Montes Taurus J. Pure Appl. Math. 2 (2), 7–37, 2020

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Generating Functions of Generalized Tribonacci and Tricobsthal Polynomials

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Abstract

In this paper, we introduce a symmetric endomorphism operator $\delta^k_{a_1a_2}$ allows us to obtain a new generating functions involving the product of generalized Tribonacci and Tricobsthal polynomials with k- Mersenne numbers.

Keywords: Symmetric functions, Generating functions, Generalized Tricobsthal Polynomials, Generalized Tribonacci polynomials

2010 MSC: 05E05, 11B39

1. Introduction

Recently, Fibonacci and Lucas numbers have investigated very largely and authors tried to developed and give some directions to mathematical calculations using these type of special numbers. One of these directions goes through to the Tribonacci and the Tribonacci-Lucas numbers. In fact Tribonacci numbers have been firstly defined by M. Feinberg in 1963 and then some important properties over this numbers have been created by [9, 11, 12]. On the other hand, Tribonacci-Lucas numbers have been introduced and investigated by author in [8, 10]. In addition, there exists another mathematical term, namely to be incomplete, on Tribonacci and Tribonacci-Lucas numbers. As a brief background, the incomplete Tribonacci and Tribonacci-Lucas numbers were introduced by authors [15, 16], and further the generating functions of these numbers were presented by authors.

For $n \geq 3$, it is known that while the Tribonacci sequence $\{T_n\}_{n\in\mathbb{N}}$ is defined by

$$\begin{cases} T_n = T_{n-1} + T_{n-2} + T_{n-3},\ n \geq 3 \\ T_0 = T_1 = 1, T_2 = 2 \end{cases}. \tag{1.1}$$

and the Tribonacci-Lucas sequence $\{K_n\}_{n\in\mathbb{N}}$ is defined by

$$\begin{cases} K_n = K_{n-1} + K_{n-2} + K_{n-3},\ n \geq 3 \\ K_0 = 3, K_1 = 1, K_2 = 3 \end{cases}. \tag{1.2}$$

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Received:12 May 2020, Accepted:4 September 2020
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There is also well known that each of the Tribonacci and Tribonaccci-Lucas numbers is actually a linear combination of

$$T_n = -\frac{(r_2+r_3-r_2r_3-2)}{(r_1-r_2)(r_1-r_3)}r_1^n + \frac{(r_1+r_3-r_1r_3-2)}{(r_1-r_2)(r_2-r_3)}r_2^n - \frac{(r_1+r_2-r_1r_2-2)}{(r_1-r_3)(r_2-r_3)}r_3^n, \tag{1.3}$$

$$K_n = r_1^n + r_2^n + r_3^n. \tag{1.4}$$

where r_1, r_2 and r_3 are roots of the characteristic equations of (1.1) and (1.2) such that

$$r_1 = \frac{1}{3}\left[1 + \sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}}\right] = 1.839\,3,$$

$$r_2 = \frac{1}{3}\left[1 + w\sqrt[3]{19+3\sqrt{33}} + w^2\sqrt[3]{19-3\sqrt{33}}\right] = -0.419\,64 + 0.606\,29i,$$

$$r_3 = \frac{1}{3}\left[1 + w^2\sqrt[3]{19+3\sqrt{33}} + w\sqrt[3]{19-3\sqrt{33}}\right] = -0.419\,64 - 0.606\,29i,$$

with $w = \frac{-1+i\sqrt{3}}{2}$. Meanwhile we note that equations in (1.3) and (1.4) are called the Binet's formulas for Tribonacci and Tribonacci-Lucas numbers, respectively.

In 1973, Hoggatt and Bicknell [10] introduced Tribonacci polynomials. The Tribonacci polynomials $\{T_n(x)\}_{n\in\mathbb{N}}$ are defined by the recurrence relation

$$\begin{cases} T_n(x) = x^2T_{n-1}(x) + xT_{n-2}(x) + T_{n-3}(x),\ n \geq 3 \\ T_0(x) = 1, T_1(x) = x^2, T_2(x) = x^4 + x \end{cases}.$$

and property $T_n(1) = T_n$. First 10^{th} Tribonacci polynomials are:

$$\begin{aligned}
T_0(x) &= 1,\\
T_1(x) &= x^2,\\
T_2(x) &= x^4 + x\\
T_3(x) &= x^6 + 2x^3 + 1,\\
T_4(x) &= x^8 + 3x^5 + 3x^2,\\
T_5(x) &= x^{10} + 4x^7 + 6x^4 + 2x,\\
T_6(x) &= x^{12} + 5x^9 + 10x^6 + 7x^3 + 1,\\
T_7(x) &= x^{14} + 6x^{11} + 15x^8 + 16x^5 + 6x^2,\\
T_8(x) &= x^{16} + 7x^{13} + 21x^{10} + 30x^7 + 19x^4 + 3x,\\
T_9(x) &= x^{18} + 8x^{15} + 28x^{12} + 50x^9 + 45x^6 + 16x^3 + 1.
\end{aligned}$$

Also, in [16], authors defined Tribonacci-Lucas polynomials, incomplete Tribonacci-Lucas numbers and incomplete Tribonacci-Lucas polynomials. The Tribonacci-Lucas polynomials are defined by

$$\begin{cases} K_n(x) = x^2K_{n-1}(x) + xK_{n-2}(x) + K_{n-3}(x),\ n \geq 3 \\ K_0(x) = 3, K_1(x) = x^2, K_2(x) = x^4 + 2x \end{cases}.$$

Definition 1.1. [18] For $n \in \mathbb{N}$, the k- Mersenne numbers, denoted by $\{M_{k,n}\}_{n\in\mathbb{N}}$ defined recursively by

$$\begin{cases} M_{k,n} = 3kM_{k,n-1} - 2M_{k,n-2} \text{ for all } n \geq 2, \\ M_{k,0} = 0,\ M_{k,1} = 1 \end{cases}. \tag{1.5}$$

The Binet's fomula is given by

$$M_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}.$$

such that $r_1 = \frac{3k+\sqrt{9k^2-8}}{2}$ and $r_2 = \frac{3k-\sqrt{9k^2-8}}{2}$ are roots of the characteristic equation of the sequence (1.5).

2. Definitions and Some Properties

In this section, we introduce a new symmetric function and give some properties of this symmetric function. We also give some more useful definitions from the literature which are used in the subsequent sections.

We shall handle functions on different sets of indeterminates (called alphabets, though we shall mostly use commutative indeterminates for the moment). A symmetric function of an alphabet A is a function of the letters which is invariant under permutation of the letters of A. Taking an extra indeterminate t, one has two fundamental series [2].

$$\lambda_z(A) = \Pi_{a\in A}(1 + ta),\ \sigma_z(A) = \frac{1}{\Pi_{a\in A}(1 - ta)},$$

the expansion of which gives the elementary symmetric functions $\Lambda_n(A)$ and the complete functions $S_n(A)$:

$$\lambda_z(A) = \sum_{n=0}^{\infty} \Lambda_n(A)t^n,\ \sigma_z(A) = \sum_{n=0}^{\infty} S_n(A)t^n.$$

Let us now start at the following definition.

Definition 2.1. Let A and B be any two alphabets, then we give $S_n(A - B)$ by the following form:

$$\frac{\Pi_{b\in B}(1 - tb)}{\Pi_{a\in A}(1 - ta)} = \sum_{n=0}^{\infty} S_n(A - B)t^n = \sigma_z(A - B), \tag{2.1}$$

with the condition $S_n(A - B) = 0$ for $n < 0$ [1].

Remark 2.2. Taking $A = \{0, 0, ...0\}$ in (2.1) gives

$$\Pi_{b\in B}(1 - tb) = \sum_{n=0}^{\infty} S_n(-B)t^n = \lambda_z(-B). \tag{2.2}$$

Further, in the case $A = \{0, 0, ...0\}$ or $B = \{0, 0, ...0\}$, we have

$$\sum_{n=0}^{\infty} S_n(A - B)t^n = \sigma_z(A) \times \lambda_z(-B). \tag{2.3}$$

Thus,

$$S_n(A - B) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-B). \tag{2.4}$$

Definition 2.3. [3] Let g be any function on $\mathbb{R}^n$, then we consider the divided difference operator as the following form

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1, \cdots, x_i, x_{i+1}, \cdots x_n) - g(x_1, \cdots x_{i-1}, x_{i+1,}x_i, x_{i+2} \cdots x_n)}{x_i - x_{i+1}}.$$

Definition 2.4. [6] Let n be positive integer and $A = \{a_1, a_2\}$ are set of given variables, then, the n -th symmetric function $S_n(a_1 + a_2)$ is defined by

$$S_n(A) = S_n(a_1 + a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2},$$

with

$$\begin{aligned}
S_0(A) &= S_0(a_1 + a_2) = 1,\\
S_1(A) &= S_1(a_1 + a_2) = a_1 + a_2,\\
S_2(A) &= S_2(a_1 + a_2) = a_1^2 + a_1a_2 + a_2^2,\\
&\vdots
\end{aligned}$$

Definition 2.5. [6] The symmetrizing operator $\delta^k_{a_1a_2}$ is defined by

$$\delta^k_{a_1a_2}f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \text{ for all } k \in \mathbb{N}_0. \tag{2.5}$$

If $f(a_1) = a_1$, the operator (2.5) gives us

$$\delta^k_{a_1a_2}f(a_1) = \frac{a_1^{k+1} - a_2^{k+1}}{a_1 - a_2} = S_k(a_1 + a_2).$$

If $f(a_2) = a_2$, the operator (2.5) gives us

$$\delta^k_{a_1a_2}f(a_2) = \frac{a_1^k a_2 - a_2^k a_1}{a_1 - a_2} = a_1a_2S_{k-2}(a_1 + a_2).$$

3. Generalized Tribonacci and Tricobsthal Polynomials

By analogy to general Fibonacci numbers (see for example [12]) let's consider conditions:

$$\begin{aligned} W_1(x) &= a, \\ W_2(x) &= b_2x^2 + b_1x + b_0, \\ W_3(x) &= c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0, \end{aligned} \tag{3.1}$$

where b_2, c_1, c_4 positive integers and others parameters are non negative integers as initial conditions for Tribonacci polynomials, then we have following definition:

Definition 3.1. The Generalized Tribonacci polynomials are defined by recurrence relation

$$W_n(x) = x^2W_{n-1}(x) + xW_{n-2}(x) + W_{n-3}(x) \text{ for } n \geq 3, \tag{3.2}$$

with initial conditions (3.1).

When $a = b_2 = c_1 = c_4 = 1$ and others parameters are zero, then one gets Tribonacci polynomials with property $W_n(1) = T_n(1) = T_n$. When one considers any polynomials related with name Fibonacci should rewrite them in analog of the Binet's formula and for considered in this section polynomials one gets following theorem.

Theorem 3.2. *The Binet's formula for the Generalized Tribonacci polynomials defined by (3.2) with initial conditions (3.1) is*

$$W_n(x) = C_{1,W}\alpha_W^{n-1} + C_{2,W}\beta_W^{n-1} + C_{3,W}\gamma_W^{n-1}, \tag{3.3}$$

where n is positive integer,

$$\begin{aligned} C_{1,W} &:= \frac{W_3(x) - (\gamma_W + \beta_W)W_2(x) + \gamma_W\beta_W W_1(x)}{(\alpha_W - \gamma_W)(\alpha_W - \beta_W)}, \\ C_{2,W} &:= \frac{W_3(x) - (\gamma_W + \alpha_W)W_2(x) + \gamma_W\alpha_W W_1(x)}{(\beta_W - \gamma_W)(\beta_W - \alpha_W)}, \\ C_{3,W} &:= \frac{W_3(x) - (\alpha_W + \beta_W)W_2(x) + \alpha_W\beta_W W_1(x)}{(\gamma_W - \alpha_W)(\gamma_W - \beta_W)}, \end{aligned}$$

and α_W, β_W, γ_W *are different solutions of characteristic equation* $y^3 - x^2y^2 - xy - 1 = 0$ *of (3.2).*

$$\begin{aligned} \alpha_W &= \frac{x^2}{3} - \frac{2^{\frac{1}{3}}(-3x - x^4)}{3\delta_W} + \frac{\delta_W}{3.2^{\frac{1}{3}}}, \\ \beta_W &= \frac{x^2}{3} + \frac{(1 + i\sqrt{3})(-3x - x^4)}{2^{\frac{2}{3}}.3\delta_W} - \frac{(1 - i\sqrt{3})(-3x - x^4)}{2^{\frac{2}{3}}.6\delta_W}, \\ \gamma_W &= \frac{x^2}{3} + \frac{(1 - i\sqrt{3})(-3x - x^4)}{2^{\frac{2}{3}}.3\delta_W} - \frac{(1 + i\sqrt{3})(-3x - x^4)}{2^{\frac{2}{3}}.6\delta_W}, \end{aligned} \tag{3.4}$$

where

$$\delta_W = \sqrt[3]{27 + 9x^3 + 2x^6 + 3\sqrt{3}\sqrt{27 + 14x^3 + 3x^6}}. \tag{3.5}$$

We omit dependency from x for parameters $\alpha_W, \beta_W, \gamma_W, \delta_W$ in theorem.

