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Interpolation Functions for New Classes Special Numbers and Polynomials via Applications of p-adic Integrals and Derivative Operator

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Abstract

The main purpose of this paper is to not only define Apostol type new classes of numbers and polynomials, but also construct generating function for two new classes of special combinatorial numbers and polynomials by applications of p-adic integrals including the Volkenborn integral and the fermionic integral. By using these generating functions, we introduce not only fundamental properties of these combinatorial numbers and polynomials, but also new identities and formulas. In general, identities and formulas obtained in this paper include the newly introduced combinatorial numbers and polynomials, Bernoulli numbers and polynomials, Euler numbers and polynomials, Apostol-Bernoulli numbers and polynomials, Apostol-Euler numbers and polynomials, Stirling numbers of the second kind, Daehee numbers, Changhee numbers, the generalized Eulerian type numbers, Eulerian polynomials, Fubini numbers, Dobinski numbers. Moreover, by applying derivative operator to the generating functions for two new classes of special combinatorial numbers, we construct interpolation functions for these numbers. We also introduce another zeta-type function which interpolates a special case of one of the newly introduced combinatorial numbers at negative integers. Very interesting results are obtained from these interpolation functions, especially a new combinatorial numbers derived. So, 4 open problems are raised involving these new numbers. Finally, we give conclusions for the results of this paper with some comments and observations.

Keywords: Generating functions, special numbers and polynomials, Bernoulli-type numbers and polynomials, Euler-type numbers and polynomials, Stirling numbers of the second kind, Daehee numbers, Fubini numbers, p-adic integral

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1. Introduction

Special numbers and polynomials have been among the most used and studied subjects of mathematics both in science and social sciences in recent years. In particular, it is easy to observe that the generating functions of these special numbers and polynomials are equally frequently used in the related topics. With the help of these functions, most of the fundamental properties of special numbers and polynomials, especially their derivative formulas, recurrent relations, can be given, including the Raabe type multiplication formula. As a result, in this study, special combinatorial numbers and polynomials consisting of two new classes and the construction of their generating functions are prepared based on the first main motivation. Not only with the aid of the Volkenborn integral and the fermionic integral, but also with the help of generating functions with their functional equations, when these two new classes of

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special numbers and polynomials are examined with detailed criticism and some special values of them are examined, it has been shown that they are related to the following very important and well-known special numbers and polynomials: combinatorial numbers and polynomials, Bernoulli numbers and polynomials, Euler numbers and polynomials, Apostol-Bernoulli numbers and polynomials, Apostol-Euler numbers and polynomials, Stirling numbers of the second kind, Daehee numbers, Changhee numbers, Eulerian polynomials, Fubini numbers, Dobinski numbers, and others.

It is well known that the study of the behavior of generating functions under integral transforms and derivative operators relies on very old work. Among these behaviors, the areas where many zeta-type functions and Dirichlet-type series are constructed are one of the most popular areas of analytic number theory. Because it is known that these zeta type functions and Dirichlet type series are used as many kinds of problems involving models for solving real world problems not only in mathematics but also in physics and engineering (*cf.* [1]–[81]).

The second motivation of this study is to construct interpolation functions of two new families of newly defined combinatorial numbers with the help of the derivative operator of interpolation functions. In addition, some basic properties of interpolation functions are examined.

Of course, some basic standard notations, formulas and definitions, which we will give in the next step, are needed to achieve the above mentioned results.

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integer numbers and the set of complex numbers, respectively, and also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We also tacitly suppose that for $z \in \mathbb{C}$, $\log z$ denotes the principal branch of the many-valued function $\mathrm{Im}(\log z)$ with the imaginary part $\log z$ constrained by

$$-\pi < \mathrm{Im}(\log z) \leq \pi.$$

Here $\log e = 1$ will be considered throughout this paper.

Therefore, for all constraints and properties on the Apostol-Bernoulli numbers and polynomials, and Apostol-Euler numbers and polynomials which are given below, exponential functions, and complex valued functions, it is recommended to first look at the reference [76]-[78], and also see the other the reference list of this paper.

$$0^n = \begin{cases} 1, & n = 0 \\ 0, & n \in \mathbb{N} \end{cases}$$

with

$$\binom{\alpha}{n} = \frac{(\alpha)_n}{n!},$$

where

$$(\alpha)_n = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)$$

with $(\alpha)_0 = 1$ and $n \in \mathbb{N}_0$.

The Apostol-Bernoulli polynomials, $\mathcal{B}_n(x;\lambda)$, are defined by the following generating function:

$$\frac{t}{\lambda e^t - 1} e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda) \frac{t^n}{n!}, \tag{1.1}$$

where λ is an arbitrary (real or complex) parameter and $|t| < 2\pi$ when $\lambda = 1$ and $|t| < \left|\log \lambda\right|$ when $\lambda \neq 1$ (*cf.* [3], [43], [74], [78]). Moreover, setting $x = 0$ in (1.1), we have the Apostol-Bernoulli numbers:

$$\mathcal{B}_n(0;\lambda) = \mathcal{B}_n(\lambda)$$

(*cf.* [3], [43], [74], [78]).

When $\lambda = 1$ in (1.1), we have

$$\mathcal{B}_n(1) = B_n$$

where B_n denotes the Bernoulli numbers (*cf.* [1]-[78]; and references therein).

The Apostol-Euler polynomials, $\mathcal{E}_n(x;\lambda)$, are defined by the following generating function:

$$\frac{2}{\lambda e^t + 1} e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n(x;\lambda) \frac{t^n}{n!}, \tag{1.2}$$

where λ is an arbitrary (real or complex) parameter and $|t| < \pi$ when $\lambda = 1$ and $|t| < \left|\log(-\lambda)\right|$ when $\lambda \neq 1$ (*cf.* [43], [74], [78]). Moreover, setting $x = 0$ in (1.2), we have the Apostol-Euler numbers:

$$\mathcal{E}_n(0;\lambda) = \mathcal{E}_n(\lambda)$$

(*cf.* [43], [74], [78]).

When $\lambda = 1$ in (1.2), we have

$$\mathcal{E}_n(1) = E_n$$

where E_n denotes the Euler numbers (*cf.* [1]-[78]; and references therein).

