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Omega Invariant of the Line Graphs of Unicyclic Graphs

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Abstract

A recently introduced graph invariant $\Omega(G)$ for a graph G is proven to have several nice applications in Graph Theory and Combinatorics. This number gives direct information on the number of components, realizability, cyclicness, connectedness, chords, loops, pendant edges, faces, bridges and the number of realizations. In this paper, we determine Ω values of the line graphs of unicyclic graphs.

Keywords: Line graph, Omega invariant, Degree sequence

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1. Introduction

Let $G = (V, E)$ be a graph of order n and size m. We denote the degree of a vertex $v \in V(G)$ by d_v. A vertex of degree one will be called a pendant vertex. If u and v are two adjacent vertices of G, then the edge e connecting these vertices will be denoted by $e = uv$. e will be said to be incident with the vertices u and v. The largest vertex degree in a graph will be denoted by Δ. A vertex which is adjacent to a pendant vertex is called a support vertex. A graph is said to be connected if there is a path between every pair of vertices and disconnected if not. For the definitions of the fundamental notions in Graph Theory, see [1], [3], [4].

In many occasions, we shall classify our graphs under consideration according to whether they have at least one cycle or not. Those graphs having no cycle will be called acyclic. For example, all trees are acyclic. The remaining graphs are called cyclic graphs. A graph having one, two, three cycles is called unicyclic, bicyclic and tricyclic, respectively.

Written with multiplicities, a degree sequence in general is

$$D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$$

where some of a_i's could be zero. Let $D = \{d_1, d_2, d_3, \cdots, \Delta\}$ be a set of non-decreasing non-negative integers. If the degree sequence of a graph G is equal to D, then it is said that G is a realization of D and D is said to be realizable.

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2. Ω invariant

We recall the definition and some fundamental properties of the number $\Omega(G)$ which was defined and studied in [2] for a given graph G or for a realizable degree sequence D having a realization G. The invariant $\Omega(D)$ was defined in [2] as a generalization of this equation as follows:

Definition 2.1. Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be the degree sequence of a graph G. The $\Omega(G)$ of the graph G is defined only in terms of the degree sequence as

$$\begin{aligned}\Omega(G) = &\; a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 \\ = &\; \sum_{i=1}^{\Delta}(i-2)a_i.\end{aligned}$$

$\Omega(G)$ of several main graph classes such as T, P_n, C_n, S_n, K_n, $T_{r,s}$ and $K_{r,s}$, where $n = r + s$ which respectively denote a tree, path, cycle, star, complete, tadpole and complete bipartite graphs with n vertices are $\Omega(C_n) = 0$, $\Omega(P_n) = -2$, $\Omega(S_n) = -2$, $\Omega(T) = -2$, $\Omega(K_n) = n(n-3)$, $\Omega(K_{r,s}) = 2[rs - (r+s)]$ and $\Omega(T_{r,s}) = 0$. Amongst these graph classes, the most useful ones in the study of Ω are the path, cycle and tree graphs. Note that the Ω of a path, star or tree is equal to −2, and again note that these are acyclic graphs. This is in fact true for all connected acyclic graphs as given in [2].

We now recall some fundamental properties of Ω.

Theorem 2.2. *[2] For any graph G,*

$$\Omega(G) = 2(m - n).$$

That is, for any graph G, $\Omega(G)$ is even. Therefore if $\Omega(D)$ is odd for a set D of non-negative integers, then D is not realizable.

The number r of independent (non-overlapping) cycles in a graph G which is also known as the cyclomatic number of G can be given in terms of $\Omega(G)$:

Corollary 2.3. *[2] Let* $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ *be realizable as a graph G with c components. The number r of faces of G is given by*

$$r = \frac{\Omega(G)}{2} + c.$$

3. Line graph

Given a simple undirected graph G, a graph obtained by associating a new vertex onto each edge of G and connecting two such new vertices with an edge iff the corresponding edges of the graph G have a vertex in common is called the line graph of G and will be denoted by $L(G)$. It is known that the order of the line graph is equal to the size of the graph, and the number of edges of the line graph is given by the formula

$$m(L(G)) = \frac{1}{2}\sum_{u \in V(G)} d_u^2 - m(G).$$

Here noticing that the sum here is equal to the first Zagreb index $M_1(G)$, this equation can be restated as

$$m(L(G)) = \frac{1}{2}M_1(G) - m(G).$$

It can be easily calculated that $L(P_n) = P_{n-1}$, $L(C_n) = C_n$ and $L(S_n) = K_{n-1}$.

In this paper, we shall study the unicyclic graphs. Therefore we recall the following result which characterizes the connected unicyclic graphs:

Lemma 3.1. *[5] Let G be a connected graph. G is unicyclic iff* $m(G) = n(G)$.

That is, the order and size of every connected unicyclic graph are equal and vice versa. i.e. the number of vertices must be equal to the number of edges. As $n(L(G)) = m(G)$, Lemma 3.1 immediately reduces the following result which characterises all the graphs G such that the orders of G and $L(G)$ are equal:

Theorem 3.2. *[5] Let G be a connected graph. G is unicyclic iff* $n(L(G)) = n(G)$.