Proof. From [12] one can check that characteristic equation of (3.2) is in the form given in theorem. From Cardano formula its roots are of the form (3.4) with (3.5). The Binet's formula will be proved inductively. It's easy to check that formula (3.3) is correct for $n \in \{1, 2, 3\}$. Let's assume that (3.3) is fulfilled for any number less or equal n and let's show that it is also true for $n + 1$. From (3.2), properties of roots of characteristic equation and assumption of induction one can write:

$$\begin{aligned} W_{n+1}(x) &= x^2W_n(x) + xW_{n-1}(x) + W_{n-2}(x) \\ &= C_{1,W}\alpha_W^{n-3}(x^2\alpha_W^2 + x\alpha_W + 1) + C_{2,W}\beta_W^{n-3}(x^2\beta_W^2 + x\beta_W + 1) \\ &\quad + C_{3,W}\gamma_W^{n-3}(x^2\gamma_W^2 + x\gamma_W + 1) \\ &= C_{1,W}\alpha_W^n + C_{2,W}\beta_W^n + C_{3,W}\gamma_W^n. \end{aligned}$$

So by mathematical induction our statement is truth. □

Corollary 3.3. *When $a = b_2 = c_1 = c_4 = 1$, then Binet's formula in theorem 3.1 is the Binet's formula for Tribonacci polynomials.*

As a result, the terms of the generalized Tribonacci polynomails are seen as the coefficients of the corresponding generating function.

Theorem 3.4. *For $n \in \mathbb{N}$, the new generating function of generalized Tribonacci polynomails is given by*

$$\begin{aligned} \sum_{n=0}^{\infty} W_n(x)\,t^n &= \frac{a + \left[(b_2 - a)\,x^2 + b_1x + b_0\right]t}{1 - x^2t - xt^2 - t^3} \\ &\quad + \frac{\left[(c_4 - b_2)\,x^4 + (c_3 - b_1)x^3 + (c_2 - b_0)x^2 + (c_1 - a)x + c_0\right]t^2}{1 - x^2t - xt^2 - t^3}. \end{aligned}$$

Proof. The ordinary generating function associated is defined by

$$G(W_n, t) = \sum_{n=0}^{\infty} W_n(x)\,t^n.$$

Using the initial conditions, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} W_n(x)t^n &= W_0(x) + W_1(x)t + W_2(x)t^2 + \sum_{n=3}^{\infty} W_n(x)t^n \\
&= a + \left(b_0 + b_1x + b_2x^2\right)t + \left(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4\right)t^2 \\
&\quad + \sum_{n=3}^{\infty}(x^2W_{n-1}(x) + xW_{n-2}(x) + W_{n-3}(x))t^n \\
&= a + \left(b_0 + b_1x + b_2x^2\right)t + \left(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4\right)t^2 + x^2t\sum_{n=2}^{\infty} W_n(x)t^n \\
&\quad + xt^2\sum_{n=1}^{\infty} W_n(x)t^n + t^3\sum_{n=0}^{\infty} W_n(x)t^n \\
&= a + \left(b_0 + b_1x + b_2x^2\right)t - ax^2t + \left(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4\right)t^2 - axt^2 \\
&\quad - \left(b_0 + b_1x + b_2x^2\right)x^2t^2 + x^2t\sum_{n=0}^{\infty} W_n(x)t^n + xt^2\sum_{n=0}^{\infty} W_n(x)t^n + t^3\sum_{n=0}^{\infty} W_n(x)t^n \\
&= a + \left[(b_2 - a)x^2 + b_1x + b_0\right]t + \left[\begin{array}{c} (c_4 - b_2)x^4 + (c_3 - b_1)x^3 + (c_2 - b_0)x^2 \\ +(c_1 - a)x + c_0 \end{array}\right]t^2 \\
&\quad + x^2t\sum_{n=0}^{\infty} W_n(x)t^n + xt^2\sum_{n=0}^{\infty} W_n(x)t^n + t^3\sum_{n=0}^{\infty} W_n(x)t^n,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\sum_{n=0}^{\infty} W_n(x)t^n(1 - x^2t - xt^2 - t^3) &= a + \left[(b_2 - a)x^2 + b_1x + b_0\right]t \\
&\quad + \left[(c_4 - b_2)x^4 + (c_3 - b_1)x^3 + (c_2 - b_0)x^2 + (c_1 - a)x + c_0\right]t^2,
\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} W_n(x)t^n = \frac{\begin{array}{c} a + \left[(b_2 - a)x^2 + b_1x + b_0\right]t \\ + \left[(c_4 - b_2)x^4 + (c_3 - b_1)x^3 + (c_2 - b_0)x^2 + (c_1 - a)x + c_0\right]t^2 \end{array}}{1 - x^2t - xt^2 - t^3}.$$

This completes the proof. □

By analogy to generalized Jacobsthal polynomials [17] and Tribonacci polynomials [12] let's introduce Tricobsthal polynomials in the following way:

Definition 3.5. The Tricobsthal polynomials are defined by recurrence formula

$$J_n^{(3)}(x) = J_{n-1}^{(3)}(x) + xJ_{n-2}^{(3)}(x) + x^2J_{n-3}^{(3)}(x) \text{ for } n \geq 3, \tag{3.6}$$

with initial conditions:

$$J_1^{(3)}(x) = 1,\ J_2^{(3)}(x) = 1 \text{ and } J_3^{(3)}(x) = x + 1. \tag{3.7}$$

The choice of initial conditions is according to property $J_n(1) = W_n(1) = T_n$ is n^{th} Tribonacci number, by analogy to Jacobsthal and Fibonacci polynomials, see [12, 17]. Analogously we can define generalized Tricobsthal polynomials:

Definition 3.6. Generalized Tricobsthal polynomials are defined by recurrence relation

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x) + x^2J_{n-3}(x) \text{ for } n \geq 3, \tag{3.8}$$

with initial conditions:

$$\begin{aligned} J_1(x) &= a, \\ J_2(x) &= b, \\ J_3(x) &= c_1x + c_0, \end{aligned} \tag{3.9}$$

where parameters c_1 is positive integers and a, b, c_0 are non-negative integers.

When $a = b = c_1 = c_0 = 1$, then one gets Tricobsthal polynomials introduced earlier. First 10^{th} Tricobsthal polynomials are presented below:

$$\begin{aligned} J_1(x) &= 1, \\ J_2(x) &= 1, \\ J_3(x) &= x+1, \\ J_4(x) &= x^2+2x+1, \\ J_5(x) &= 3x^2+3x+1, \\ J_6(x) &= 2x^3+6x^2+4x+1, \\ J_7(x) &= x^4+7x^3+10x^2+5x+1, \\ J_8(x) &= 6x^4+16x^3+15x^2+6x+1, \\ J_9(x) &= 3x^5+19x^4+30x^3+21x^2+7x+1, \\ J_{10}(x) &= x^6+16x^5+45x^4+50x^3+28x^2+8x+1. \end{aligned}$$

By analogy to previous section below we present the Binet's formula for considered polynomials.

Theorem 3.7. *The Binet's formula for generalized Tricobsthal polynomials defined by (3.8) with initial conditions (3.9) is*

$$J_n(x) = C_{1,J}\alpha_J^{n-1} + C_{2,J}\beta_J^{n-1} + C_{3,J}\gamma_J^{n-1}, \tag{3.10}$$

where n is positive integer, $x \neq 0$ *and*

$$\begin{aligned} C_{1,J} &: = \frac{J_3(x) - (\gamma_J + \beta_J)J_2(x) + \gamma_J\beta_J J_1(x)}{(\alpha_J - \gamma_J)(\alpha_J - \beta_J)}, \\ C_{2,J} &: = \frac{J_3(x) - (\gamma_J + \alpha_J)J_2(x) + \gamma_J\alpha_J J_1(x)}{(\beta_J - \gamma_J)(\beta_J - \alpha_J)}, \\ C_{3,J} &: = \frac{J_3(x) - (\alpha_J + \beta_J)J_2(x) + \alpha_J\beta_J J_1(x)}{(\gamma_J - \alpha_J)(\gamma_J - \beta_J)}, \end{aligned}$$

where α_J*,* β_J*,* γ_J *are different solutions of characteristic equation* $y^3 - y^2 - xy - x^2 = 0$ *of (3.8).*

$$\begin{aligned} \alpha_J &= \frac{1}{3} + \frac{8(3x+1)}{3\sqrt[3]{4}\delta_J} + \frac{\delta_J}{3.2^{\frac{1}{3}}}, \\ \beta_J &= \frac{1}{3} - \frac{(1+i\sqrt{3})(3x+1)}{\sqrt[3]{4}3\delta_J} - \frac{(1-i\sqrt{3})\delta_J}{\sqrt[3]{2}6}, \\ \gamma_J &= \frac{1}{3} - \frac{(1-i\sqrt{3})(3x+1)}{\sqrt[3]{4}3\delta_J} - \frac{(1+i\sqrt{3})\delta_J}{\sqrt[3]{2}6}, \end{aligned}$$

where $\delta_J = \sqrt{27x^2 + 3\sqrt{3}\sqrt{27x^4 + 14x^3 + 3x^2} + 9x + 2}$.

We omit dependency from x for parameters $\alpha_J, \beta_J, \gamma_J, \delta_J$ in theorem. Prove of this theorem goes in the same way like for the generalized Tribonacci polynomials.

Corollary 3.8. *The Binet's formula in theorem 3.7 is the Binet's formula for the Tricobsthal polynomials if* $a = b = c_1 = c_0 = 1$.

As a result, the terms of the generalized Tricobstal polynomails are seen as the coefficients of the corresponding generating function.

Theorem 3.9. *For* $n \in \mathbb{N}$, *the new generating function of generalized the Tricobstal polynomails is given by*

$$\sum_{n=0}^{\infty} J_n(x)\, t^n = \frac{a + (b-a)\,t + [(c_1 - a)\,x + c_0 - b]\,t^2}{1 - t - xt^2 - x^2t^3}.$$

Proof. The ordinary generating function associated is defined by

$$G(J_n, t) = \sum_{n=0}^{\infty} J_n(x)\, t^n.$$

Using the initial conditions, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} J_n(x)\, t^n &= J_0(x) + J_1(x)\,t + J_2(x)t^2 + \sum_{n=3}^{\infty} J_n(x)\, t^n \\
&= a + bt + (c_0 + c_1 x)\, t^2 + \sum_{n=3}^{\infty} (J_{n-1}(x) + xJ_{n-2}(x) + x^2 J_{n-3}(x))t^n \\
&= a + bt + (c_0 + c_1 x)\, t^2 + t\sum_{n=2}^{\infty} J_n(x)\, t^n + xt^2 \sum_{n=1}^{\infty} J_n(x)t^n + x^2t^3 \sum_{n=0}^{\infty} J_n(x)\, t^n \\
&= a + bt + (c_0 + c_1 x)\, t^2 - at - bt^2 - axt^2 \\
&\quad + t\sum_{n=0}^{\infty} J_n(x)\, t^n + xt^2 \sum_{n=0}^{\infty} J_n(x)\, t^n + x^2t^3 \sum_{n=0}^{\infty} J_n(x)\, t^n \\
&= a + [b-a]\,t + [(c_1 - a)\,x + c_0 - b]\,t^2 \\
&\quad + t\sum_{n=0}^{\infty} J_n(x)\, t^n + xt^2 \sum_{n=0}^{\infty} J_n(x)t^n + x^2t^3 \sum_{n=0}^{\infty} J_n(x)\, t^n,
\end{aligned}$$

which is equivalent to

$$\sum_{n=0}^{\infty} J_n(x)\, t^n (1 - t - xt^2 - x^2t^3) = a + [b-a]\,t + [(c_1 - a)\,x + c_0 - b]\,t^2,$$

therefore

$$\sum_{n=0}^{\infty} J_n(x)\, t^n = \frac{a + [b-a]\,t + [(c_1 - a)\,x + c_0 - b]\,t^2}{1 - t - xt^2 - x^2t^3}.$$

This completes the proof. □

4. Main Results

In this part, we are now in a position to provide theorem. At the mean time we derive the new generating functions of the products of some known numbers.

Theorem 4.1. *Let A and E be two alphabets, respectively,* $\{a_1, a_2\}$ *and* $\{e_1, e_2, e_3\}$ *,then we have*

$$\sum_{n=0}^{\infty} S_n(E)\, S_{n+k-1}(A)\, t^n = \frac{\sum_{n=0}^{\infty} S_n(-E)\delta_{a_1a_2}^k\left(a_2^n\right)t^n}{\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-E)a_2^n t^n\right)}.$$

Proof. Applying the operator $\delta_{a_1a_2}^k$ to the series $f(a_1t) = \sum_{n=0}^{\infty} S_n(E)a_1^n t^n$, we have

$$\begin{aligned}
\delta_{a_1a_2}^k f(a_1t) &= \delta_{a_1a_2}^k\left(\sum_{n=0}^{\infty} S_n(E)a_1^n t^n\right) \\
&= \frac{\sum_{n=0}^{\infty} S_n(E)a_1^{n+k}t^n - \sum_{n=0}^{\infty} S_n(E)a_2^{n+k}t^n}{a_1-a_2} \\
&= \sum_{n=0}^{\infty} S_n(E)\frac{a_1^{n+k}-a_2^{n+k}}{a_1-a_2}t^n \\
&= \sum_{n=0}^{\infty} S_n(E)\, S_{n+k-1}(A)\, t^n.
\end{aligned}$$

On the other part, since $\sum_{n=0}^{\infty} S_n(E)a_1^n t^n = \frac{1}{\prod_{e\in E}(1-ea_1t)}$ we have

$$\begin{aligned}
\delta_{a_1a_2}^k f(a_1t) &= \delta_{a_1a_2}^k\left(\frac{1}{\prod_{e\in E}(1-ea_1t)}\right) \\
&= \frac{\frac{a_1^k}{\prod_{e\in E}(1-ea_1t)} - \frac{a_2^k}{\prod_{e\in E}(1-ea_2t)}}{a_1-a_2} \\
&= \frac{a_1^k\prod_{e\in E}(1-ea_2t) - a_2^k\prod_{e\in E}(1-ea_1t)}{(a_1-a_2)\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1-ea_2t)}.
\end{aligned}$$

Using the fact that: $\sum_{n=0}^{\infty} S_n(-E)a_1^n t^n = \prod_{e\in E}(1-ea_1t)$, then

$$\begin{aligned}
\delta_{a_1a_2}^k f(a_1t) &= \frac{a_1^k\sum_{n=0}^{\infty} S_n(-E)a_2^n t^n - a_2^k\sum_{n=0}^{\infty} S_n(-E)a_1^n t^n}{(a_1-a_2)\left(\sum_{n=0}^{\infty} S_n(-E)a_2^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n t^n\right)} \\
&= \frac{\sum_{n=0}^{\infty} S_n(-E)\frac{a_1^ka_2^n-a_2^ka_1^n}{a_1-a_2}t^n}{\left(\sum_{n=0}^{\infty} S_n(-E)a_2^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n t^n\right)} \\
&= \frac{\sum_{n=0}^{\infty} S_n(-E)\delta_{a_1a_2}^k\left(a_2^n\right)t^n}{\left(\sum_{n=0}^{\infty} S_n(-E)a_2^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-E)a_1^n t^n\right)}.
\end{aligned}$$

This completes the proof. □

4.1. Ordinary Generating Functions

In this part, we consider theorem 4.1 in order to derive a generating functions of generalized Tribonacci and Tricobsthal Polynomials if $E = \{e_1, e_2, e_3\}, A = \{1, 0\}$ and $k = 1$.

