq 0.$$

To satisfy the boundary conditions

$$H(t, -d) = 0, \qquad H(t, d) = 0, \tag{7.44}$$

we define

$$\varphi_n = 0, k_n = (2n+1) \cdot \frac{\pi}{2d}, n = 0, 1, 2, 3, \dots. \tag{7.45}$$

We obtain

$$\cos\left[(2n+1) \cdot \frac{\pi \cdot z}{2d}\right] = \begin{cases} \cos\left[(2n+1) \cdot \frac{\pi}{2}\right] = 0 & \text{at} \quad z = d, \\ \cos\left[-(2n+1) \cdot \frac{\pi}{2}\right] = 0 & \text{at} \quad z = -d. \end{cases} \tag{7.46}$$

$$u = \sum_{n=0}^{\infty} [u_{1,n} \cdot ch(a(k_n) \cdot t) + u_{2,n} \cdot sh(a(k_n) \cdot t)] \cdot \exp(-\kappa_m \cdot {k_n}^2 \cdot t) \cdot \tag{7.47}$$

$$\sin\left[(2n+1) \cdot \frac{\pi \cdot z}{2d}\right],$$

$$H = \sum_{n=0}^{\infty} [H_{1,n} \cdot ch(a(k_n) \cdot t) + H_{2,n} \cdot sh(a(k_n) \cdot t)] \cdot \exp(-\kappa_m \cdot {k_n}^2 \cdot t) \cdot$$

$$\cos\left[(2n+1) \cdot \frac{\pi \cdot z}{2d}\right].$$

We assume that the upper and low wall moves with the fluid velocity. So the boundary conditions for the fluid velocity are satisfied.

We satisfy the initial condition for the sum of the stationary and non-stationary velocity

$$v'\left(\frac{z}{d}\right) - u(0, z) = 0. \tag{7.48}$$

To simplify the expressions, we accept the stationary velocity in the form

$$v'\left(\frac{z}{d}\right) = \frac{U \cdot sh\left(M\frac{z}{d}\right)}{shM}. \tag{7.49}$$

We have

$$\frac{U \cdot sh\left(M\frac{z}{d}\right)}{shM} - \sum_{n=0}^{\infty} u_{1,n} \cdot \sin\left[(2n+1) \cdot \frac{\pi \cdot z}{2d}\right] = 0, \tag{7.50}$$

We orthonormalize the set of functions $\sin\left[(2n+1) \cdot \frac{\pi \cdot z}{2d}\right]$, $n = 0, 1, 2, 3, \dots$ using the Gram–Schmidt process and obtain the algebraic equations for the coefficients $u_{1,n}$. We can use another way to obtain the values of these coefficients. We expand the left part of this expression to the Fourier series using the functions $\sin\left[m \cdot \frac{\pi \cdot z}{d}\right]$, $m = 1, 2, 3, \dots$ which are orthogonal on the interval [–*d*,*d*]. As a result, we obtain the system with the coefficients $u_{1,n}$.

From the expressions (7.42) we obtain that

$$|a(k_0)| < |a(k_n)|, \quad n = 1, 2, 3, \dots \tag{7.51}$$

We have three forms of the solutions:

$$\begin{aligned} &(1)\ \ a(k_0) < \kappa_m \cdot {k_0}^2, \ \ v \to 0, \ \ \mathrm{H} \to 0, \ \text{ if t} \to \infty, v \text{ and } H \text{ is decreased over time,} \\ &(2)\ \ a(k_0) = \kappa_m \cdot {k_0}^2, \ \ v \text{ and } H \text{ is stationary and unstable,} \\ &(3)\ \ a(k_0) > \kappa_m \cdot {k_0}^2, \ \ v \to \infty, \ \ H \to \infty, \ \text{ if } t \to \infty. v \text{ and } H \text{ blows up.} \end{aligned} \tag{7.52}$$

If

$$a(k_0) = \kappa_m \cdot {k_0}^2, a(k_0) = \left(\frac{r_0}{2} + \sqrt{\frac{{G_0}^2}{4\pi\rho} \cdot {k_0}^2 + \frac{{r_0}^2}{4}}\right), r_0 = (\kappa_m - \kappa) \cdot {k_0}^2 + v_0 \cdot \alpha, \tag{7.53}$$

we obtain

$$\frac{{G_0}^2}{4\pi\rho \cdot \kappa_m \cdot \kappa} = {k_0}^2 - \frac{v_0 \cdot \alpha}{\kappa}. \tag{7.54}$$

We use

$$\frac{{G_0}^2}{4\pi\rho \cdot \kappa_m \cdot \kappa} = \frac{Ha^2}{d^2}, \quad \alpha = \frac{1}{L}, \quad Re' = \frac{v_0 \cdot d}{\kappa}, \quad Re_c = \frac{L}{d}. \tag{7.55}$$

We obtain the obvious condition for the stationary unstable flow in the system (see (7.17)):

$${k_0}^2 = Ha^2 + \frac{Re'}{Re_c}. \tag{7.56}$$

8. Conclusion

The results of this work indicate that the experimentally observed instability of Couette and Poiseuille flow can be explained by the methods of modern mathematical analysis. The paradoxes of symmetry and turbulence for Couette and Poiseuille flow are overcome.