We also need the following well-known relation between the Apostol-Euler numbers, $\mathcal{E}_n(\lambda)$ and the Apostol-Bernoulli numbers, $\mathcal{B}_n(\lambda)$:

$$\mathcal{E}_n(\lambda) = -\frac{2}{n+1}\mathcal{B}_{n+1}(-\lambda) \tag{1.3}$$

(*cf.* [71, Eq. (1.28)], [78]).

Let $a, b, c \in \mathbb{R}^+$ $(a \neq b)$, $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $u \in \mathbb{C}\backslash\{\lambda\}$. The generalized Eulerian type polynomials, $\mathcal{H}_n(x;u;a,b,c;\lambda)$, are defined by the following generating function:

$$F_\lambda(t,x;u,a,b,c;\lambda) = \frac{(a^t-u)\,c^{xt}}{\lambda b^t-u} = \sum_{n=0}^{\infty}\mathcal{H}_n(x;u;a,b,c;\lambda)\frac{t^n}{n!}, \tag{1.4}$$

where $|t| < \frac{2\pi}{|\log b|}$ when $\lambda = u$; $\left|t\log b + \log\left(\frac{\lambda}{u}\right)\right| < 2\pi$ when $\lambda \neq u$ (*cf.* [56, Eq. (13)]; and also see [57], [67], [38]).

When $x = 0$ in (1.5), we have the generalized Eulerian type numbers:

$$\mathcal{H}_n(0;u;a,b,c;\lambda) = \mathcal{H}_n(u;a,b,c;\lambda),$$

which is defined by means of the following generating function:

$$\frac{a^t-u}{\lambda b^t-u} = \sum_{n=0}^{\infty}\mathcal{H}_n(u;a,b,c;\lambda)\frac{t^n}{n!}, \tag{1.5}$$

(*cf.* [56]; and also see [57], [67], [38]).

In the special case when $\lambda = a = 1$ and $b = c = e$, the generalized Eulerian type polynomials are reduced to the Eulerian polynomials (or Frobenius Euler polynomials) which are defined by means of the following generating function:

$$\frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty}H_n(x;u)\frac{t^n}{n!} \tag{1.6}$$

(*cf.* [8], [9], [10], [11], [32], [38], [55]). Moreover, setting $x = 0$ in (1.6), we have the Eulerian numbers:

$$H_n(0;u) = H_n(u)$$

(*cf.* [8], [9], [10], [11], [32], [38], [55]).

Substituting $u = -1$ and $x = 0$ into (1.6), we have

$$H_n(0;-1) = H_n(-1) = E_n$$

(*cf.* [8], [9], [10], [11], [32], [38], [55]).

The Stirling numbers of the second kind, $S_2(n,k)$, are defined by the following generating function:

$$\frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty}S_2(n,k)\frac{t^n}{n!} \tag{1.7}$$

(*cf.* [1]-[78]; and references therein).

By using (1.7), an explicit formula for the numbers $S_2(n,k)$ is given by:

$$S_2(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n \tag{1.8}$$

with $k > n$, $S_2(n,k) = 0$ (*cf.* [1]-[78]).

The Stirling numbers of the second kind are also given by the following generating function including falling factorial:

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k \tag{1.9}$$

(*cf.* [1]-[78]; and references therein).

The Fubini numbers, $w_g(n)$, are defined by means of the following generating function:

$$\frac{1}{2-e^t} = \sum_{n=0}^{\infty} w_g(n)\frac{t^n}{n!}, \tag{1.10}$$

(*cf.* [16], [18], [19], [20], [21]).

Combining (1.7) with (1.10), the following relation is derived:

$$w_g(n) = \sum_{j=0}^{n} j!S_2(n,j), \tag{1.11}$$

(*cf.* [16], [18], [20], [21]).

The Dobinski numbers, $D(n)$, are defined by means of the following generating function:

$$F_d(t) = \frac{1}{e^{e^t-1}} = \sum_{n=0}^{\infty} D(n)\frac{t^n}{n!}, \tag{1.12}$$

(*cf.* [19, Eq. (3.15)]).

The Dobinski numbers are related to the exponential numbers (or the Bell numbers) and other combinatorial numbers. The exponential numbers, which not only occur often in probability, but also are associated with that of the Poisson-Charlier polynomial, are defined by

$$F_e(t) = e^{e^t-1} = \sum_{n=0}^{\infty} B(n)\frac{t^n}{n!}$$

(*cf.* [16], [31], [52], [58]). Using the above generating functions, one has the followingw well-known results:

$$B(n) = \sum_{j=0}^{n} S_2(n,j)$$

(*cf.* [16], [31], [52], [58]). Since

$$F_e(t)F_d(t) = 1,$$

for $n \in \mathbb{N}$, we have

$$\sum_{j=0}^{n}\binom{n}{j}B(n-j)D(j) = 0.$$

In [65] and [66], we defined the λ-Apostol-Daehee numbers of higher order. Generating function for the λ-Apostol-Daehee numbers, $\mathfrak{D}_n(\lambda)$ is given by:

$$\frac{\log\lambda + \log(1+\lambda t)}{\lambda(1+\lambda t)-1} = \sum_{n=0}^{\infty}\mathfrak{D}_n(\lambda)\frac{t^n}{n!} \tag{1.13}$$

(*cf.* [65], [66]). Recently, the λ-Apostol-Daehee numbers have been also studied by many authors such as [15], [35], [37], [65], [66], [72].

By using (1.13), we have

$$
\begin{aligned}
\mathfrak{D}_0(\lambda) &= \frac{\log\lambda}{\lambda-1},\\
\mathfrak{D}_1(\lambda) &= -\frac{\lambda^2\log\lambda}{(\lambda-1)^2}+\frac{\lambda}{\lambda-1},\\
\mathfrak{D}_2(\lambda) &= \frac{2\lambda^4\log\lambda}{(\lambda-1)^3}+\frac{\lambda^2(1-3\lambda)}{(\lambda-1)^2},\\
\mathfrak{D}_3(\lambda) &= -\frac{6\lambda^6\log\lambda}{(\lambda-1)^4}+\frac{\lambda^3\left(11\lambda^2-7\lambda+2\right)}{(\lambda-1)^3},
\end{aligned}
$$

and so on (*cf.* [37], [65], [66], [72]).