Lemma 4.2. *Given an alphabet $E = \{e_1, e_2, e_3\}$, we have*

$$\sum_{n=0}^{\infty} S_n(E)\, t^n = \frac{1}{(1-e_1 t)(1-e_2 t)(1-e_3 t)}. \tag{4.1}$$

with

$$\begin{aligned}(1-e_1 t)(1-e_2 t)(1-e_3 t) &= 1-(e_1+e_2+e_3)t+(e_1e_2+e_1e_3+e_2e_3)t^2-e_1e_2e_3t^3\\ &= 1+S_1(-E)\,t+S_2(-E)\,t^2+S_3(-E)\,t^3.\end{aligned}$$

Proposition 4.3. *Given an alphabet $E = \{e_1, e_2, e_3\}$, we have*

$$\sum_{n=0}^{\infty} S_{n-1}(E)\, t^n = \frac{t}{(1-e_1 t)(1-e_2 t)(1-e_3 t)}. \tag{4.2}$$

Proof. By applying the operator $\partial_{e_1 e_2}$to the identity $\sum\limits_{n=0}^{\infty} e_1^n t^n = \frac{1}{(1-e_1 t)}$, we have

$$\begin{aligned}\partial_{e_1 e_2}\sum_{n=0}^{\infty} e_1^n t^n &= \partial_{e_1 e_2}\frac{1}{1-e_1 t}\\ \sum_{n=0}^{\infty}\partial_{e_1 e_2} e_1^n t^n &= \frac{t}{(1-e_1 t)(1-e_2 t)}\\ \sum_{n=0}^{\infty} S_{n-1}(e_1+e_2)t^n &= \frac{t}{(1-e_1 t)(1-e_2 t)}.\end{aligned} \tag{4.3}$$

By applying the operator $\delta^1_{e_2 e_3}$to the identity (4.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\delta^1_{e_2 e_3} S_{n-1}(e_1+e_2)t^n &= \delta^1_{e_1 e_2}\frac{t}{(1-e_1 t)(1-e_2 t)}\\ \sum_{n=0}^{\infty} S_{n-1}(e_1+e_2+e_3)t^n &= \frac{t}{(1-e_1 t)}\delta^1_{e_2 e_3}\frac{1}{1-e_2 t}\\ \sum_{n=0}^{\infty} S_{n-1}(e_1+e_2+e_3)t^n &= \frac{t}{(1-e_1 t)}\frac{1}{(1-e_2 t)(1-e_3 t)},\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} S_{n-1}(e_1+e_2+e_3)t^n = \frac{t}{(1-e_1 t)(1-e_2 t)(1-e_3 t)}.$$

This completes the proof. □

Proposition 4.4. *Given an alphabet $E = \{e_1, e_2, e_3\}$, we have*

$$\sum_{n=0}^{\infty} S_{n-2}(E)\, t^n = \frac{t^2}{(1-e_1 t)(1-e_2 t)(1-e_3 t)}. \tag{4.4}$$

Proof. To have formula (4.4), it suffices to apply the operator $\partial_{e_2e_3}$to the identity (4.3), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\partial_{e_2e_3}S_{n-1}(e_1+e_2)t^n &= \partial_{e_2e_3}\frac{t}{(1-e_1t)(1-e_2t)}\\
\sum_{n=0}^{\infty}S_{n-2}(e_1+e_2+e_3)t^n &= \frac{t}{(1-e_1t)}\partial_{e_2e_3}\frac{1}{(1-e_2t)}\\
\sum_{n=0}^{\infty}S_{n-2}(e_1+e_2+e_3)t^n &= \frac{t}{(1-e_1t)}\frac{t}{(1-e_2t)(1-e_3t)}\\
\sum_{n=0}^{\infty}S_{n-2}(e_1+e_2+e_3)t^n &= \frac{t}{(1-e_1t)}\frac{t}{(1-e_2t)(1-e_3t)}\\
\sum_{n=0}^{\infty}S_{n-2}(e_1+e_2+e_3)t^n &= \frac{t^2}{(1-e_1t)(1-e_2t)(1-e_3t)}.
\end{aligned}$$

This completes the proof. □

Setting $\begin{cases} S_1(-E)=-x^2,\\ S_2(-E)=-x,\\ S_3(-E)=-1,\end{cases}$ in (4.1) , (4.2) and (4.4) , we obtain

$$\sum_{n=0}^{\infty}S_n(E)t^n=\frac{1}{1-x^2t-xt^2-t^3}, \tag{4.5}$$

$$\sum_{n=0}^{\infty}S_{n-1}(E)t^n=\frac{t}{1-x^2t-xt^2-t^3}, \tag{4.6}$$

$$\sum_{n=0}^{\infty}S_{n-2}(E)t^n=\frac{t^2}{1-x^2t-xt^2-t^3}. \tag{4.7}$$

Multiplying the equation (4.5) by (a) and adding it from (4.6) multiplying by $\left[b_0+b_1x+(b_2-a)x^2\right]$ and adding it from (4.7) multiplying by $\left[c_0+(c_1-a)x+(c_2-b_0)x^2+(c_3-b_1)x^3+(c_4-b_2)x^4\right]$, we have

$$\sum_{n=0}^{\infty}\begin{pmatrix} aS_n(E)+\left[b_0+b_1x+(b_2-a)x^2\right]S_{n-1}(E)\\ +\left[c_0+(c_1-a)x+(c_2-b_0)x^2+(c_3-b_1)x^3+(c_4-b_2)x^4\right]S_{n-2}(E)\end{pmatrix}t^n$$

$$=\frac{\begin{array}{c} a+\left[b_0+b_1x+(b_2-a)x^2\right]t+\\ \left[c_0+(c_1-a)x+(c_2-b_0)x^2+(c_3-b_1)x^3+(c_4-b_2)x^4\right]t^2\end{array}}{1-x^2t-xt^2-t^3}. \tag{4.8}$$

Formula (4.8) is the new generating function of the generalized Tribonacci polynomials. We note that in the generalized Tribonacci polynomials can be written as

$$aS_n(E)+\left[b_0+b_1x+(b_2-a)x^2\right]S_{n-1}(E)+\begin{bmatrix} c_0+(c_1-a)x\\ +(c_2-b_0)x^2+(c_3-b_1)x^3+(c_4-b_2)x^4\end{bmatrix}S_{n-2}(E)=W_n(x).$$

- Put $a=b_2=c_1=c_4=1$ in (4.8), we have the following Corollary.

Corollary 4.5. *For $n\in\mathbb{N}$, the generating function of Tribonacci polynomials is given by*

$$\sum_{n=0}^{\infty}T_n(x)t^n=\frac{1}{1-x^2t-xt^2-t^3},\ \textit{with } T_n(x)=S_n(E).$$

- Put $a = 3$, $b_2 = c_4 = 1$and $c_1 = 2$ in (4.8), we have the following Corollary.

Corollary 4.6. *For $n \in \mathbb{N}$, the generating function of Tribonacci Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} K(x)t^n = \frac{3 - 2x^2t - xt^2}{1 - x^2t - xt^2 - t^3}, \textit{with } K_n(x) = 3S_n(E) - 2x^2S_{n-1}(E) - xS_{n-2}(E).$$

Setting $\begin{cases} S_1(-E) = -1, \\ S_2(-E) = -x, \\ S_3(-E) = -x^2, \end{cases}$ in (4.1) , (4.2) and (4.4) , we obtain

$$\sum_{n=0}^{\infty} S_n(E)\, t^n = \frac{1}{1 - t - xt^2 - x^2t^3}, \tag{4.9}$$

$$\sum_{n=0}^{\infty} S_{n-1}(E)\, t^n = \frac{t}{1 - t - xt^2 - x^2t^3}, \tag{4.10}$$

$$\sum_{n=0}^{\infty} S_{n-2}(E)\, t^n = \frac{t^2}{1 - t - xt^2 - x^2t^3}. \tag{4.11}$$

Multiplying the equation (4.9) by (a) and adding it from (4.10) multiplying by $(b - a)$ and adding it from (4.11) multiplying by $[c_0 - b + (c_1 - a)x]$, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} [aS_n(E) + (b-a)S_{n-1}(E) + [c_0 - b + (c_1 - a)x]\, S_{n-2}(E)]\, t^n \\ = \; &\frac{a + (b-a)t + [c_0 - b + (c_1 - a)x]\, t^2}{1 - t - xt^2 - x^2t^3}, \end{aligned} \tag{4.12}$$

formula (4.12) is the new generating function of the generalized Tricobsthal polynomials. We note that in the generalized Tricobsthal polynomials can be written as

$$aS_n(E) + (b-a)\, S_{n-1}(E) + [c_0 - b + (c_1 - a)x]\, S_{n-2}(E) = J_n(x).$$

According to $a = b = c_1 = c_0 = 1$ in (4.12), we conclude the following Corollary.

Corollary 4.7. *For $n \in \mathbb{N}$, the generating function of Tricobsthal polynomials is given by*

$$\sum_{n=0}^{\infty} J_n^{(3)}(x)t^n = \frac{1}{1 - t - xt^2 - x^2t^3}, \textit{with } J_n^{(3)}(x) = S_n(E).$$

4.2. Generating Functions of Binary Products of Generalized Tibonacci and Tricobsthal Polynomials with Special Numbers

In this part, we consider theorem (4.1) in order to derive a new generating function for the products of Generalized Tibonacci and Tricobsthal Polynomials with Special Numbers if $k = 0$, $k = 1$ and $E = \{e_1, e_2, e_3\}$, $A = \{a_1, -a_2\}$.

Lemma 4.8. *Let $A = \{a_1, -a_2\}$ and $E = \{e_1, e_2, e_3\}$ two alphabets, we have*

$$\sum_{n=0}^{\infty} S_n(E)S_{n-1}(A)t^n = \frac{-S_1(-E)\, t - (a_1 - a_2)S_2(-E)\, t^2 - ((a_1 - a_2)^2 + a_1a_2)S_3(-E)\, t^3}{\prod_{e\in E}(1 - ea_1t) \prod_{e\in E}(1 + ea_2t)}. \tag{4.13}$$

Lemma 4.9. *Let $A = \{a_1, -a_2\}$ and $E = \{e_1, e_2, e_3\}$ two alphabets, we have*

$$\sum_{n=0}^{\infty} S_n(E) S_n(A) t^n = \frac{1 + a_1 a_2 S_2(-E)\, t^2 + a_1 a_2 (a_1 - a_2) S_3(-E)\, t^3}{\prod_{e\in E}(1 - e a_1 t) \prod_{e\in E}(1 + e a_2 t)}. \tag{4.14}$$

Proposition 4.10. *Given two alphabets $E = \{e_1, e_2, e_3\}$ and $A = \{a_1, -a_2\}$ we have*

$$\sum_{n=0}^{\infty} S_{n-1}(E) S_n(A) t^n = \frac{(a_1 - a_2)t - S_1(-E)\, a_1 a_2 t^2 - S_3(-E)\, a_1^2 a_2^2 t^4}{\prod_{e\in E}(1 - e a_1 t) \prod_{e\in E}(1 + e a_2 t)}. \tag{4.15}$$