These paradoxes can be solved if we introduce the additional dissipative terms in the Navier-Stokes equations. As a result, in the boundary layer we obtain the solutions which satisfy the condition of fluid adhesion to the bounding surfaces and describe the non-linear effects governed by the ratio of the Reynolds number and critical Reynolds number (β). The solutions are presented as the power series which coefficients are dependent on the ratio β. The symmetric Couette and Poiseuille flows are described by the first and the second term of the power series and observed at the small β. At the growth of β the other terms of series become relatively large and the symmetric flows are replaced by the laminar asymmetric one. The separation of flows and loss of stability is observed at the large β.

The crucial problem of the theory of hydrodynamic flow is a mathematical model of its space-time symmetry breaking and the development of turbulence. Spontaneous symmetry breaking is one of the fundamental ideas of modern physics [2].

According to the results of solving the problem on the establishment of Couette flow which one of its rigid boundary moved suddenly and one held stationary, this flow is observed only at the low Reynolds numbers, because in this case the flow perturbation caused by the suddenly shifted boundary decreases rapidly in time. This perturbation is described by laminar flows, but it does not obtain the spatial symmetry inherent to Couette flow. At the large Reynolds numbers this decline is slowed down. The perturbation is almost stationary if the Reynolds number is close to the critical value. Spontaneous breaking of the spatial and time symmetry of flow occurs when the Reynolds number takes the critical value. Under this condition, the solution is presented as a superposition of steady and laminar unsteady flow and has time symmetry. But the steady flow is unstable. It spontaneously blows up under the influence of fluctuations with small energy, loses symmetry, and the turbulence develops.

The formulas for estimating the critical Reynolds number explain Poiseuille flow breaking with increasing the hydraulic diameter of a pipe. The Reynolds number grows, the critical Reynolds number drops if this diameter increases.

According to [10], the Korteweg–de Vries is the simplest unidirectional wave equation including dispersion and nonlinearity, but without taking into account dissipation.

We obtained the solitary wave equations which based on dissipation and don't consider dispersion. The soliton is described as a monochromatic nonlinear wave without use of the wave packet model. It is a dissipative structure [27], i.e. a steady state system which organizes chaotic move of molecules in the result of the counteraction of nonlinearity and dissipation. This dissipative structure is a subject of turbulence. The turbulence can be described as the non-linear resonance using the expansion of the velocity into the Fourier series which is different from an ordinary wave packet [11].

According to [11], the stochastic dynamics of the system with weak dissipation is although different from the Hamiltonian dynamics, but this difference is small. Therefore, the universal nature of turbulence is most likely not related to the existence of dissipative factors, although dissipation always exists for turbulent movements.

Our conclusion is that influence of dissipation on turbulence is more essential than believed until now.

The space symmetry of the plane flow of an electrically conducting viscous fluid is destroyed at growth of the value $\frac{\beta}{M^2}$. As a result, the magnetic field is increased. This increase is infinite, if $\frac{\beta}{M^2} > 1$. The non-stationary fluid flow is stationary but unstable together with the magnetic field at $a(k_0) = \kappa_m \cdot {k_0}^2$. The fluid flow and the magnetic field exponentially blows up over time if $a(k_0) > \kappa_m \cdot {k_0}^2$. The series divergent over space and time variable are sensitive to small fluctuations in the initial conditions. The perturbations of the solution grow from small to great values. The flow becomes irregular.

Thus, the modified Navier-Stokes equations give a mathematical model of the magnetic dynamo for plane-parallel incompressible electrically conducting viscous fluid flows. As a result, we have disagreement with the antidynamo theorem installed for these flows [29, 30, 31].

Acknowledgments

The author received no direct funding for this research.