For $n \in \mathbb{N}$, combining the equation (1.13) with the following well-known series

$$\log(1+t)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^n}{n},$$

$|t|<1$, we have

$$
\begin{aligned}
&\mathfrak{D}_0(\lambda)+\sum_{n=1}^{\infty}\mathfrak{D}_n(\lambda)\frac{t^n}{n!}\\
=\ &\frac{\log\lambda}{\lambda-1}+\frac{\log\lambda}{\lambda-1}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\lambda^2}{\lambda-1}t\right)^n+\frac{1}{\lambda-1}\sum_{n=1}^{\infty}(-1)^{n+1}\lambda^n\frac{t^n}{n}\\
&+\frac{1}{\lambda-1}\sum_{n=1}^{\infty}(-1)^{n+1}\lambda^n\frac{t^n}{n}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\lambda^2}{\lambda-1}t\right)^n
\end{aligned}
$$

assuming that $|\lambda t|<1$. By applying the Cauchy multiplication rule to the right-hand side of the above equation for two series product, after some elementay calculation, we arrive at the following explicit formula for the numbers $\mathfrak{D}_n(\lambda)$:

For $n=0$, we have

$$\mathfrak{D}_0(\lambda)=\frac{\log\lambda}{\lambda-1}$$

and for $n\geq 1$, we have

$$\mathfrak{D}_n(\lambda)=(-1)^{n+1}n!\left(\frac{\lambda^n}{n(\lambda-1)}-\frac{\lambda^{2n}\log\lambda}{(\lambda-1)^{n+1}}+\sum_{k=1}^{n}\frac{\lambda^{n+k}}{(n-k)(\lambda-1)^{k+1}}\right).$$

For another explicit formulas for the numbers $\mathfrak{D}_n(\lambda)$, see [37], [65], [66], [72]). Among others, in [37], Kucukoglu and Simsek gave the following explicit formula for the numbers $\mathfrak{D}_n(\lambda)$:

$$\mathfrak{D}_n(\lambda)=n!(-1)^n\left(\frac{\lambda^2}{\lambda-1}\right)^n\left(\frac{\log\lambda}{\lambda-1}-\frac{1}{\lambda}\sum_{k=0}^{n-1}\frac{1}{k+1}\left(\frac{\lambda-1}{\lambda}\right)^k\right) \tag{1.14}$$

(*cf.* [37]).

When $\lambda=1$ in (1.13), we have

$$\mathfrak{D}_n(1)=D_n$$

where D_n denotes the Daehee numbers (*cf.* [17], [23]; and see also [15], [37], [65], [66], [72]).

By using (1.13), one has a computation formula for the Daehee numbers D_n as follows:

$$D_n = (-1)^n \frac{n!}{n+1} \tag{1.15}$$

(*cf.* [23], [49]; and see also [71]).

The Changhee numbers, Ch_n, are defined by the following generating function:

$$\frac{2}{2+t} = \sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!} \tag{1.16}$$

(*cf.* [25], [33]).

By using (1.16), one has a computation formula for the Changhee numbers Ch_n as follows:

$$Ch_n = (-1)^n \frac{n!}{2^n} \tag{1.17}$$

(*cf.* [25], see also [33], [71]).

In [56, Eq. (37)], we defined the numbers $\mathrm{Y}_n(u;a)$ and the polynomials $\mathrm{Y}_n(x,u;a)$ by means of the following generating functions, respectively:

$$G_Y(t,u,a) = \frac{1}{a^t - u} = \sum_{n=0}^{\infty} \mathrm{Y}_n(u;a) \frac{t^n}{n!} \tag{1.18}$$

and

$$H_Y(x,t,u,a) = G_Y(t,u,a)a^{xt} = \sum_{n=0}^{\infty} \mathrm{Y}_n(x,u;a) \frac{t^n}{n!} \tag{1.19}$$

where $a \geq 1$; $u \neq 0$, $u \neq 1$ and $\left| t \log a + \log\left(\frac{1}{u}\right) \right| < 2\pi$.

Putting $a = 1$ in (1.18), we have

$$\mathrm{Y}_0(u;1) = \frac{1}{1-u}$$

(*cf.* [56, p. 22]). Substituting $a = 1$ into (1.19), we obtain

$$\mathrm{Y}_0(x,u;1) = \frac{1}{1-u}.$$

These numbers and polynomials have been also studied in the following references [1], [56], [67].

When we substitute $x = 0$ into (1.19), we have

$$\mathrm{Y}_n(0,u;a) = \mathrm{Y}_n(u;a).$$

It is time to give brief summary of this paper as follows:

In Section 2, we define Apostol type new classes of numbers and polynomials. We give some preperties of these numbers.

In Section 3, we construct two new functions by applications of p-adic integrals including Volkenborn and fermionic integrals which yields generating function for two new families of special combinatorial numbers and polynomials.

In Section 4 and Section 5, we introduce the aforementioned two new families of special combinatorial numbers and polynomials with their generating functions. Moreover, we give some fundamental properties of these combinatorial numbers and polynomials, and derive identities and formulas for these combinatorial numbers and polynomials. In Section 4, we also introduce a presumably new zeta-type function which interpolates a special case of one of the newly introduced combinatorial numbers at negative integers. Interpolation functions of these combinatorial numbers are defined with the aid of the derivative operator. Some properties of these functions are studied.

In Section 6, new numbers including combinatorial sums are defined, and some open problems are raised involving these new numbers.

In Section 7, we conclude the results of this paper with some comments and observations.

2. Apostol type new classes of numbers and polynomials

Here, we define the Apostol type of the numbers $\mathrm{Y}_n(u;a)$ and the polynomials $\mathrm{Y}_n(x,u;a)$ by means of the following generating functions. These numbers and polynomials denote by $\mathcal{Y}_n(u;a,\lambda)$ and $\mathcal{Y}_n(x,u;a,\lambda)$, respectively:

$$g_Y(t,u,a,\lambda) = \frac{1}{\lambda a^t - u} = \sum_{n=0}^{\infty} \mathcal{Y}_n(u;a,\lambda)\frac{t^n}{n!} \tag{2.1}$$

and

$$h_Y(x,t,u,a,\lambda) = g_Y(t,u,a,\lambda)a^{xt} = \sum_{n=0}^{\infty} \mathcal{Y}_n(x,u;a,\lambda)\frac{t^n}{n!}, \tag{2.2}$$

where $a \geq 1$; $u \neq 0, 1$.

Let us briefly give some well-known numbers to which the numbers $\mathcal{Y}_n(u;a,\lambda)$ are related.