Proof. Suffices to apply the operator $\delta^1_{a_1 a_2}$to the identity $\sum_{n=0}^{\infty} S_{n-1}(E)\, a_1^n t^n = \frac{a_1 t}{\prod_{e\in E}(1 - e a_1 t)}$, we obtain

$$\begin{aligned}
\delta^1_{a_1 a_2} \sum_{n=0}^{\infty} S_{n-1}(E)\, a_1^n t^n &= \delta^1_{a_1 a_2} \frac{a_1 t}{(1 - e_1 a_1 t)(1 - e_2 a_1 t)(1 - e_3 a_1 t)} \\
\sum_{n=0}^{\infty} S_{n-1}(E)\, \delta^1_{a_1 a_2} a_1^n t^n &= \frac{\frac{a_1^2 t}{\prod_{e\in E}(1 - e a_1 t)} - \frac{a_2^2 t}{\prod_{e\in E}(1 - e a_2 t)}}{a_1 - a_2} \\
\sum_{n=0}^{\infty} S_{n-1}(E)\, S_n(a_1 + a_2) t^n &= \frac{\sum_{n=0}^{\infty} S_n(-E) \frac{a_1^2 a_2{}^n - a_2^2 a_1^n}{a_1 - a_2} t^{n+1}}{\prod_{e\in E}(1 - e a_1 t) \prod_{e\in E}(1 - e a_2 t)} \\
\sum_{n=0}^{\infty} S_{n-1}(E)\, S_n(a_1 + a_2) t^n &= \frac{S_1(a_1 + a_2)t + a_1 a_2 S_1(-E) t^2 - a_1^2 a_2^2 S_3(-E) t^4}{\prod_{e\in E}(1 - e a_1 t) \prod_{e\in E}(1 - e a_2 t)},
\end{aligned}$$

by replacing a_2 by $[-a_2]$, we obtien

$$\sum_{n=0}^{\infty} S_{n-1}(E) S_n(A) t^n = \frac{(a_1 - a_2)t - S_1(-E)\, a_1 a_2 t^2 - S_3(-E)\, a_1^2 a_2^2 t^4}{\prod_{e\in E}(1 - e a_1 t) \prod_{e\in E}(1 + e a_2 t)}.$$

This completes the proof. □

Proposition 4.11. *Let $A = \{a_1, -a_2\}$ and $E = \{e_1, e_2, e_3\}$ two alphabets, we have*

$$\sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}(A) t^n = \frac{t + a_1 a_2 S_2(-E)\, t^3 + a_1 a_2 (a_1 - a_2) S_3(-E)\, t^4}{\prod_{e\in E}(1 - e a_1 t) \prod_{e\in E}(1 + e a_2 t)}. \tag{4.16}$$

Proposition 4.12. *Given two alphabets $E = \{e_1, e_2, e_3\}$ and $A = \{a_1, -a_2\}$ we have*

$$\sum_{n=0}^{\infty} S_{n-2}(E) S_{n-1}(A) t^n = \frac{(a_1 - a_2)t^2 - S_1(-E)\, a_1 a_2 t^3 - S_3(-E)\, a_1^2 a_2^2 t^5}{\prod_{e\in E}(1 - e a_1 t) \prod_{e\in E}(1 + e a_2 t)}. \tag{4.17}$$

Proof. Suffices to apply the operator $\partial_{a_1a_2}$to the identity $\sum_{n=0}^{\infty} S_{n-2}(E)\,a_1^n t^n = \frac{a_1^2t^2}{\prod_{e\in E}(1-ea_1t)}$, we obtain

$$\begin{aligned}
\partial_{a_1a_2}\sum_{n=0}^{\infty} S_{n-2}(E)a_1^n t^n &= \partial_{a_1a_2}\frac{a_1^2t^2}{\prod_{e\in E}(1-ea_1t)}\\
\sum_{n=0}^{\infty} S_{n-2}(E)\partial_{a_1a_2}a_1^n t^n &= \frac{\frac{a_1^2t^2}{\prod_{e\in E}(1-ea_1t)}-\frac{a_2^2t^2}{\prod_{e\in E}(1-ea_2t)}}{a_1-a_2}\\
\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1+a_2)t^n &= \frac{\sum_{n=0}^{\infty} S_n(-E)\frac{a_1^2a_2^n-a_2^2a_1^n}{a_1-a_2}t^{n+2}}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1-ea_2t)}\\
\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1+a_2)t^n &= \frac{S_1(a_1+a_2)t^2+a_1a_2S_1(-E)t^3-a_1^2a_2^2S_3(-E)t^5}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1-ea_2t)},
\end{aligned}$$

by substitution a_2 by $[-a_2]$, we have

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(A)t^n = \frac{(a_1-a_2)t^2-S_1(-E)\,a_1a_2t^3-S_3(-E)\,a_1^2a_2^2t^5}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1+ea_2t)}.$$

This completes the proof. □

Proposition 4.13. *Given two alphabets $E=\{e_1,e_2,e_3\}$ and $A=\{a_1,-a_2\}$, we have*

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_n(A)t^n = \frac{\left((a_1-a_2)^2+a_1a_2\right)t^2-a_1a_2(a_1-a_2)S_1(-E)\,t^3+S_2(-E)\,a_1^2a_2^2t^4}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1+ea_2t)}. \tag{4.18}$$

Proof. Suffices to apply the operator $\delta^1_{a_1a_2}$to the identity $\sum_{n=0}^{\infty} S_{n-2}(E)\,a_1^n t^n = \frac{a_1^2t^2}{\prod_{e\in E}(1-ea_1t)}$, we obtain

$$\begin{aligned}
\delta^1_{a_1a_2}\sum_{n=0}^{\infty} S_{n-2}(E)a_1^n t^n &= \delta^1_{a_1a_2}\frac{a_1^2t^2}{\prod_{e\in E}(1-ea_1t)}\\
\sum_{n=0}^{\infty} S_{n-2}(E)\delta^1_{a_1a_2}a_1^n t^n &= \delta^1_{a_1a_2}\frac{a_1^2t^2}{\prod_{e\in E}(1-ea_1t)}\\
\sum_{n=0}^{\infty} S_{n-2}(E)S_n(a_1+a_2)t^n &= \frac{\frac{a_1^3z^2}{\prod_{e\in E}(1-ea_1t)}-\frac{a_2^3z^2}{\prod_{e\in E}(1-ea_2t)}}{a_1-a_2}\\
\sum_{n=0}^{\infty} S_{n-2}(E)S_n(a_1+a_2)z^n &= \frac{\sum_{n=0}^{\infty} S_n(-E)\frac{a_1^3a_2^n+a_2^3a_1^n}{a_1-a_2}t^{n+2}}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1-ea_2t)}\\
\sum_{n=0}^{\infty} S_{n-2}(E)S_n(a_1+a_2)z^n &= \frac{S_2(a_1+a_2)t^2+a_1a_2S_1(a_1+a_2)S_1(-E)t^3+a_1^2a_2^2S_2(-E)t^4}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1-ea_2t)},
\end{aligned}$$

replace a_2 by$[-a_2]$ we have

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_n(A)t^n = \frac{\left((a_1-a_2)^2+a_1a_2\right)t^2-a_1a_2(a_1-a_2)S_1(-E)\,t^3+S_2(-E)\,a_1^2a_2^2t^4}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1+ea_2t)}.$$

This completes the proof. □

Theorem 4.14. *For $n \in \mathbb{N}$, the new generating function for the combined Generalized Tribonacci polynomials and k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} W_n(x)F_{k,n}t^n = \frac{\begin{array}{c} a + k\left(b_0 + b_1x + (b_2 - a)x^2\right)t \\ +\left(\begin{array}{c} k^2\left(c_0 + (c_1 - a)x + (c_2 - b_0)x^2 + (c_3 - b_1)x^3 + (c_4 - b_2)x^4\right) \\ +\left(c_0 + c_1x + c_2x^2 + c_3x^3 + (c_4 - a)x^3\right) \end{array}\right)t^2 \\ -k(a - c_0x^2 - (c_1 - a)x^3 - (c_2 - b_0)x^4 - (c_3 - b_1)x^5 - (c_4 - b_2)x^6)t^3 \\ +\left(\begin{array}{c} b_0 + (b_1 - c_0)x + (b_2 - c_1)x^2 - (c_2 - b_0)x^3 \\ -(c_3 - b_1)x^4 - (c_4 - b_2)x^5 \end{array}\right)t^4 \end{array}}{1 - kx^2t - x(x^3 + 2 + k^2)t^2 - k(k^2 + x^3 + 3)t^3 - x^2\left(k^2 + 1\right)t^4 + kxt^5 - t^6}. \quad (4.19)$$

Proof. We have

$$F_{k,n} = S_n\left(a_1 + [-a_2]\right), \text{ (see [7]).}$$

We see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} W_n(x)F_{k,n}t^n \\
= &\sum_{n=0}^{\infty}\left(\begin{array}{c} aS_n(E) + \left(b_0 + b_1x + (b_2 - a)x^2\right)S_{n-1}(E) \\ +\left(c_0 + (c_1 - a)x + (c_2 - b_0)x^2 + (c_3 - b_1)x^3 + (c_4 - b_2)x^4\right)S_{n-2}(E) \end{array}\right)S_n\left(a_1 + [-a_2]\right)t^n \\
= &\ a\sum_{n=0}^{\infty} S_n(E)S_n\left(a_1 + [-a_2]\right)t^n + \left(b_0 + b_1x + (b_2 - a)x^2\right)\sum_{n=0}^{\infty} S_{n-1}(E)S_n\left(a_1 + [-a_2]\right)t^n \\
&+\left(c_0 + (c_1 - a)x + (c_2 - b_0)x^2 + (c_3 - b_1)x^3 + (c_4 - b_2)x^4\right)\sum_{n=0}^{\infty} S_{n-2}(E)S_n\left(a_1 + [-a_2]\right)t^n.
\end{aligned}$$

Using the formulas (4.14), (4.15) and (4.18) we have

$$\sum_{n=0}^{\infty} W_n(x)F_{k,n}t^n$$

$$= a\left(\frac{1 - xa_1a_2t^2 - a_1a_2(a_1-a_2)t^3}{\begin{array}{c}1 - x^2(a_1-a_2)t - x(a_1a_2\left(x^3+2\right) + (a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2 + a_1a_2(x^3+3))t^3\\ -a_1a_2x^2\left((a_1-a_2)^2 + a_1a_2\right)t^4 + a_1^2a_2^2(a_1-a_2)xt^5 - a_1^3a_2^3t^6\end{array}}\right)$$

$$+\left(b_0 + b_1x + (b_2-a)x^2\right)\left(\frac{(a_1-a_2)t + x^2a_1a_2t^2 + a_1^2a_2^2t^4}{\begin{array}{c}1 - x^2(a_1-a_2)t - x(a_1a_2\left(x^3+2\right) + (a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2 + a_1a_2(x^3+3))t^3\\ -a_1a_2x^2\left((a_1-a_2)^2 + a_1a_2\right)t^4 + a_1^2a_2^2(a_1-a_2)xt^5 - a_1^3a_2^3t^6\end{array}}\right)$$

$$+\left(\begin{array}{c}c_0 + (c_1-a)x + (c_2-b_0)x^2\\ +(c_3-b_1)x^3 + (c_4-b_2)x^4\end{array}\right)\left(\frac{\left((a_1-a_2)^2 + a_1a_2\right)t^2 + x^2a_1a_2(a_1-a_2)t^3 - a_1^2a_2^2xt^4}{\begin{array}{c}1 - x^2(a_1-a_2)t - x(a_1a_2\left(x^3+2\right) + (a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2 + a_1a_2(x^3+3))t^3\\ -a_1a_2x^2\left((a_1-a_2)^2 + a_1a_2\right)t^4 + a_1^2a_2^2(a_1-a_2)xt^5 - a_1^3a_2^3t^6\end{array}}\right)$$

$$= \frac{\begin{array}{c}a + \left(b_0 + b_1x + (b_2-a)x^2\right)(a_1-a_2)t\\ +\left(\begin{array}{c}\left(c_0 + (c_1-a)x + (c_2-b_0)x^2 + (c_3-b_1)x^3 + (c_4-b_2)x^4\right)(a_1-a_2)^2\\ +a_1a_2\left(c_0 + c_1x + c_2x^2 + c_3x^3 + (c_4-a)x^4\right)\end{array}\right)t^2\\ -(a_1-a_2)a_1a_2(a - c_0x^2 - (c_1-a)x^3 - (c_2-b_0)x^4 - (c_3-b_1)x^5 - (c_4-b_2)x^6)t^3\\ +a_1^2a_2^2\left(\begin{array}{c}b_0 + (b_1-c_0)x + (b_2-c_1)x^2 - (c_2-b_0)x^3\\ -(c_3-b_1)x^4 - (c_4-b_2)x^5\end{array}\right)t^4\end{array}}{\begin{array}{c}1 - x^2(a_1-a_2)t - x(a_1a_2\left(x^3+2\right) + (a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2 + a_1a_2(x^3+3))t^3 - a_1a_2x^2\left((a_1-a_2)^2 + a_1a_2\right)t^4 + a_1^2a_2^2(a_1-a_2)xt^5 - a_1^3a_2^3t^6\end{array}},$$

therefore

$$\sum_{n=0}^{\infty} W_n(x)F_{k,n}t^n = \frac{\begin{array}{c}a + k\left(b_0 + b_1x + (b_2-a)x^2\right)t\\ +\left(\begin{array}{c}k^2\left(c_0 + (c_1-a)x + (c_2-b_0)x^2 + (c_3-b_1)x^3 + (c_4-b_2)x^4\right)\\ +c_0 + c_1x + c_2x^2 + c_3x^3 + (c_4-a)x^4\end{array}\right)t^2\\ -k(a - c_0x^2 - (c_1-a)x^3 - (c_2-b_0)x^4 - (c_3-b_1)x^5 - (c_4-b_2)x^6)t^3\\ +\left(\begin{array}{c}b_0 + (b_1-c_0)x + (b_2-c_1)x^2 - (c_2-b_0)x^3\\ -(c_3-b_1)x^4 - (c_4-b_2)x^5\end{array}\right)t^4\end{array}}{1 - kx^2t - x(x^3+2+k^2)t^2 - k(k^2+x^3+3)t^3 - x^2\left(k^2+1\right)t^4 + kxt^5 - t^6}.$$

This completes the proof. □

- Set $a = b_2 = c_1 = c_4 = 1$ in (4.19), we have the following proposition.