By Theorem 3.2, G and $L(G)$ have the same number of vertices whenever the graph G is unicyclic. Of course, the line graph, having many faces, is rarely unicyclic. Here, we start with a characterization of all graphs having the property that $n(L(G)) = n(G)$. Normally even if the graph is unicyclic or acyclic, its line graph may have a large number of faces. In [5], the following result was given about the structure of the line graph of a tree:

Theorem 3.3. *Let G be a connected simple acyclic graph with degree sequence* $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$. *Then its line graph* $L(G)$ *consists of* a_2 *times* K_2*'s,* a_3 *times* K_3*'s,* $\cdots$*,* a_Δ *times* K_Δ*'s where* K_r *and* K_s *have a unique common vertex in* $L(G)$ *for* $d_{v_i} = r$ *and* $d_{v_j} = s$ *iff* v_i *and* v_j *are adjacent in G.*

Finally, we must recall the following result relating the size of $L(G)$ and the size of G:

Theorem 3.4. *[5] For any graph G,*

$$m(L(G)) + m(G) = \frac{M_1(G)}{2} \tag{3.1}$$

where $M_1(G)$ *denotes the first Zagreb index of G.*

Example 3.5. Let G be the tree in Fig. 1 with $m(G) = 7$ and $n(G) = 8$.

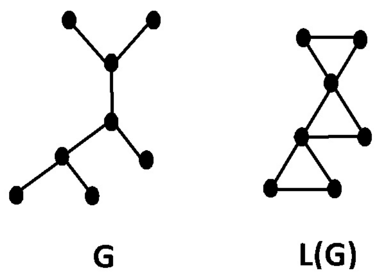

Figure 1 A tree and its line graph

Note that the degree sequence of G is $\{1^{(5)}, 3^{(3)}\}$ and hence the first Zagreb index of G is $M_1(G) = 5 \cdot 1^2 + 3 \cdot 3^2 = 32$. As $m(G) = 7$ and $m(L(G)) = 9$, the equality in Eqn. (1) holds.

4. Line graphs of unicyclic graphs

Recall that for any connected unicyclic graph G, the omega of G must be $\Omega(G) = 0$. Now we calculate the number of faces and following this, the omega invariant of the line graphs of the unicyclic graphs. First we characterize unicyclic regular graphs by means of omega invariant:

Theorem 4.1. *A connected k-regular graph is unicyclic iff* $k = 2$.

Proof. ($\Rightarrow$) Let G be a k-regular unicyclic graph. Note that the degree sequence of G would be $\{k^n\}$ where n is the number of vertices and also $r = 1$. The omega of G is $\Omega(G) = (k-2)n$. As G is connected, by Corollary 2.3, $r = \frac{\Omega}{2} + 1 = \frac{(k-2)n}{2} + 1 = 1$. Therefore k must be 2.
($\Leftarrow$) Let G be a $k = 2$-regular graph. Then the degree sequence of G would be $\{2^n\}$. Hence $\Omega(G) = (2-2)n = 0$ implying that $r = \frac{0}{2} + 1 = 1$. Therefore G must be unicyclic. □

This means that if a unicyclic graph is k-regular, k must be 2. Similarly, if a graph is k-regular and if $k \geq 3$, then it cannot be unicyclic. For example, no 3-regular graph can be unicyclic.

Theorem 4.2. *Let G be a unicyclic graph. Then the omega value of the line graph $L(G)$ of G is*

$$\Omega(L(G)) = 2\sum_{i=3}^{\Delta} a_i T_{i-2}$$

where T_i is the i-th triangular number given by $T_i = \dfrac{i(i+1)}{2}$.

Proof. Let G be a unicyclic graph. By Theorem 3.3, $L(G)$ consists of complete graphs K_r around each vertex v_i of degree $r \geq 2$ while there is also a cycle connecting the new vertices on the edges of the unique cycle of G. K_n has $\dfrac{n(n-3)}{2} + 1$ regions surrounded by the edges of K_n. As each edge of $L(G)$ belongs to exactly one of those complete graphs, we get

$$\begin{aligned} r(L(G)) &= \textstyle\sum_{i=3}^{\Delta} a_i \left(\dfrac{i(i-3)}{2} + 1\right) + 1 \\ &= \textstyle\sum_{i=3}^{\Delta} a_i \left(\dfrac{(i-1)(i-2)}{2}\right) + 1 \\ &= \textstyle\sum_{i=3}^{\Delta} a_i T_{i-2} + 1. \end{aligned}$$

By Corollary 2.3, the result can be deduced. □

Note that we can easily say that the omega invariants of the line graphs of two important classes of unicyclic graphs, namely cycle and tadpole graphs, are 0 and 2 according to Theorem 4.2.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, NY, 2008.
[2] S. Delen and I. N. Cangul, *A New Graph Invariant*, Turkish Journal of Analysis and Number Theory **6 (1)**, 30-33, 2018.
[3] R. Diestel, *Graph Theory*, Springer, GTM, 2010.
[4] L. R. Foulds, *Graph Theory Applications*, Springer Universitext, 1992.
[5] H. Ozden, F. Ersoy Zihni, F. Ozen Erdogan, I. N. Cangul, G. Srivastava and H. M. Srivastava, *Independence Number of Graphs and Line Graphs of Trees by Means of Omega Invariant*, Revista de la Real Academia de Ciencias Exactas Fsicas y Naturales. Serie A. Matemticas **114**, 91, 2020.