Putting $a = 1$ in (2.1), we obtain

$$\mathcal{Y}_0(u;a,\lambda) = \frac{1}{\lambda - u}.$$

We also note that

$$\mathcal{Y}_n(u;a,\lambda) = \frac{1}{\lambda}\mathrm{Y}_n\left(\frac{u}{\lambda};a\right). \tag{2.3}$$

When $a = e$, for $n \in \mathbb{N}$, we also obtain

$$\mathcal{Y}_{n-1}(u;e,\lambda) = \frac{1}{nu}\mathcal{B}_n\left(\frac{u}{\lambda}\right). \tag{2.4}$$

Combining (2.4) with (1.3), we get

$$\mathcal{Y}_n(u;e,\lambda) = -\frac{1}{2u}\mathcal{E}_n\left(-\frac{u}{\lambda}\right),$$

$$\mathcal{Y}_n(u;e,\lambda) = \frac{1}{\lambda - u}H_n\left(\frac{u}{\lambda}\right).$$

Substituting $u = -1$ and $\lambda = 1$ into above equation, we have

$$\mathcal{Y}_n(-1;e,1) = \frac{1}{2}E_n.$$

Let also us give some formulas for the numbers $\mathcal{Y}_n(u;a,\lambda)$ and polynomials $\mathcal{Y}_n(x,u;a,\lambda)$.

Using (2.1), we get

$$\lambda\sum_{n=0}^{\infty}\mathcal{Y}_n(u;a,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}(\log a)^n\frac{t^n}{n!} - u\sum_{n=0}^{\infty}\mathcal{Y}_n(u;a,\lambda)\frac{t^n}{n!} = 1.$$

Using the Cauchy rule for the product of two converging series to the left hand side of the previous equation, the following equation is found after some algebraic calculations:

$$\lambda\sum_{n=0}^{\infty}\sum_{v=0}^{n}\binom{n}{v}\mathcal{Y}_v(u;a,\lambda)(\log a)^{n-v}\frac{t^n}{n!} - u\sum_{n=0}^{\infty}\mathcal{Y}_n(u;a,\lambda)\frac{t^n}{n!} = 1.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.1. *Let*

$$\mathcal{Y}_0(u;a,\lambda) = \frac{1}{\lambda - u}.$$

For $n \geq 1$, *we have*

$$\mathcal{Y}_n(u;a,\lambda) = \frac{\lambda}{u}\sum_{v=0}^{n}\binom{n}{v}\mathcal{Y}_v(u;a,\lambda)(\log a)^{n-v}. \tag{2.5}$$

With the help of the recurrence relation given by Eq. (2.5), for $n = 1, 2, 3, \ldots$, some values of the numbers $\mathcal{Y}_n(u; a, \lambda)$ are found as follows:

$$\begin{aligned}
\mathcal{Y}_1(u; a, \lambda) &= -\frac{\lambda \log a}{(\lambda - u)^2}, \\
\mathcal{Y}_2(u; a, \lambda) &= \frac{\lambda (\log a)^2}{(\lambda - u)^3}(u + 2 - \lambda),
\end{aligned}$$

and so on.

Combining (2.1) with (2.2), we get

$$\sum_{n=0}^{\infty} \mathcal{Y}_n(x, u; a, \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{Y}_n(u; a, \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} (x \log a)^n \frac{t^n}{n!}.$$

Using the Cauchy rule for the product of two converging series to the right hand side of the previous equation, the following equation is found after some algebraic calculations:

$$\sum_{n=0}^{\infty} \mathcal{Y}_n(x, u; a, \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{v=0}^{n} \binom{n}{v} \mathcal{Y}_v(u; a, \lambda) (x \log a)^{n-v} \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.2. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{Y}_n(x, u; a, \lambda) = \sum_{v=0}^{n} \binom{n}{v} \mathcal{Y}_v(u; a, \lambda) (x \log a)^{n-v}. \tag{2.6}$$

Using Eq. (2.5) and Eq. (2.6), for $n = 1, 2, 3, \ldots$, some values of the polynomials $\mathcal{Y}_n(x, u; a, \lambda)$ are found as follows:

$$\begin{aligned}
\mathcal{Y}_0(x, u; a, \lambda) &= \frac{1}{\lambda - u}, \\
\mathcal{Y}_1(x, u; a, \lambda) &= \frac{\log a}{\lambda - u} x - \frac{\lambda \log a}{(\lambda - u)^2}, \\
\mathcal{Y}_2(x, u; a, \lambda) &= \frac{(\log a)^2}{\lambda - u} x^2 - \frac{2\lambda (\log a)^2}{(\lambda - u)^2} x + \frac{\lambda (\log a)^2}{(\lambda - u)^3}(u + 2 - \lambda),
\end{aligned}$$

and so on.

3. Two new families of special combinatorial numbers and polynomials derived from *p*-adic integrals

In this section, we introduce two new families special combinatorial numbers and polynomials derived from p-adic integrals which are briefly given as follows:

Let $\mathbb{Z}_p$ denote the set of p-adic integers. Let $\mathbb{K}$ be a field with a complete valuation. Let $f \in C^1\left(\mathbb{Z}_p \to \mathbb{K}\right)$, set of continuous derivative functions.

The Volkenborn integral (bosonic p-adic integral) of the function f on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N - 1} f(x), \tag{3.1}$$

where

$$\mu_1(x) = \frac{1}{p^N}$$

(*cf.* [26], [27], [53], [71]; and the references cited therein).

By using (3.1), the Bernoulli numbers B_n is also given by the p-adic integral of the function x^n as follows:

$$\int_{\mathbb{Z}_p} x^n d\mu_1(x) = B_n \tag{3.2}$$

(*cf.* [26], [27], [53], [71]; and the references cited therein).

With the help of (3.1), Kim et al. [23] gave the Daehee numbers D_n by the following p-adic integral of the function $(x)_n$ as follows:

$$\int_{\mathbb{Z}_p} (x)_n \, d\mu_1(x) = \frac{(-1)^n n!}{n+1} = D_n. \tag{3.3}$$

Recently, by using (3.1) and (3.3), many properties and applications of the function $(x)_n$ were given by Simsek [71].

The fermionic p-adic integral of function f on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(x) \, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x), \tag{3.4}$$

where

$$\mu_{-1}(x) = (-1)^x$$

(*cf.* [28], [29], [30], [32], [71]; and the references therein).