Proposition 4.15. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci polynomials and k-Fibonacci* *numbers is given by*

$$\sum_{n=0}^{\infty} T_n(x)F_{k,n}t^n = \frac{1+xt^2-kt^3}{1-kx^2t-x(x^3+2+k^2)t^2-k(k^2+x^3+3)t^3-x^2\left(k^2+1\right)t^4+kxt^5-t^6}. \tag{4.20}$$

- Let $a = 3$, $b_2 = c_4 = 1$, $c_1 = 2$ in (4.19), we have the following proposition.

Proposition 4.16. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci-Lucas polynomials and k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} K_n(x)F_{k,n}t^n = \frac{3-2kt^2+(2-2x^3-k^2)xt^2-k(3+x^3)t^3-x^2t^4}{1-kx^2t-x(x^3+2+k^2)t^2-k(k^2+x^3+3)t^3-x^2\left(k^2+1\right)t^4+kxt^5-t^6}. \tag{4.21}$$

- Put $k = 1$ in the relationships (4.20) and (4.21), we obtain the following corollaries.

Corollary 4.17. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci polynomials and Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} T_n(x)F_nt^n = \frac{1+xt^2-t^3}{1-x^2t-x(x^3+3)t^2-(x^3+4)t^3-2x^2t^4+xt^5-t^6}.$$

Corollary 4.18. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci-Lucas polynomials and Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} K_n(x)F_nt^n = \frac{3-2t^2+(1-2x^3)xt^2-(3+x^3)t^3-x^2t^4}{1-x^2t-x(x^3+3)t^2-(x^3+4)t^3-2x^2t^4+xt^5-t^6}.$$

Theorem 4.19. *For $n \in \mathbb{N}$, the new generating function for the combined Generalized Tribonacci polynomials and k-Pell numbers is given by*

$$\sum_{n=0}^{\infty} W_n(x)P_{k,n}t^n = \frac{\begin{array}{c}\left(b_0+b_1x+b_2x^2\right)t+2\left(c_0+c_1x+(c_2-b_0)x^2+(c_3-b_1)x^3+(c_4-b_2)x^4\right)t^2\\ +\left(k\left(\begin{array}{c}a-b_0x+(c_0-b_1)x^2+(c_1-b_2)x^3+(c_2-b_0)x^4\\ +(c_3-b_1)x^5+(c_4-b_2)x^6\end{array}\right)+4a\right)t^3\\ -2k\left(b_0+b_1x+(b_2-a)x^2\right)t^4+4\left(\begin{array}{c}c_0+(c_1-a)x+(c_2-b_0)x^2\\ +(c_3-b_1)x^3+(c_4-b_2)x^4\end{array}\right)t^5\end{array}}{1-4x^2t-x(k(x^3+2)+4)t^2-2(4+k(x^3+3))t^3-kx^2(4+k)t^4+2k^2xt^5-k^3t^6}. \tag{4.22}$$

Proof. We have

$$P_{k,n} = S_{n-1}(a_1+[-a_2]), \text{ (see [7]).}$$

We see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} W_n(x)P_{k,n}t^n\\
= &\sum_{n=0}^{\infty}\left(\begin{array}{c}aS_n(E)+\left(b_0+b_1x+(b_2-a)x^2\right)S_{n-1}(E)\\ +\left(c_0+(c_1-a)x+(c_2-b_0)x^2+(c_3-b_1)x^3+(c_4-b_2)x^4\right)S_{n-2}(E)\end{array}\right)S_{n-1}(a_1+[-a_2])t^n\\
= &\ a\sum_{n=0}^{\infty} S_n(E)S_{n-1}(a_1+[-a_2])t^n+\left(b_0+b_1x+(b_2-a)x^2\right)\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1+[-a_2])t^n\\
&+\left(c_0+(c_1-a)x+(c_2-b_0)x^2+(c_3-b_1)x^3+(c_4-b_2)x^4\right)\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1+[-a_2])t^n.
\end{aligned}$$

Using the formulas (4.13), (4.16) and (4.17) we have

$$\sum_{n=0}^{\infty} W_n(x)P_{k,n}t^n$$

$$= a\left(\frac{x^2t + x(a_1-a_2)t^2 + (a_1a_2 + (a_1-a_2)^2)t^3}{\begin{array}{c}1 - x^2(a_1-a_2)t - x(a_1a_2\left(x^3+2\right) + (a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2 + a_1a_2(x^3+3))t^3\\ -a_1a_2x^2\left((a_1-a_2)^2 + a_1a_2\right)t^4 + a_1^2a_2^2(a_1-a_2)xt^5 - a_1^3a_2^3t^6\end{array}}\right)$$

$$+\left(b_0 + b_1x + (b_2-a)x^2\right)\left(\frac{t - xa_1a_2t^3 - a_1a_2(a_1-a_2)t^4}{\begin{array}{c}1 - x^2(a_1-a_2)t - x(a_1a_2\left(x^3+2\right) + (a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2 + a_1a_2(x^3+3))t^3\\ -a_1a_2x^2\left((a_1-a_2)^2 + a_1a_2\right)t^4 + a_1^2a_2^2(a_1-a_2)xt^5 - a_1^3a_2^3t^6\end{array}}\right)$$

$$+\left(\begin{array}{c}c_0 + (c_1-a)x + (c_2-b_0)x^2\\ +(c_3-b_1)x^3 + (c_4-b_2)x^4\end{array}\right)\left(\frac{(a_1-a_2)t^2 + x^2a_1a_2t^3 + a_1^2a_2^2t^5}{\begin{array}{c}1 - x^2(a_1-a_2)t - x(a_1a_2\left(x^3+2\right) + (a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2 + a_1a_2(x^3+3))t^3\\ -a_1a_2x^2\left((a_1-a_2)^2 + a_1a_2\right)t^4 + a_1^2a_2^2(a_1-a_2)xt^5 - a_1^3a_2^3t^6\end{array}}\right)$$

$$= \frac{\begin{array}{c}\left(b_0 + b_1x + b_2x^2\right)t + \left(c_0 + c_1x + (c_2-b_0)x^2 + (c_3-b_1)x^3 + (c_4-b_2)x^4\right)(a_1-a_2)t^2\\ +\left(a_1a_2\left(\begin{array}{c}a - b_0x + (c_0-b_1)x^2 + (c_1-b_2)x^3 + (c_2-b_0)x^4\\ +(c_3-b_1)x^5 + (c_4-b_2)x^6\end{array}\right) + a(a_1-a_2)^2\right)t^3\\ -\left(b_0 + b_1x + (b_2-a)x^2\right)a_1a_2(a_1-a_2)t^4 + a_1^2a_2^2\left(\begin{array}{c}c_0 + (c_1-a)x + (c_2-b_0)x^2\\ +(c_3-b_1)x^3 + (c_4-b_2)x^4\end{array}\right)t^5\end{array}}{\begin{array}{c}1 - x^2(a_1-a_2)t - x(a_1a_2\left(x^3+2\right) + (a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2 + a_1a_2(x^3+3))t^3 - a_1a_2x^2\left((a_1-a_2)^2 + a_1a_2\right)t^4 + a_1^2a_2^2(a_1-a_2)xt^5 - a_1^3a_2^3t^6\end{array}},$$

therefore

$$\sum_{n=0}^{\infty} W_n(x)P_{k,n}t^n = \frac{\begin{array}{c}\left(b_0 + b_1x + b_2x^2\right)t + 2\left(c_0 + c_1x + (c_2-b_0)x^2 + (c_3-b_1)x^3 + (c_4-b_2)x^4\right)t^2\\ +\left(k\left(\begin{array}{c}a - b_0x + (c_0-b_1)x^2 + (c_1-b_2)x^3 + (c_2-b_0)x^4\\ +(c_3-b_1)x^5 + (c_4-b_2)x^6\end{array}\right) + 4a\right)t^3\\ -2k\left(b_0 + b_1x + (b_2-a)x^2\right)t^4 + 4\left(\begin{array}{c}c_0 + (c_1-a)x + (c_2-b_0)x^2\\ +(c_3-b_1)x^3 + (c_4-b_2)x^4\end{array}\right)t^5\end{array}}{1 - 4x^2t - x(k\left(x^3+2\right) + 4)t^2 - 2(4 + k(x^3+3))t^3 - kx^2\left(4+k\right)t^4 + 2k^2xt^5 - k^3t^6}.$$

This completes the proof. □

- Set $a = b_2 = c_1 = c_4 = 1$ in (4.22), we have the following proposition.

Proposition 4.20. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci polynomials and k-Pell numbers is given by*

$$\sum_{n=0}^{\infty} T_n(x)P_{k,n}t^n = \frac{x^2t + 2xt^2 + (k+4)t^3}{1 - 4x^2t - x(k\left(x^3+2\right) + 4)t^2 - 2(4 + k(x^3+3))t^3 - kx^2\left(4+k\right)t^4 + 2k^2xt^5 - k^3t^6}. \tag{4.23}$$

- Let $a = 3$, $b_2 = c_4 = 1$, $c_1 = 2$ in (4.22), we have the following proposition.

Proposition 4.21. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci-Lucas polynomials and k-Pell numbers is given by*

$$\sum_{n=0}^{\infty} K_n(x)P_{k,n}t^n = \frac{x^2t + 4xt^2 + (k(3 + x^3) + 12)t^3 + 4kx^2t^4 - 4xt^5}{1 - 4x^2t - x(k(x^3 + 2) + 4)t^2 - 2(4 + k(x^3 + 3))t^3 - kx^2(4 + k)t^4 + 2k^2xt^5 - k^3t^6}. \tag{4.24}$$

- Put $k = 1$ in the relationships (4.23) and (4.24), we obtain the following corollaries.

Corollary 4.22. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci polynomials and Pell numbers is given by*

$$\sum_{n=0}^{\infty} T_n(x)P_nt^n = \frac{x^2t + 2xt^2 + 5t^3}{1 - 4x^2t - x(x^3 + 6)t^2 - 2(x^3 + 7)t^3 - 5x^2t^4 + 2xt^5 - t^6}.$$

Corollary 4.23. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci-Lucas polynomials and Pell numbers is given by*

$$\sum_{n=0}^{\infty} K_n(x)P_nt^n = \frac{x^2t + 4xt^2 + (x^3 + 15)t^3 + 4x^2t^4 - 4xt^5}{1 - 4x^2t - x(x^3 + 6)t^2 - 2(x^3 + 7)t^3 - 5x^2t^4 + 2xt^5 - t^6}.$$

Theorem 4.24. *For $n \in \mathbb{N}$, the new generating function for the combined Generalized Tribonacci polynomials and k-Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} W_n(x)J_{k,n}t^n = \frac{\begin{array}{c}\left(b_0 + b_1x + b_2x^2\right)t + k\left(c_0 + c_1x + (c_2 - b_0)x^2 + (c_3 - b_1)x^3 + (c_4 - b_2)x^4\right)t^2 \\ +\left(2\left(\begin{array}{c} a - b_0x + (c_0 - b_1)x^2 + (c_1 - b_2)x^3 + (c_2 - b_0)x^4 \\ +(c_3 - b_1)x^5 + (c_4 - b_2)x^6\end{array}\right) + ak^2\right)t^3 \\ -2k\left(b_0 + b_1x + (b_2 - a)x^2\right)t^4 + 4\left(\begin{array}{c} c_0 + (c_1 - a)x + (c_2 - b_0)x^2 \\ +(c_3 - b_1)x^3 + (c_4 - b_2)x^4\end{array}\right)t^5\end{array}}{1 - kx^2t - x(2(x^3 + 2) + k^2)t^2 - k(k^2 + 2(x^3 + 3))t^3 - 2x^2(k^2 + 2)t^4 + 4kxt^5 - 8t^6}. \tag{4.25}$$

Proof. We have

$$J_{k,n} = S_{n-1}(a_1 + [-a_2]), \text{ (see [7])}.$$

We see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} W_n(x)J_{k,n}t^n \\
&= \sum_{n=0}^{\infty}\left(\begin{array}{c} aS_n(E) + \left(b_0 + b_1x + (b_2 - a)x^2\right)S_{n-1}(E) \\ +\left(c_0 + (c_1 - a)x + (c_2 - b_0)x^2 + (c_3 - b_1)x^3 + (c_4 - b_2)x^4\right)S_{n-2}(E)\end{array}\right)S_{n-1}(a_1 + [-a_2])t^n \\
&= a\sum_{n=0}^{\infty} S_n(E)S_{n-1}(a_1 + [-a_2])t^n + \left(b_0 + b_1x + (b_2 - a)x^2\right)\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1 + [-a_2])t^n \\
&\quad + \left(c_0 + (c_1 - a)x + (c_2 - b_0)x^2 + (c_3 - b_1)x^3 + (c_4 - b_2)x^4\right)\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1 + [-a_2])t^n.
\end{aligned}$$