References

[1] G. Birkhoff, *Hydrodynamics, a study in logic, fact and similitude*, Revised Edition, Princeton Univ. Press, Princeton, N. J., 1960, ISBN 978-0691625911.
[2] M. A. Goldshtik, N. Stern, N. I. Jaworski, *Viscous flows with paradoxical properties*, Science, 1989, ISBN 5-02-028637-336.
[3] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flows* (2nd ed.), Mathematics and Its Applications, 2 (Revised Second ed.), New York-London-Paris-Montreux-Tokyo-Melbourne: Gordon and Breach,pp. XVIII+224, 1969, MR 0254401, Zbl 0184.52603.
[4] C. H. Fefferman, *Existence and smoothness of the Navier-Stokes equation*, Cambridge, MA, 1-5, 2000, Retrieved from http://claymath.org/Millenium_Prize_Problems/Navier-Stokes_Equations, ClayMathematics Institute.
[5] P. Constantin, *Some open problems and research directions in the mathematical study of fluid dynamics*, in mathematics unlimited-2001 and beyond, Berlin: Springer Verlag, 353-360, 2001.
[6] A. V. Zhirkin, *Existence and properties of the Navier-Stokes equations*, Cogent Mathematics, 2016, 3: 1190308, http://dx.doi.org/10.1080/23311835.2016.1190308.
[7] R. L. Salmon, *Lectures on geophysical fluid dynamics*, Oxford University Press, 23-26, 1998, ISBN 0-19-510808-6.
[8] E. M. Lifshitz, L. P. Pitaevskii, *Physical kinetics* (Vol. 10, 1st ed.) Pergamon Press, 1981, ISBN 978-0-7506-2635-4.
[9] G. K. Batchelor, *An introduction to fluid dynamics*, Cambridge University Press, Cambridge, 2000, ISBN 0-521-66396-2.
[10] E. Infeld, G. Rowlands, *Nonlinear waves, solitons and chaos* 2nd ed., Cambridge University Press, 2000, ISBN 978-1-1391-7128-1, https://doi.org/10.1017/CBO9781139171281.
[11] G. M. Zaslavskii, R. Z. Sagdeev, *Introduction to nonlinear physics: from the pendulum to turbulence and chaos* (in Russian), Nauka, Moscow, 1988; English transl.: R . Z. Sagdeev, D. A. Usikov and G. M. Zaslavsky, *Introduction to nonlinear physics*, Harwood, New York, 1988, ISBN 978-3-71-864832-0.
[12] L. D. Landau, E. M. Lifshitz, *Fluid mechanics* (Vol. 6, 2nd ed.) Butterworth Heinemann, 1987, ISBN 978-0-080-33933-7.
[13] L. D. Landau, E. M. Lifshitz, *Statistical physics* (Vol. 5, 3rd ed.) Butterworth Heinemann, 1980, ISBN 978-0-7506-3372-7.
[14] A. Einstein, *Zur Elektrodynamik bewegter Körper*, Annalen der Physik, 30 June 1905, **17 (10)**: 891-921, Bibcode:1905AnP...322..891E. English translation: On the electrodynamics of moving bodies, Translation by George Barker Jeffery and Wilfrid Perrett in The Principle of Relativity, London: Methuen and Company, Ltd., 1923.
[15] M. N. Kogan, *Rarefied gas dynamics*, Springer Science+Business Media, LLC, 1970, ISBN: 978-1-4899-6189-71.
[16] G. M. Fihtengolts, *Course of differential and integral calculus* (Vol. 3) Fizmatlit, 2003, in Russian, ISBN 5-9221-0155-2, Retrieved from http://www.ebook300.com.
[17] K. Chandrasekharan, *Elliptic functions*, Springer-Verlag, 1980, ISBN 0-387-15295-4.
[18] M. Tabor, *Chaos and integrability in nonlinear dynamics*, John Wiley and Sons, New York, 1989, ISBN 978-0-47-182728-3.
[19] G. M. Fihtengolts, *Course of differential and integral calculus* (Vol. 2) Fizmatlit, 2003, in Russian, ISBN: 5-9221-0155-2. Retrieved from http://www.ebook300.com.
[20] F. White, *Fluid mechanics* Section 7.1 (7th edition), McGraw-Hill Higher Education, 2010, ISBN 978-0-07-742241-7.
[21] N. N. Bogoliubov, Y. A. Mitropolski, *Asymptotic methods in the theory of nonlinear oscillations*, New York, NY: Gordon and Breach, 1961, ISBN 978-0677200507.
[22] L. D. Landau, E. M. Lifshitz, *Mechanics* (Vol. 1, 3rd ed.) Oxford: Butterworth Heinemann, 1976, ISBN 978-0-750-62896-9.
[23] D. J. Korteweg, G. de Vries, *On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves*, Philosophical Magazine, **39**: 422-443, 1985, https://doi.org/10.1080/14786449508620739.
[24] A. C. Newell, *Solitons in mathematics and physics*, SIAM, 1985, ISBN 978-0-89871-196-7.
[25] N. M. Ryskin, D. I. Trubetskov, *Nonlinear waves*, Fizmatlit, Moscow, 2000 (in Russian), ISBN 978-5-9710-3927-3.
[26] B. B. Kadomtsev, V. I. Petviashvili, *On the stability of solitary waves in weakly dispersive media*, Soviet Physics Doklady, **Vol. 15**, 539-541, 1970, Bibcode:1970SPhD...15..539K.
[27] G. Nicolis, I. Prigogine, *Self-organization in nonequilibrium systems*, John Wiley and Sons, Inc., 1979, ISBN 0-471-02401-5.
[28] L. D. Landau, E. M. Lifshitz, *Electrodynamics of continuous media* (Vol. 8, 2nd ed.) Butterworth Heinemann, Oxford, 1976, ISBN 978-0-7506-2634-7.
[29] O. G. Cowling, *Magnetohydrodynamics*, Interscience Publishers, New York and London, 1957.
[30] Y. B. Zeldovich, *Magnetic field at the two-dimensional motion of conductive turbulent fluid*, JETP, **31**, 154 - 156, 1956.
[31] H. K. Moffatt, *Magnetic field generation in electrically conducting fluids*, Cambridge University Press, Cambridge, London-New York-Melbourne, 1978, ISBN 9780521275835.