Using (3.4), the Euler numbers E_n is also given by the fermionic p-adic integral of the function x^n as follows:

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n \tag{3.5}$$

(*cf.* [28], [29], [30], [32], [71]; and the references therein).

With the help of (3.4), Kim et al. [25] gave the Changhee numbers Ch_n by the following fermionic p-adic integral of the function $(x)_n$ as follows:

$$\int_{\mathbb{Z}_p} (x)_n \, d\mu_{-1}(x) = \frac{(-1)^n n!}{2^n} = Ch_n. \tag{3.6}$$

Recently, by using (3.4) and (3.6), many properties and applications of the function $(x)_n$ were also given by Simsek [71].

3.1. Construction of generating function for combinatorial numbers denoted by $y_{8,n}(\lambda; a)$

Here, by applying the Volkenborn integral to the following function

$$f(t, x; \lambda, a) = \left(\lambda + a^t\right)^x, \qquad \left(\lambda, x, t \in \mathbb{Z}_p\right) \tag{3.7}$$

we construct the following function which is used to define generating function for the numbers $y_{8,n}(\lambda; a)$:

$$\int_{\mathbb{Z}_p} \left(\lambda + a^t\right)^x d\mu_1(x) = \frac{\log(\lambda + a^t)}{a^t + \lambda - 1}. \tag{3.8}$$

By applying Mahler's theorem, (proved by K. Mahler (1958) [53]) to (3.8), we have many interesting results since continuous p-adic-valued function f on $\mathbb{Z}_p$ in can be written as in terms of polynomials. By using Binomial theorem (3.8), we get

$$\sum_{m=0}^{\infty} \left(\frac{a^t}{\lambda}\right)^m \int_{\mathbb{Z}_p} \binom{x}{m} \lambda^x d\mu_1(x) = \frac{\log(\lambda + a^t)}{a^t + \lambda - 1}.$$

Combining the above equation with (3.3) in the case when $\lambda = 1$, we have the following generating functions:

$$\sum_{m=0}^{\infty} D_m \frac{a^{tm}}{m!} = \frac{\log\left(1+a^t\right)}{a^t}.$$

Remark 3.1. The above series can be also studied in different manners with the aid of following well-known generating functions for the harmonic numbers H_m:

$$\sum_{m=1}^{\infty} H_m t^m = \frac{\log\left(1-t\right)}{t-1} \tag{3.9}$$

(*cf.* [16, 51, 78, 79]).

3.2. Construction of generating function for combinatorial numbers denoted by $y_{9,n}(\lambda; a)$

Here, by applying the fermionic integral to (3.7) on $\mathbb{Z}_p$, we construct the following function which is used to define generating function for the numbers $y_{9,n}(\lambda; a)$:

$$\int_{\mathbb{Z}_p} \left(\lambda + a^t\right)^x d\mu_{-1}(x) = \frac{2}{a^t + \lambda}. \tag{3.10}$$

By using Binomial theorem (3.10), we get

$$\sum_{m=0}^{\infty} \left(\frac{a^t}{\lambda}\right)^m \int_{\mathbb{Z}_p} \binom{x}{m} \lambda^x d\mu_{-1}(x) = \frac{2}{a^t + \lambda}.$$

Combining the above equation with (3.6) in the case when $\lambda = 1$, we have the following generating functions:

$$\sum_{m=0}^{\infty} Ch_m \frac{a^{tm}}{m!} = \frac{2}{a^t + 1}.$$

4. Generating functions for new classes of combinatorial numbers $y_{8,n}(\lambda; a)$ and polynomials $y_{8,n}(x, \lambda; a)$ derived from the p-adic integral (3.8)

In this section, by aid of equation (3.8), we define the combinatorial numbers $y_{8,n}(\lambda; a)$ and the combinatorial polynomials $y_{8,n}(x, \lambda; a)$ respectively as follows:

$$K_1(t; a, \lambda) := \frac{\log\left(\lambda + a^t\right)}{a^t + \lambda - 1} = \sum_{n=0}^{\infty} y_{8,n}(\lambda; a) \frac{t^n}{n!}, \tag{4.1}$$

and

$$\begin{aligned} K_2(t, x; a, \lambda) &:= a^{xt} K_1(t; a, \lambda) \\ &= \sum_{n=0}^{\infty} y_{8,n}(x, \lambda; a) \frac{t^n}{n!}. \end{aligned} \tag{4.2}$$

For functions $K_1(t; a, \lambda)$ and $K_2(t, x; a, \lambda)$, when $\lambda \neq 0$, we asume that

$$\left|\frac{a^t}{\lambda}\right| < 1$$

and

$$\left|t \log a + \log\left(\frac{1}{\lambda - 1}\right)\right| < \pi.$$

When $\lambda = 0$, using (4.1),we have

$$\frac{t \log a}{e^{t \log a} - 1} = \sum_{n=0}^{\infty} y_{8,n}(0; a) \frac{t^n}{n!}.$$

Combining the above function with (1.1), we get

$$y_{8,n}(0; a) = (\log a)^n B_n.$$

Here, we note that all the constraints given in [76]-[78] also apply to the generating functions given in equation (4.1) and equation (4.2).

Before giving the results of this section and the following sections, we would like to give a brief explanation for the relevant numbers given in the index of the relevant numbers. That is, we now give brief information about notations and index for the above special combinatorial numbers and polynomials (and also for numbers and polynomials in the next section) is given as follows:

The author has recently defined many different Peters and Boole type combinatorial numbers and polynomials. He gave some notations for these numbers and polynomials. For instance, in order to distinguish them from each other, these polynomials are labeled by the following symbols:

$y_{j,n}(x; \lambda, q)$, $j = 1, 2, \ldots, 9$, and also $Y_n(x; \lambda)$. Therefore, for the numbers $y_{9,n}(\lambda; a)$ the number 9 is only used for index representation for these polynomials (*cf.* [54]-[73]).

Here, by using these generating functions, we give not only fundamental properties of these polynomials and numbers, but also new identities and formulas including these numbers and polynomials, the Daehee numbers, the Stirling numbers of the second kind, the Apostol-Bernoulli numbers, the Apostol-Euler numbers, the λ-Apostol-Daehee numbers and the numbers $\mathrm{Y}_n(\lambda; a)$. In addition, we introduce a presumably new zeta-type function which interpolates the numbers $y_{8,m}(1; a)$ at negative integers.