Using the formulas (4.13), (4.16) and (4.17) we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} W_n(x)J_{k,n}t^n \\
&= a\left(\frac{x^2t + x(a_1-a_2)t^2 + (a_1a_2 + (a_1-a_2)^2)t^3}{\begin{array}{c}1 - x^2(a_1-a_2)t - x(a_1a_2\left(x^3+2\right) + (a_1-a_2)^2)t^2 \\ -(a_1-a_2)((a_1-a_2)^2 + a_1a_2(x^3+3))t^3 \\ -a_1a_2x^2\left((a_1-a_2)^2 + a_1a_2\right)t^4 + a_1^2a_2^2(a_1-a_2)xt^5 - a_1^3a_2^3t^6\end{array}}\right) \\
&+\left(b_0 + b_1x + (b_2-a)x^2\right)\left(\frac{t - xa_1a_2t^3 - a_1a_2(a_1-a_2)t^4}{\begin{array}{c}1 - x^2(a_1-a_2)t - x(a_1a_2\left(x^3+2\right) + (a_1-a_2)^2)t^2 \\ -(a_1-a_2)((a_1-a_2)^2 + a_1a_2(x^3+3))t^3 \\ -a_1a_2x^2\left((a_1-a_2)^2 + a_1a_2\right)t^4 + a_1^2a_2^2(a_1-a_2)xt^5 - a_1^3a_2^3t^6\end{array}}\right) \\
&+\left(\begin{array}{c}c_0 + (c_1-a)x + (c_2-b_0)x^2 \\ +(c_3-b_1)x^3 + (c_4-b_2)x^4\end{array}\right)\left(\frac{(a_1-a_2)t^2 + x^2a_1a_2t^3 + a_1^2a_2^2t^5}{\begin{array}{c}1 - x^2(a_1-a_2)t - x(a_1a_2\left(x^3+2\right) + (a_1-a_2)^2)t^2 \\ -(a_1-a_2)((a_1-a_2)^2 + a_1a_2(x^3+3))t^3 \\ -a_1a_2x^2\left((a_1-a_2)^2 + a_1a_2\right)t^4 + a_1^2a_2^2(a_1-a_2)xt^5 - a_1^3a_2^3t^6\end{array}}\right) \\
&= \frac{\begin{array}{c}\left(b_0 + b_1x + b_2x^2\right)t + \left(c_0 + c_1x + (c_2-b_0)x^2 + (c_3-b_1)x^3 + (c_4-b_2)x^4\right)(a_1-a_2)t^2 \\ +\left(a_1a_2\left(\begin{array}{c}a - b_0x + (c_0-b_1)x^2 + (c_1-b_2)x^3 + (c_2-b_0)x^4 \\ +(c_3-b_1)x^5 + (c_4-b_2)x^6\end{array}\right) + a(a_1-a_2)^2\right)t^3 \\ -\left(b_0 + b_1x + (b_2-a)x^2\right)a_1a_2(a_1-a_2)t^4 + a_1^2a_2^2\left(\begin{array}{c}c_0 + (c_1-a)x + (c_2-b_0)x^2 \\ +(c_3-b_1)x^3 + (c_4-b_2)x^4\end{array}\right)t^5\end{array}}{\begin{array}{c}1 - x^2(a_1-a_2)t - x(a_1a_2\left(x^3+2\right) + (a_1-a_2)^2)t^2 \\ -(a_1-a_2)((a_1-a_2)^2 + a_1a_2(x^3+3))t^3 - a_1a_2x^2\left((a_1-a_2)^2 + a_1a_2\right)t^4 + a_1^2a_2^2(a_1-a_2)xt^5 - a_1^3a_2^3t^6\end{array}},
\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} W_n(x)J_{k,n}t^n = \frac{\begin{array}{c}\left(b_0 + b_1x + b_2x^2\right)t + k\left(c_0 + c_1x + (c_2-b_0)x^2 + (c_3-b_1)x^3 + (c_4-b_2)x^4\right)t^2 \\ +\left(2\left(\begin{array}{c}a - b_0x + (c_0-b_1)x^2 + (c_1-b_2)x^3 + (c_2-b_0)x^4 \\ +(c_3-b_1)x^5 + (c_4-b_2)x^6\end{array}\right) + ak^2\right)t^3 \\ -2k\left(b_0 + b_1x + (b_2-a)x^2\right)t^4 + 4\left(\begin{array}{c}c_0 + (c_1-a)x + (c_2-b_0)x^2 \\ +(c_3-b_1)x^3 + (c_4-b_2)x^4\end{array}\right)t^5\end{array}}{1 - kx^2t - x(2(x^3+2) + k^2)t^2 - k(k^2 + 2(x^3+3))t^3 - 2x^2(k^2+2)t^4 + 4kxt^5 - 8t^6}.$$

This completes the proof. □

- Set $a = b_2 = c_1 = c_4 = 1$ in (4.25), we have the following proposition.

Proposition 4.25. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci polynomials and k-Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} T_n(x)J_{k,n}t^n = \frac{x^2t + kxt^2 + \left(2 + k^2\right)t^3}{1 - kx^2t - x(2(x^3+2) + k^2)t^2 - k(k^2 + 2(x^3+3))t^3 - 2x^2(k^2+2)t^4 + 4kxt^5 - 8t^6}. \quad (4.26)$$

- Let $a = 3$, $b_2 = c_4 = 1$, $c_1 = 2$ in (4.25), we have the following proposition.

Proposition 4.26. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci- Lucas polynomials and k-Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} K_n(x)J_{k,n}t^n = \frac{x^2t + 2kxt^2 + \left(6 + 2x^3 + 3k^2\right)t^3 + 4kx^2t^4 - 4xt^5}{1 - kx^2t - x(2\left(x^3 + 2\right) + k^2)t^2 - k(k^2 + 2(x^3 + 3))t^3 - 2x^2\left(k^2 + 2\right)t^4 + 4kxt^5 - 8t^6}. \tag{4.27}$$

- Put $k = 1$ in the relationships (4.26) and (4.27), we obtain the following corollaries.

Corollary 4.27. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci polynomials and Jacobcsthal numbers is given by*

$$\sum_{n=0}^{\infty} T_n(x)J_nt^n = \frac{x^2t + xt^2 + 3t^3}{1 - x^2t - x(2x^3 + 5)t^2 - (2x^3 + 7)t^3 - 6x^2t^4 + 4xt^5 - 8t^6}.$$

Corollary 4.28. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci-Lucas polynomials and Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} K_n(x)J_nt^n = \frac{x^2t + 2xt^2 + \left(9 + 2x^3\right)t^3 + 4x^2t^4 - 4xt^5}{1 - x^2t - x(2x^3 + 5)t^2 - (2x^3 + 7)t^3 - 6x^2t^4 + 4xt^5 - 8t^6}.$$

Theorem 4.29. *For $n \in \mathbb{N}$, the new generating function for the combined Generalized Tribonacci polynomials and k-Mersenne numbers is given by*

$$\sum_{n=0}^{\infty} W_n(x)M_{k,n}t^n = \frac{\begin{array}{c}\left(b_0 + b_1x + b_2x^2\right)t + 3k\left(c_0 + c_1x + (c_2 - b_0)x^2 + (c_3 - b_1)x^3 + (c_4 - b_2)x^4\right)t^2 \\ -\left(2\left(\begin{array}{c} a - b_0x + (c_0 - b_1)x^2 + (c_1 - b_2)x^3 + (c_2 - b_0)x^4 \\ + (c_3 - b_1)x^5 + (c_4 - b_2)x^6 \end{array}\right) - 9ak^2\right)t^3 \\ +6k\left(b_0 + b_1x + (b_2 - a)x^2\right)t^4 + 4\left(\begin{array}{c} c_0 + (c_1 - a)x + (c_2 - b_0)x^2 \\ + (c_3 - b_1)x^3 + (c_4 - b_2)x^4 \end{array}\right)t^5\end{array}}{\begin{array}{c}1 - 3kx^2t - (9k^2 - 2\left(x^3 + 2\right))xt^2 - 3k(9k^2 - 2\left(x^3 + 3\right))t^3 \\ +2(9k^2 - 2)x^2t^4 + 12kxt^5 + 8t^6\end{array}}. \tag{4.28}$$

Proof. We have

$$M_{k,n} = S_{n-1}\left(a_1 + [-a_2]\right), \text{ (see [18]).}$$

We see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} W_n(x)M_{k,n}t^n \\
= \;& \sum_{n=0}^{\infty}\left(\begin{array}{c} aS_n(E) + \left(b_0 + b_1x + (b_2 - a)x^2\right)S_{n-1}(E) \\ + \left(c_0 + (c_1 - a)x + (c_2 - b_0)x^2 + (c_3 - b_1)x^3 + (c_4 - b_2)x^4\right)S_{n-2}(E) \end{array}\right)S_{n-1}\left(a_1 + [-a_2]\right)t^n \\
= \;& a\sum_{n=0}^{\infty} S_n(E)S_{n-1}\left(a_1 + [-a_2]\right)t^n + \left(b_0 + b_1x + (b_2 - a)x^2\right)\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}\left(a_1 + [-a_2]\right)t^n \\
& + \left(c_0 + (c_1 - a)x + (c_2 - b_0)x^2 + (c_3 - b_1)x^3 + (c_4 - b_2)x^4\right)\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}\left(a_1 + [-a_2]\right)t^n.
\end{aligned}$$

Using the formulas (4.13), (4.16) and (4.17) we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} W_n(x)M_{k,n}t^n \\
&= a\left(\frac{x^2t+x(a_1-a_2)t^2+(a_1a_2+(a_1-a_2)^2)t^3}{\begin{array}{c}1-x^2(a_1-a_2)t-x(a_1a_2\left(x^3+2\right)+(a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2+a_1a_2(x^3+3))t^3\\ -a_1a_2x^2\left((a_1-a_2)^2+a_1a_2\right)t^4+a_1^2a_2^2(a_1-a_2)xt^5-a_1^3a_2^3t^6\end{array}}\right)\\
&+\left(b_0+b_1x+(b_2-a)\,x^2\right)\left(\frac{t-xa_1a_2t^3-a_1a_2(a_1-a_2)t^4}{\begin{array}{c}1-x^2(a_1-a_2)t-x(a_1a_2\left(x^3+2\right)+(a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2+a_1a_2(x^3+3))t^3\\ -a_1a_2x^2\left((a_1-a_2)^2+a_1a_2\right)t^4+a_1^2a_2^2(a_1-a_2)xt^5-a_1^3a_2^3t^6\end{array}}\right)\\
&+\left(\begin{array}{c}c_0+(c_1-a)\,x+(c_2-b_0)\,x^2\\ +(c_3-b_1)\,x^3+(c_4-b_2)\,x^4\end{array}\right)\left(\frac{(a_1-a_2)t^2+x^2a_1a_2t^3+a_1^2a_2^2t^5}{\begin{array}{c}1-x^2(a_1-a_2)t-x(a_1a_2\left(x^3+2\right)+(a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2+a_1a_2(x^3+3))t^3\\ -a_1a_2x^2\left((a_1-a_2)^2+a_1a_2\right)t^4+a_1^2a_2^2(a_1-a_2)xt^5-a_1^3a_2^3t^6\end{array}}\right)\\
&= \frac{\begin{array}{c}\left(b_0+b_1x+b_2x^2\right)t+\left(c_0+c_1x+(c_2-b_0)\,x^2+(c_3-b_1)\,x^3+(c_4-b_2)\,x^4\right)(a_1-a_2)t^2\\ +\left(a_1a_2\left(\begin{array}{c}a-b_0x+(c_0-b_1)\,x^2+(c_1-b_2)\,x^3+(c_2-b_0)\,x^4\\ +(c_3-b_1)\,x^5+(c_4-b_2)\,x^6\end{array}\right)+a(a_1-a_2)^2\right)t^3\\ -\left(b_0+b_1x+(b_2-a)\,x^2\right)a_1a_2(a_1-a_2)t^4+a_1^2a_2^2\left(\begin{array}{c}c_0+(c_1-a)\,x+(c_2-b_0)\,x^2\\ +(c_3-b_1)\,x^3+(c_4-b_2)\,x^4\end{array}\right)t^5\end{array}}{\begin{array}{c}1-x^2(a_1-a_2)t-x(a_1a_2\left(x^3+2\right)+(a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2+a_1a_2(x^3+3))t^3-a_1a_2x^2\left((a_1-a_2)^2+a_1a_2\right)t^4+a_1^2a_2^2(a_1-a_2)xt^5-a_1^3a_2^3t^6\end{array}},
\end{aligned}
$$

therefore

$$
\sum_{n=0}^{\infty} W_n(x)M_{k,n}t^n = \frac{\begin{array}{c}\left(b_0+b_1x+b_2x^2\right)t+3k\left(c_0+c_1x+(c_2-b_0)\,x^2+(c_3-b_1)\,x^3+(c_4-b_2)\,x^4\right)t^2\\ -\left(2\left(\begin{array}{c}a-b_0x+(c_0-b_1)\,x^2+(c_1-b_2)\,x^3+(c_2-b_0)\,x^4\\ +(c_3-b_1)\,x^5+(c_4-b_2)\,x^6\end{array}\right)-9ak^2\right)t^3\\ +6k\left(b_0+b_1x+(b_2-a)\,x^2\right)t^4+4\left(\begin{array}{c}c_0+(c_1-a)\,x+(c_2-b_0)\,x^2\\ +(c_3-b_1)\,x^3+(c_4-b_2)\,x^4\end{array}\right)t^5\end{array}}{\begin{array}{c}1-3kx^2t-(9k^2-2\left(x^3+2\right))xt^2-3k(9k^2-2\left(x^3+3\right))t^3\\ +2(9k^2-2)x^2t^4+12kxt^5+8t^6\end{array}}.
$$

This completes the proof. □

- Set $a=b_2=c_1=c_4=1$ in (4.28), we have the following proposition.

Proposition 4.30. *For $n\in\mathbb{N}$, the new generating function for the combined Tribonacci polynomials and k-Mersenne* *numbers is given by*

$$\sum_{n=0}^{\infty} T_n(x)M_{k,n}t^n = \frac{x^2t + 3kxt^2 - \left(2 - 9k^2\right)t^3}{\begin{array}{c}1 - 3kx^2t - (9k^2 - 2\left(x^3 + 2\right))xt^2 - 3k(9k^2 - 2\left(x^3 + 3\right))t^3 \\ +2(9k^2 - 2)x^2t^4 + 12kxt^5 + 8t^6\end{array}}. \tag{4.29}$$

- Let $a = 3$, $b_2 = c_4 = 1$, $c_1 = 2$ in (4.28), we have the following proposition.

Proposition 4.31. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci-Lucas polynomials and k-Mersenne numbers is given by*

$$\sum_{n=0}^{\infty} K_n(x)M_{k,n}t^n = \frac{x^2t + 6kxt^2 - \left(6 + 2x^3 - 27k^2\right)t^3 - 12kx^2t^4 - 4xt^5}{\begin{array}{c}1 - 3kx^2t - (9k^2 - 2\left(x^3 + 2\right))xt^2 - 3k(9k^2 - 2\left(x^3 + 3\right))t^3 \\ +2(9k^2 - 2)x^2t^4 + 12kxt^5 + 8t^6\end{array}}. \tag{4.30}$$

- Put $k = 1$ in the relationships (4.29) and (4.30), we obtain the following corollaries.

Corollary 4.32. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci polynomials and Mersenne numbers is given by*

$$\sum_{n=0}^{\infty} T_n(x)M_nt^n = \frac{x^2t + 3xt^2 + 7t^3}{1 - 3x^2t - (5 - 2x^3)xt^2 - 3(3 - 2x^3)t^3 + 14x^2t^4 + 12xt^5 + 8t^6}.$$

Corollary 4.33. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci-Lucas polynomials and Mersenne numbers is given by*

$$\sum_{n=0}^{\infty} K_n(x)M_nt^n = \frac{x^2t + 6xt^2 + \left(21 - 2x^3\right)t^3 - 12x^2t^4 - 4xt^5}{1 - 3x^2t - (5 - 2x^3)xt^2 - 3(3 - 2x^3)t^3 + 14x^2t^4 + 12xt^5 + 8t^6}.$$

Theorem 4.34. *For $n \in \mathbb{N}$, the new generating function for the combined Generalized Tricobasthal polynomials and k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} J_n(x)F_{k,n}t^n = \frac{\begin{array}{c}a + k(b-a)t + \left[k^2\left(c_0 - b + (c_1 - a)x\right) + c_0 - a + c_1x\right]t^2 + k\left(c_0 - b + (c_1 - a)x - ax^2\right)t^3 \\ -\left(c_0 - b + (c_1 - b)\,x\right)xt^4\end{array}}{1 - kt - (1 + 2x + k^2x)t^2 - k\left(k^2x + 3x + 1\right)xt^3 - \left(k^2 + 1\right)x^2t^4 + kx^3t^5 - x^4t^6}. \tag{4.31}$$

Proof. We have

$$F_{k,n} = S_n\left(a_1 + [-a_2]\right), \text{ (see [7]).}$$

We see that

$$\begin{aligned}\sum_{n=0}^{\infty} J_n(x)F_{k,n}t^n &= \sum_{n=0}^{\infty}(aS_n(E) + (b-a)\,S_{n-1}(E) + (c_0 - b + (c_1 - a)\,x)\,S_{n-2}(E))S_{n-1}\left(a_1 + [-a_2]\right)t^n \\ &= a\sum_{n=0}^{\infty} S_n(E)S_n\left(a_1 + [-a_2]\right)t^n + (b-a)\sum_{n=0}^{\infty} S_{n-1}(E)S_n\left(a_1 + [-a_2]\right)t^n \\ &\quad + \left(c_0 - b + (c_1 - a)\,x\right)\sum_{n=0}^{\infty} S_{n-2}(E)S_n\left(a_1 + [-a_2]\right)t^n,\end{aligned}$$

by the formulas (4.14), (4.15) and (4.18) we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} J_n(x)M_{k,n}t^n \\
&= a\left(\frac{1-a_1a_2xt^2-a_1a_2(a_1-a_2)x^2t^3}{\begin{array}{c}1-x^2(a_1-a_2)t-x(a_1a_2\left(x^3+2\right)+(a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2+a_1a_2(x^3+3))t^3\\ -a_1a_2x^2\left((a_1-a_2)^2+a_1a_2\right)t^4+a_1^2a_2^2(a_1-a_2)xt^5-a_1^3a_2^3t^6\end{array}}\right)\\
&+(b-a)\left(\frac{(a_1-a_2)t+a_1a_2t^2+a_1^2a_2^2x^2t^4}{\begin{array}{c}1-(a_1-a_2)t-((1+2x)a_1a_2+(a_1-a_2)^2x)t^2\\ -(a_1-a_2)\left((a_1-a_2)^2x+a_1a_2(1+3x)\right)xt^3\\ -a_1a_2\left((a_1-a_2)^2+a_1a_2\right)x^2t^4\\ +a_1^2a_2^2(a_1-a_2)x^3t^5-a_1^3a_2^3x^4t^6\end{array}}\right)\\
&+(c_0-b+(c_1-a)x)\left(\frac{\left((a_1-a_2)^2+a_1a_2\right)t^2+a_1a_2(a_1-a_2)t^3-a_1^2a_2^2xt^4}{\begin{array}{c}1-(a_1-a_2)t-((1+2x)a_1a_2+(a_1-a_2)^2x)t^2\\ -(a_1-a_2)\left((a_1-a_2)^2x+a_1a_2(1+3x)\right)xt^3\\ -a_1a_2\left((a_1-a_2)^2+a_1a_2\right)x^2t^4\\ +a_1^2a_2^2(a_1-a_2)x^3t^5-a_1^3a_2^3x^4t^6\end{array}}\right)\\
&= \frac{\begin{array}{c}a+(b-a)(a_1-a_2)t+\left[(c_0-b+(c_1-a)x)(a_1-a_2)^2+a_1a_2(c_0-a+c_1x)\right]t^2\\ +a_1a_2(a_1-a_2)\left(c_0-b+(c_1-a)x-ax^2\right)t^3\\ -a_1^2a_2^2(c_0-b+(c_1-b)x)xt^4\end{array}}{\begin{array}{c}1-(a_1-a_2)t-((1+2x)a_1a_2+(a_1-a_2)^2x)t^2\\ -(a_1-a_2)\left((a_1-a_2)^2x+a_1a_2(1+3x)\right)xt^3\\ -a_1a_2\left((a_1-a_2)^2+a_1a_2\right)x^2t^4+a_1^2a_2^2(a_1-a_2)x^3t^5-a_1^3a_2^3x^4t^6\end{array}},
\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} J_n(x)F_{k,n}t^n = \frac{\begin{array}{c}a+k(b-a)t+\left[k^2(c_0-b+(c_1-a)x)+c_0-a+c_1x\right]t^2\\ +k\left(c_0-b+(c_1-a)x-ax^2\right)t^3-(c_0-b+(c_1-b)x)xt^4\end{array}}{1-kt-(1+2x+k^2x)t^2-k(k^2x+3x+1)xt^3-(k^2+1)x^2t^4+kx^3t^5-x^4t^6}.$$

This completes the proof. □

- Put $a=b=c_1=c_0=1$ in (4.31) we have the following proposition.

Proposition 4.35. *For $n\in\mathbb{N}$, the new generating function for the combined Tricobasthal polynomials and k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} J_n^{(3)}(x)F_{k,n} = \frac{1+xt^2-kx^2t^3}{1-kt-(1+2x+k^2x)t^2-k(k^2x+3x+1)xt^3-(k^2+1)x^2t^4+kx^3t^5-x^4t^6}. \quad (4.32)$$

- Put $k = 1$ in the relationships (4.32),we obtain the following corollary.

Corollary 4.36. *For $n \in \mathbb{N}$, the new generating function for the combined Tricobasthal polynomials and Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} J_n^{(3)}(x) F_n = \frac{1 + xt^2 - x^2t^3}{1 - t - (1+3x)t^2 - (4x+1)\, xt^3 - 2x^2t^4 + x^3t^5 - x^4t^6}.$$

Theorem 4.37. *For $n \in \mathbb{N}$, the new generating function for the combined Generalized Tricobasthal polynomials and k-Pell numbers is given by*

$$\sum_{n=0}^{\infty} J_n(x) P_{k,n} t^n = \frac{\begin{array}{c} bt + 2\,(c_0 - b + c_1x)\, t^2 + \left(4ax^2 + k\left(c_0 - b + (c_1 - b)\, x + ax^2\right)\right) t^3 \\ -2\,(b-a)\, kx^2t^4 + k^2\,(c_0 - b + (c_1 - a)\, x)\, x^2t^5 \end{array}}{1 - 2t - (k\,(1+2x) + 4x)t^2 - 2\,(4x + k(1+3x))\, xt^3 - k\,(4+k)\, x^2t^4 + 2k^2x^3t^5 - k^3x^4t^6}. \quad (4.33)$$

Proof. We have

$$P_{k,n} = S_{n-1}\,(a_1 + [-a_2])\,, \text{ (see [7]).}$$

We see that

$$\begin{aligned} \sum_{n=0}^{\infty} J_n(x) P_{k,n} t^n &= \sum_{n=0}^{\infty} (aS_n(E) + (b-a)\, S_{n-1}(E) + (c_0 - b + (c_1 - a)\, x)\, S_{n-2}(E)) S_{n-1}\,(a_1 + [-a_2])\, t^n \\ &= a \sum_{n=0}^{\infty} S_n(E) S_{n-1}\,(a_1 + [-a_2])\, t^n + (b-a) \sum_{n=0}^{\infty} S_{n-1}(E) S_{n-1}\,(a_1 + [-a_2])\, t^n \\ &\quad + (c_0 - b + (c_1 - a)\, x) \sum_{n=0}^{\infty} S_{n-2}(E) S_{n-1}\,(a_1 + [-a_2])\, t^n, \end{aligned}$$

by the formulas (4.13), (4.16) and (4.17) we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} J_n(x)P_{k,n}t^n \\
=\ & a\left(\frac{t+(a_1-a_2)xt^2+(a_1a_2+(a_1-a_2)^2)x^2t^3}{\begin{array}{c}1-x^2(a_1-a_2)t-x(a_1a_2\left(x^3+2\right)+(a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2+a_1a_2(x^3+3))t^3\\ -a_1a_2x^2\left((a_1-a_2)^2+a_1a_2\right)t^4+a_1^2a_2^2(a_1-a_2)xt^5-a_1^3a_2^3t^6\end{array}}\right)\\
&+(b-a)\left(\frac{t-a_1a_2xt^3-a_1a_2(a_1-a_2)x^2t^4}{\begin{array}{c}1-(a_1-a_2)t-((1+2x)\,a_1a_2+(a_1-a_2)^2x)t^2\\ -(a_1-a_2)\left((a_1-a_2)^2x+a_1a_2(1+3x)\right)xt^3\\ -a_1a_2\left((a_1-a_2)^2+a_1a_2\right)x^2t^4\\ +a_1^2a_2^2(a_1-a_2)x^3t^5-a_1^3a_2^3x^4t^6\end{array}}\right)\\
&+(c_0-b+(c_1-a)\,x)\left(\frac{(a_1-a_2)t^2+a_1a_2t^3+a_1^2a_2^2x^2t^5}{\begin{array}{c}1-(a_1-a_2)t-((1+2x)\,a_1a_2+(a_1-a_2)^2x)t^2\\ -(a_1-a_2)\left((a_1-a_2)^2x+a_1a_2(1+3x)\right)xt^3\\ -a_1a_2\left((a_1-a_2)^2+a_1a_2\right)x^2t^4\\ +a_1^2a_2^2(a_1-a_2)x^3t^5-a_1^3a_2^3x^4t^6\end{array}}\right)\\
=\ & \frac{\begin{array}{c}bt+(a_1-a_2)\,(c_0-b+c_1x)\,t^2+\left(a(a_1-a_2)^2x^2+a_1a_2\left(c_0-b+(c_1-b)\,x+ax^2\right)\right)t^3\\ -(b-a)\,a_1a_2(a_1-a_2)x^2t^4+a_1^2a_2^2\,(c_0-b+(c_1-a)\,x)\,x^2t^5\end{array}}{\begin{array}{c}1-(a_1-a_2)t-((1+2x)\,a_1a_2+(a_1-a_2)^2x)t^2\\ -(a_1-a_2)\left((a_1-a_2)^2x+a_1a_2(1+3x)\right)xt^3\\ -a_1a_2\left((a_1-a_2)^2+a_1a_2\right)x^2t^4+a_1^2a_2^2(a_1-a_2)x^3t^5-a_1^3a_2^3x^4t^6\end{array}},
\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} J_n(x)P_{k,n}t^n=\frac{\begin{array}{c}bt+2\,(c_0-b+c_1x)\,t^2+\left(4ax^2+k\left(c_0-b+(c_1-b)\,x+ax^2\right)\right)t^3\\ -2\,(b-a)\,kx^2t^4+k^2\,(c_0-b+(c_1-a)\,x)\,x^2t^5\end{array}}{1-2t-(k\,(1+2x)+4x)t^2-2\,(4x+k(1+3x))\,xt^3-k\,(4+k)\,x^2t^4+2k^2x^3t^5-k^3x^4t^6}.$$

This completes the proof. □

- Put $a=b=c_1=c_0=1$ in (4.33) we have the following proposition.