By using (4.1) and (4.2), for $x = 0$, we have

$$y_{8,n}(0, \lambda; a) = y_{8,n}(\lambda; a).$$

With the help of (4.1) and (4.2), we also get

$$\sum_{n=0}^{\infty} y_{8,n}(x, \lambda; a) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (x \log a)^n \frac{t^n}{n!} \sum_{n=0}^{\infty} y_{8,n}(\lambda; a) \frac{t^n}{n!}.$$

Using the Cauchy product in the above equation yields

$$\sum_{n=0}^{\infty} y_{8,n}(x, \lambda; a) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} (x \log a)^{n-j} y_{8,j}(\lambda; a) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, a computation formula for the polynomials $y_{8,n}(x, \lambda; a)$ is obtained as in the following theorem:

Theorem 4.1. *Let $n \in \mathbb{N}_0$. Then we have*

$$y_{8,n}(x, \lambda; a) = \sum_{j=0}^{n} \binom{n}{j} (x \log a)^{n-j} y_{8,j}(\lambda; a). \tag{4.3}$$

Setting $a = 1$ in (4.1), we have

$$K_1(t; 1, \lambda) = \frac{\log(\lambda + 1)}{\lambda} = \sum_{n=0}^{\infty} y_{8,n}(\lambda; 1) \frac{t^n}{n!}.$$

From the previous equation, the following relations are derived:

$$y_{8,0}(\lambda; 1) = \frac{\log(\lambda + 1)}{\lambda}, \tag{4.4}$$

and for $n \geq 1$,

$$y_{8,n}(\lambda; 1) = 0.$$

Moreover, combining (4.4) with (1.13), we also obtain

$$y_{8,0}(\lambda; 1) = \frac{\log(\lambda + 1)}{\lambda} = \sum_{n=0}^{\infty} D_n \frac{\lambda^n}{n!}.$$

Thus, we see that $y_{8,0}(\lambda; 1)$ gives us the generating function for the Daehee numbers.

By combining (1.18) with (4.1), we get the following functional equation:

$$G_Y(t, 1-\lambda, a) \log\left(\lambda + a^t\right) = \sum_{n=0}^{\infty} y_{8,n}(\lambda; a) \frac{t^n}{n!}. \tag{4.5}$$

By using (4.5), we get

$$\sum_{n=0}^{\infty} \mathrm{Y}_n(1-\lambda; a) \frac{t^n}{n!} \left(\log(\lambda) + \log\left(1 + \frac{a^t}{\lambda}\right)\right) = \sum_{n=0}^{\infty} y_{8,n}(\lambda; a) \frac{t^n}{n!}.$$

Assuming that $|\frac{a^t}{\lambda}| < 1$, we have

$$\begin{aligned} \sum_{m=0}^{\infty} y_{8,m}(\lambda; a) \frac{t^m}{m!} &= \log\lambda \sum_{m=0}^{\infty} \mathrm{Y}_m(1-\lambda; a) \frac{t^m}{m!} + \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \mathrm{Y}_{m-k}(1-\lambda; a) \\ &\times \sum_{n=0}^{k} \sum_{j=0}^{n+1} \binom{n+1}{j} (-1)^n \frac{j! S_2(k,j) (\log a)^k}{(n+1)\lambda^{n+1}} \frac{t^m}{m!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 4.2. *Let $\lambda \neq 0, 1$ and $m \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} y_{8,m}(\lambda; a) &= \mathrm{Y}_m(1-\lambda; a) \log(\lambda) + \sum_{k=0}^{m} \binom{m}{k} \mathrm{Y}_{m-k}(1-\lambda; a) \\ &\times \sum_{n=0}^{k} \sum_{j=0}^{n+1} \binom{n+1}{j} (-1)^n \frac{j! S_2(k,j) (\log a)^k}{(n+1)\lambda^{n+1}}. \end{aligned} \tag{4.6}$$

Combining (4.6) with (1.15), after some elementary calculations, we arrive at the following theorem:

Corollary 4.3. *Let $\lambda \neq 0, 1$ and $m \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} & y_{8,m}(\lambda; a) - \mathrm{Y}_m(1-\lambda; a) \log(\lambda) \\ = & \sum_{k=0}^{m} \binom{m}{k} \mathrm{Y}_{m-k}(1-\lambda; a) \sum_{n=0}^{k} (n+1) \sum_{j=0}^{n+1} \frac{S_2(k,j) (\log a)^k D_n}{\lambda^{n+1}(n+1-j)!}. \end{aligned} \tag{4.7}$$

Combining (4.7) with (2.3), we arrive at the following result:

Corollary 4.4. *Let $\lambda \neq 0, 1$ and $m \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} & y_{8,m}(\lambda; a) - \lambda \mathcal{Y}_m\left(\lambda - \lambda^2; a, \lambda\right) \log(\lambda) \\ = & \sum_{k=0}^{m} \binom{m}{k} \lambda \mathcal{Y}_{m-k}\left(\lambda - \lambda^2; a, \lambda\right) \sum_{n=0}^{k} (n+1) \sum_{j=0}^{n+1} \frac{S_2(k,j) (\log a)^k D_n}{\lambda^{n+1}(n+1-j)!}. \end{aligned} \tag{4.8}$$

Setting $a = e$ in (4.1), we have

$$K_1(t; e, \lambda) = \frac{\log(\lambda + e^t)}{e^t + \lambda - 1} = \sum_{n=0}^{\infty} y_{8,n}(\lambda; e)\frac{t^n}{n!}.$$

By combining the above equation with (1.2), we get

$$\sum_{n=0}^{\infty} y_{8,n}(\lambda; e)\frac{t^n}{n!} = \frac{1}{2(\lambda-1)}\sum_{n=0}^{\infty}\mathcal{E}_n\left(\frac{1}{\lambda-1}\right)\frac{t^n}{n!}\left(\log(\lambda) + \log\left(\frac{1}{\lambda}e^t + 1\right)\right).$$

From the above equation, we have

$$\begin{aligned}\sum_{n=0}^{\infty} y_{8,n}(\lambda; e)\frac{t^n}{n!} &= \frac{\log(\lambda)}{2(\lambda-1)}\sum_{n=0}^{\infty}\mathcal{E}_n\left(\frac{1}{\lambda-1}\right)\frac{t^n}{n!}\\ &+\frac{1}{2(\lambda-1)}\sum_{n=0}^{\infty}\mathcal{E}_n\left(\frac{1}{\lambda-1}\right)\frac{t^n}{n!}\sum_{m=0}^{\infty}(-1)^m\frac{e^{(m+1)t}}{(m+1)\lambda^{m+1}}.\end{aligned}$$

After some elementary calculations in the above equation, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} y_{8,n}(\lambda; e)\frac{t^n}{n!} &= \frac{\log(\lambda)}{2(\lambda-1)}\sum_{n=0}^{\infty}\mathcal{E}_n\left(\frac{1}{\lambda-1}\right)\frac{t^n}{n!}\\ &+\frac{1}{2(\lambda-1)}\sum_{n=0}^{\infty}\mathcal{E}_n\left(\frac{1}{\lambda-1}\right)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)\lambda^{m+1}}\\ &\times\sum_{j=0}^{m+1} j!\binom{m+1}{j}\frac{(e^t-1)^j}{j!}.\end{aligned}$$

Combining the previous equation with (1.7), we have

$$\begin{aligned}\sum_{n=0}^{\infty} y_{8,n}(\lambda; e)\frac{t^n}{n!} &= \frac{\log(\lambda)}{2(\lambda-1)}\sum_{n=0}^{\infty}\mathcal{E}_n\left(\frac{1}{\lambda-1}\right)\frac{t^n}{n!}\\ &+\frac{1}{2(\lambda-1)}\sum_{n=0}^{\infty}\mathcal{E}_n\left(\frac{1}{\lambda-1}\right)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)\lambda^{m+1}}\\ &\times\sum_{j=0}^{m+1} j!\binom{m+1}{j}\sum_{n=0}^{\infty}S_2(n, j)\frac{t^n}{n!}.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{n=0}^{\infty} y_{8,n}(\lambda; e)\frac{t^n}{n!} &= \frac{\log(\lambda)}{2(\lambda-1)}\sum_{n=0}^{\infty}\mathcal{E}_n\left(\frac{1}{\lambda-1}\right)\frac{t^n}{n!} + \frac{1}{2(\lambda-1)}\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\\ &\times\sum_{m=0}^{k}\sum_{j=0}^{m+1}\frac{(-1)^m j!\binom{m+1}{j}}{(m+1)\lambda^{m+1}}\mathcal{E}_{n-k}\left(\frac{1}{\lambda-1}\right)S_2(k, j).\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 4.5. *Let $\lambda \neq 0, 1$ and $n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}y_{8,n}(\lambda; e) &= \frac{\mathfrak{D}_0(\lambda)}{2}\mathcal{E}_n\left(\frac{1}{\lambda-1}\right) + \frac{1}{2(\lambda-1)}\sum_{k=0}^{n}\binom{n}{k}\\ &\times\sum_{m=0}^{k}\sum_{j=0}^{m+1}\frac{(-1)^m j!\binom{m+1}{j}}{(m+1)\lambda^{m+1}}\mathcal{E}_{n-k}\left(\frac{1}{\lambda-1}\right)S_2(k, j).\end{aligned} \tag{4.9}$$

By combining the equation (4.9) with (1.3), after some elementary calculations, we arrive at the following theorem:

Theorem 4.6. *Let $\lambda \neq 0, 1$ and $n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} y_{8,n}(\lambda; e) &= -\frac{\mathfrak{D}_0(\lambda)}{n+1}\mathcal{B}_{n+1}\left(\frac{1}{1-\lambda}\right) \\ &\quad -\frac{1}{(\lambda-1)}\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}\frac{(-1)^m}{(m+1)\lambda^{m+1}} \\ &\quad \times\sum_{j=0}^{m+1} j!\binom{m+1}{j}\frac{\mathcal{B}_{n-k+1}\left(\frac{1}{1-\lambda}\right)S_2(k,j)}{n-k+1}. \end{aligned}$$

Substituting $\lambda = 1$ into (4.1), and using (1.13), we have

$$\sum_{m=0}^{\infty} y_{8,m}(1; a)\frac{t^m}{m!} = \sum_{n=0}^{\infty} D_n \frac{a^{nt}}{n!}.$$

From the above equation, we get

$$\sum_{m=0}^{\infty} y_{8,m}(1; a)\frac{t^m}{m!} = \sum_{n=0}^{\infty}\frac{D_n}{n!}\sum_{m=0}^{\infty}\frac{(nt)^m}{m!}(\log a)^m.$$

Therefore

$$\sum_{m=0}^{\infty} y_{8,m}(1; a)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{D_n}{n!}(n\log a)^m\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we have following theorem:

Theorem 4.7. *Let $m \in \mathbb{N}_0$. Then we have*

$$y_{8,m}(1; a) = \sum_{n=0}^{\infty}\frac{D_n}{n!}(n\log a)^m. \tag{4.10}$$

Theorem 4.8. *Let $m \in \mathbb{N}_0$. Then we have*

$$y_{8,m}(1; a) = \sum_{n=0}^{\infty}\frac{D_n}{n!}\sum_{j=0}^{m}\binom{n}{j}j!S_2(m, j)(\log a)^m.$$

Proof. By combining (4.10) and (1.9), we have

$$y_{8,m}(1; a) = \sum_{n=0}^{\infty}\frac{D_n}{n!}\sum_{j=0}^{m}\binom{n}{j}j!S_2(m, j)(\log a)^m.$$

Thus proof of theorem is completed. □

4.1. Interpolation function for the numbers $y_{8,n}(\lambda; a)$

Here, using the following derivative operator in the following equation

$$\frac{\partial^k}{\partial t^k} f(t)|_{t=0} \tag{4.11}$$

to the generating function for the numbers $y_{8,n}(\lambda; a)$, we define unification of zeta type function which is interpolates the numbers $y_{8,n}(\lambda; a)$ at negative integers. We give some properties of this interpolation function.