Proposition 4.38. *For $n\in\mathbb{N}$, the new generating function for the combined Tricobasthal polynomials and k-Pell numbers is given by*

$$\sum_{n=0}^{\infty} J_n^{(3)}(x)P_{k,n}=\frac{t+2xt^2+(4+k)\,x^2t^3}{1-2t-(k\,(1+2x)+4x)t^2-2\,(4x+k(1+3x))\,xt^3-k\,(4+k)\,x^2t^4+2k^2x^3t^5-k^3x^4t^6}. \tag{4.34}$$

- Put $k = 1$ in the relationships (4.34),we obtain the following corollary.

Corollary 4.39. *For $n \in \mathbb{N}$, the new generating function for the combined Tricobasthal polynomials and Pell numbers is given by*

$$\sum_{n=0}^{\infty} J_n^{(3)}(x)P_n = \frac{t + 2xt^2 + 5x^2t^3}{1 - 2t - (1 + 6x)t^2 - 2(7x + 1)xt^3 - 5x^2t^4 + 2x^3t^5 - x^4t^6}.$$

Theorem 4.40. *For $n \in \mathbb{N}$, the new generating function for the combined Generalized Tricobasthal polynomials and k-Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} J_n(x)J_{k,n}t^n = \frac{\begin{array}{c} bt + k(c_0 - b + c_1x)t^2 + \left(ak^2x^2 + 2\left(c_0 - b + (c_1 - b)x + ax^2\right)\right)t^3 \\ -2(b - a)kx^2t^4 + 4(c_0 - b + (c_1 - a)x)x^2t^5 \end{array}}{1 - kt - (2(1 + 2x) + k^2x)t^2 - k(k^2x + 2(1 + 3x))xt^3 - 2(k^2 + 2)x^2t^4 + 4kx^3t^5 - 8x^4t^6}. \quad (4.35)$$

Proof. We have

$$J_{k,n} = S_{n-1}(a_1 + [-a_2]), \text{ (see [7]).}$$

We see that

$$\begin{aligned} \sum_{n=0}^{\infty} J_n(x)J_{k,n}t^n &= \sum_{n=0}^{\infty}(aS_n(E) + (b - a)S_{n-1}(E) + (c_0 - b + (c_1 - a)x)S_{n-2}(E))S_{n-1}(a_1 + [-a_2])t^n \\ &= a\sum_{n=0}^{\infty} S_n(E)S_{n-1}(a_1 + [-a_2])t^n + (b - a)\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(a_1 + [-a_2])t^n \\ &\quad + (c_0 - b + (c_1 - a)x)\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(a_1 + [-a_2])t^n, \end{aligned}$$

by the formulas (4.13), (4.16) and (4.17) we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} J_n(x)J_{k,n}t^n \\
= \; & a\left(\frac{t+(a_1-a_2)xt^2+(a_1a_2+(a_1-a_2)^2)x^2t^3}{\begin{array}{c}1-x^2(a_1-a_2)t-x(a_1a_2\left(x^3+2\right)+(a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2+a_1a_2(x^3+3))t^3\\ -a_1a_2x^2\left((a_1-a_2)^2+a_1a_2\right)t^4+a_1^2a_2^2(a_1-a_2)xt^5-a_1^3a_2^3t^6\end{array}}\right) \\
&+(b-a)\left(\frac{t-a_1a_2xt^3-a_1a_2(a_1-a_2)x^2t^4}{\begin{array}{c}1-(a_1-a_2)t-((1+2x)\,a_1a_2+(a_1-a_2)^2x)t^2\\ -(a_1-a_2)\left((a_1-a_2)^2x+a_1a_2(1+3x)\right)xt^3\\ -a_1a_2\left((a_1-a_2)^2+a_1a_2\right)x^2t^4\\ +a_1^2a_2^2(a_1-a_2)x^3t^5-a_1^3a_2^3x^4t^6\end{array}}\right) \\
&+(c_0-b+(c_1-a)\,x)\left(\frac{(a_1-a_2)t^2+a_1a_2t^3+a_1^2a_2^2x^2t^5}{\begin{array}{c}1-(a_1-a_2)t-((1+2x)\,a_1a_2+(a_1-a_2)^2x)t^2\\ -(a_1-a_2)\left((a_1-a_2)^2x+a_1a_2(1+3x)\right)xt^3\\ -a_1a_2\left((a_1-a_2)^2+a_1a_2\right)x^2t^4\\ +a_1^2a_2^2(a_1-a_2)x^3t^5-a_1^3a_2^3x^4t^6\end{array}}\right) \\
= \; & \frac{\begin{array}{c}bt+(a_1-a_2)\,(c_0-b+c_1x)\,t^2+\left(a(a_1-a_2)^2x^2+a_1a_2\left(c_0-b+(c_1-b)\,x+ax^2\right)\right)t^3\\ -(b-a)\,a_1a_2(a_1-a_2)x^2t^4+a_1^2a_2^2\,(c_0-b+(c_1-a)\,x)\,x^2t^5\end{array}}{\begin{array}{c}1-(a_1-a_2)t-((1+2x)\,a_1a_2+(a_1-a_2)^2x)t^2\\ -(a_1-a_2)\left((a_1-a_2)^2x+a_1a_2(1+3x)\right)xt^3\\ -a_1a_2\left((a_1-a_2)^2+a_1a_2\right)x^2t^4+a_1^2a_2^2(a_1-a_2)x^3t^5-a_1^3a_2^3x^4t^6\end{array}},
\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} J_n(x)J_{k,n}t^n = \frac{\begin{array}{c}bt+k\,(c_0-b+c_1x)\,t^2+\left(ak^2x^2+2\left(c_0-b+(c_1-b)\,x+ax^2\right)\right)t^3\\ -2\,(b-a)\,kx^2t^4+4\,(c_0-b+(c_1-a)\,x)\,x^2t^5\end{array}}{1-kt-(2\,(1+2x)+k^2x)t^2-k\,(k^2x+2(1+3x))\,xt^3-2\,(k^2+2)\,x^2t^4+4kx^3t^5-8x^4t^6}.$$

This completes the proof. □

- Put $a = b = c_1 = c_0 = 1$ in (4.35) we have the following proposition.

Proposition 4.41. *For $n \in \mathbb{N}$, the new generating function for the combined Tricobasthal polynomials and k-Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} J_n^{(3)}(x)J_{k,n} = \frac{t+kxt^2+\left(k^2+2\right)x^2t^3}{1-kt-(2\,(1+2x)+k^2x)t^2-k\,(k^2x+2(1+3x))\,xt^3-2\,(k^2+2)\,x^2t^4+4kx^3t^5-8x^4t^6}. \tag{4.36}$$

- Put $k = 1$ in the relationships (4.36),we obtain the following corollary.

Corollary 4.42. *For $n \in \mathbb{N}$, the new generating function for the combined Tricobasthal polynomials and Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} J_n^{(3)}(x)J_n = \frac{t + xt^2 + 3x^2t^3}{1 - t - (2 + 3x)t^2 - (7x + 2)\, xt^3 - 6x^2t^4 + 4x^3t^5 - 8x^4t^6}.$$

Theorem 4.43. *For $n \in \mathbb{N}$, the new generating function for the combined Generalized Tricobasthal polynomials and k-Mersenne numbers is given by*

$$\sum_{n=0}^{\infty} J_n(x)M_{k,n}t^n = \frac{\begin{array}{c} bt + 3k\,(c_0 - b + c_1x)\, t^2 + \left(9ak^2x^2 - 2\left(ax^2 + (c_1 - b)\, x + c_0 - b\right)\right) t^3 + 6k\,(b - a)\, x^2t^4 \\ +4\,(c_0 - b + (c_1 - a)\, x)\, x^2t^5 \end{array}}{\begin{array}{c} 1 - 3kt - (9k^2x - (2x + 1))t^2 - 3kx(9k^2x - 2\,(3x + 1))t^3 \\ +2(9k^2 - 2)x^2t^4 + 12kx^3t^5 + 8x^4t^6 \end{array}}. \quad (4.37)$$

Proof. We have

$$M_{k,n} = S_{n-1}\,(a_1 + [-a_2])\,, \text{ (see [18])}.$$

We see that

$$\begin{aligned} \sum_{n=0}^{\infty} J_n(x)M_{k,n}t^n &= \sum_{n=0}^{\infty}(aS_n(E) + (b - a)\, S_{n-1}(E) + (c_0 - b + (c_1 - a)\, x)\, S_{n-2}(E))S_{n-1}\,(a_1 + [-a_2])\, t^n \\ &= a\sum_{n=0}^{\infty} S_n(E)S_{n-1}\,(a_1 + [-a_2])\, t^n + (b - a)\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}\,(a_1 + [-a_2])\, t^n \\ &\quad + (c_0 - b + (c_1 - a)\, x)\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}\,(a_1 + [-a_2])\, t^n, \end{aligned}$$

by the formulas (4.13), (4.16) and (4.17) we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} J_n(x)M_{k,n}t^n \\
&= a\left(\frac{t+(a_1-a_2)xt^2+(a_1a_2+(a_1-a_2)^2)x^2t^3}{\begin{array}{c}1-x^2(a_1-a_2)t-x(a_1a_2\left(x^3+2\right)+(a_1-a_2)^2)t^2\\ -(a_1-a_2)((a_1-a_2)^2+a_1a_2(x^3+3))t^3\\ -a_1a_2x^2\left((a_1-a_2)^2+a_1a_2\right)t^4+a_1^2a_2^2(a_1-a_2)xt^5-a_1^3a_2^3t^6\end{array}}\right)\\
&+(b-a)\left(\frac{t-a_1a_2xt^3-a_1a_2(a_1-a_2)x^2t^4}{\begin{array}{c}1-(a_1-a_2)t-((1+2x)a_1a_2+(a_1-a_2)^2x)t^2\\ -(a_1-a_2)\left((a_1-a_2)^2x+a_1a_2(1+3x)\right)xt^3\\ -a_1a_2\left((a_1-a_2)^2+a_1a_2\right)x^2t^4\\ +a_1^2a_2^2(a_1-a_2)x^3t^5-a_1^3a_2^3x^4t^6\end{array}}\right)\\
&+(c_0-b+(c_1-a)x)\left(\frac{(a_1-a_2)t^2+a_1a_2t^3+a_1^2a_2^2x^2t^5}{\begin{array}{c}1-(a_1-a_2)t-((1+2x)a_1a_2+(a_1-a_2)^2x)t^2\\ -(a_1-a_2)\left((a_1-a_2)^2x+a_1a_2(1+3x)\right)xt^3\\ -a_1a_2\left((a_1-a_2)^2+a_1a_2\right)x^2t^4\\ +a_1^2a_2^2(a_1-a_2)x^3t^5-a_1^3a_2^3x^4t^6\end{array}}\right)\\
&= \frac{\begin{array}{c}bt+(a_1-a_2)(c_0-b+c_1x)t^2+\left(a(a_1-a_2)^2x^2+a_1a_2\left(c_0-b+(c_1-b)x+ax^2\right)\right)t^3\\ -(b-a)a_1a_2(a_1-a_2)x^2t^4+a_1^2a_2^2(c_0-b+(c_1-a)x)x^2t^5\end{array}}{\begin{array}{c}1-(a_1-a_2)t-((1+2x)a_1a_2+(a_1-a_2)^2x)t^2\\ -(a_1-a_2)\left((a_1-a_2)^2x+a_1a_2(1+3x)\right)xt^3\\ -a_1a_2\left((a_1-a_2)^2+a_1a_2\right)x^2t^4+a_1^2a_2^2(a_1-a_2)x^3t^5-a_1^3a_2^3x^4t^6\end{array}},
\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} J_n(x)M_{k,n}t^n = \frac{\begin{array}{c}bt+3k(c_0-b+c_1x)t^2+\left(9ak^2x^2-2\left(ax^2+(c_1-b)x+c_0-b\right)\right)t^3+6k(b-a)x^2t^4\\ +4(c_0-b+(c_1-a)x)x^2t^5\end{array}}{\begin{array}{c}1-3kt-(9k^2x-(2x+1))t^2-3kx(9k^2x-2(3x+1))t^3\\ +2(9k^2-2)x^2t^4+12kx^3t^5+8x^4t^6\end{array}}.$$

This completes the proof. □

- Put $a=b=c_1=c_0=1$ in (4.37) we have the following proposition.

Proposition 4.44. *For $n\in\mathbb{N}$, the new generating function for the combined Tricobasthal polynomials and k-Mersenne numbers is given by*

$$\sum_{n=0}^{\infty} J_n^{(3)}(x)M_{k,n} = \frac{t+3kxt^2+\left(9k^2-2\right)x^2t^3}{\begin{array}{c}1-3kt-(9k^2x-(2x+1))t^2-3kx(9k^2x-2(3x+1))t^3\\ +2(9k^2-2)x^2t^4+12kx^3t^5+8x^4t^6\end{array}}. \tag{4.38}$$

- Put $k = 1$ in the relationships (4.38),we obtain the following corollary.

Corollary 4.45. *For $n \in \mathbb{N}$, the new generating function for the combined Tricobasthal polynomials and Mersenne numbers is given by*

$$\sum_{n=0}^{\infty} J_n^{(3)}(x) M_n = \frac{t + 3xt^2 + 7x^2t^3}{1 - 3t - (7x-1)t^2 - 3x(3x-2)t^3 + 14x^2t^4 + 12x^3t^5 + 8x^4t^6}.$$

5. Conclusion

In this paper, by making use of theorem (4.1), we have derived some new generating functions for generalized Tribonacci and Tricobsthal polynomials. It would be interesting to apply the methods shown in the paper to families of other special polynomials.

Acknowledgments

Authors are grateful to the Editor-In-Chief of the Journal and the anonymous reviewers for theirconstructive comments which improved the quality and the presentation of the paper.

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