By applying the derivative operator in equation (4.11) to both sides of the equation (5.1), we get

$$y_{8,k}(\lambda;a)=\frac{\partial^k}{\partial t^k}\left\{\frac{\log(\lambda+a^t)}{a^t+\lambda-1}\right\}|_{t=0}\,.$$

Assuming that $|\frac{a^t}{\lambda}|<1$ ($\lambda\neq 0,1$) to guarantee the convergence range of the following power series, the above equation reduces to the following relation:

$$y_{8,k}(\lambda;a)=\frac{\partial^k}{\partial t^k}\left\{\sum_{n=0}^{\infty}(-1)^n\frac{a^{tn}\log\lambda}{(\lambda-1)^{n+1}}+\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n a^{t(n+1)}}{(j+1)\lambda^{j+1}(\lambda-1)^{n+1-j}}\right\}|_{t=0}\,.$$

Therefore,

$$y_{8,k}(\lambda;a)=\log\lambda\sum_{n=0}^{\infty}(-1)^n\frac{n^k(\log a)^k}{(\lambda-1)^{n+1}}+\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n(n+1)^k(\log a)^k}{(j+1)\lambda^{j+1}(\lambda-1)^{n+1-j}}. \tag{4.12}$$

With the help of analytic continuation technique applied to Lerch type zeta function, using (4.12), we arrive at the following definition of the interpolation function for the number $y_{8,k}(\lambda;a)$:

Definition 4.9. Let $a\geq 1$. For $\lambda\in\mathbb{C}\setminus\{0,1\}$ ($|\frac{1}{\lambda-1}|<1$) and $s\in\mathbb{C}$, a unification of zeta type function $\mathcal{Z}_1(s;a,\lambda)$ is defined by

$$\begin{aligned}\mathcal{Z}_1(s;a,\lambda) &= \frac{\log\lambda}{(\log a)^s}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s(\lambda-1)^{n+1}}\\ &\quad+\frac{1}{(\log a)^s}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\frac{(-1)^n}{(j+1)\lambda^{j+1}(\lambda-1)^{n+1-j}}\right)\frac{1}{(n+1)^s}\end{aligned}$$

where $\lambda\in\mathbb{C}\setminus\{0,1\}$ ($|\frac{1}{\lambda-1}|<1$; $\mathrm{Re}(s)>1$).

Substituting $a=e$ into (4.13), we have

$$\mathcal{Z}_1(s;e,\lambda)=\log\lambda\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s(\lambda-1)^{n+1}}+\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{(j+1)\lambda^{j+1}(\lambda-1)^{n+1-j}(n+1)^s}.$$

Putting $\lambda=2$ and $\mathrm{Re}(s)>1$, then we have

$$\mathcal{Z}_1(s;e,2)=\log 2\sum_{n=1}^{\infty}\frac{1}{n^s}+\frac{1}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{(j+1)2^j(n+1)^s}, \tag{4.13}$$

Note that the function $\mathcal{Z}_1(s;a,\lambda)$ has the following property: This function is analytic continuation, except $s=1$ and $\lambda=2$ in whole complex plane. Combining (4.12) with (4.13) at negative integer, the following Theorem show that the $\mathcal{Z}_1(s;a,\lambda)$ interpolates the numbers $y_{8,k}(\lambda;a)$ at negative integer.

Theorem 4.10. *Let $\lambda\in\mathbb{C}$ ($|\frac{1}{\lambda}|<1$) and $k\in\mathbb{N}$. Then we have*

$$\mathcal{Z}_1(-k;a,\lambda)=y_{8,k}(\lambda;a)\,.$$

Let's define a new zeta type function below for $\lambda=1$ in the equation (4.1) without applying the derivative operator. This new zata type function interpolates the numbers $y_{8,m}(1;a)$ at negative integers.

Assuming that $|a^t|<1$. Combining (4.10) with (1.15), after some elementary calculations, we obtain

$$\sum_{m=0}^{\infty}y_{8,m}(1;a)\frac{t^m}{m!}=\sum_{n=0}^{\infty}(-1)^n\frac{a^{tn}}{n+1}.$$

Therefore

$$\sum_{m=0}^{\infty}y_{8,m}(1;a)\frac{t^m}{m!}=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}\sum_{m=0}^{\infty}(n\log a)^m\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we have following theorem:

Theorem 4.11. *Let $m \in \mathbb{N}_0$. Then we have*

$$y_{8,m}(1;a) = (\log a)^m \sum_{n=0}^{\infty} (-1)^n \frac{n^m}{n+1}. \tag{4.14}$$

Remark 4.12. By substituting $m = 0$ into (4.14), we get

$$y_{8,0}(1;a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \log(2).$$

By using (4.14) yields the following ***presumably new zeta-type function*** which interpolates the numbers $y_{8,m}(1;a)$ at negative integers:

Definition 4.13. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. Let $a \in (1,\infty)$. We define

$$\mathcal{Z}_2(s,a) = \frac{1}{(\log a)^s} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\, n^s}. \tag{4.15}$$

The function $\mathcal{Z}_2(s,a)$ interpolates the numbers $y_{8,m}(1;a)$ at negative integers. That is, substituting $s = -m$, ($m \in \mathbb{N}$), into (4.15), using (4.14), with the help of analytic continuation technique applied to Lerch type zeta function, we arrive at the following theorem:

Theorem 4.14. *Let $m \in \mathbb{N}$. Then we have*

$$\mathcal{Z}_2(-m,a) = y_{8,m}(1;a).$$

5. Generating functions for new classes combinatorial numbers $y_{9,n}(\lambda;a)$ and polynomials $y_{9,n}(x,\lambda;a)$ derived from the p-adic integral (3.10)

In this section, by the help of (3.10), we construct generating functions for the numbers $y_{9,n}(\lambda;a)$ and the polynomials $y_{9,n}(x,\lambda;a)$ respectively as follows:

$$Y_1(t;a,\lambda) := \frac{2}{a^t+\lambda} = \sum_{n=0}^{\infty} y_{9,n}(\lambda;a)\frac{t^n}{n!}, \tag{5.1}$$

and

$$\begin{aligned} Y_2(t,x;a,\lambda) &:= a^{tx} Y_1(t;a,\lambda) \\ &= \sum_{n=0}^{\infty} y_{9,n}(x,\lambda;a)\frac{t^n}{n!}. \end{aligned} \tag{5.2}$$

For functions $Y_1(t;a,\lambda)$ and $Y_2(t,x;a,\lambda)$, when $\lambda \neq 0$, we asume that

$$\left| t\log a + \log\left(\frac{1}{\lambda}\right)\right| < \pi.$$

When $\lambda = 0$, using (5.1),we have

$$y_{9,n}(0;a) = 2(-\log a)^n.$$

Using (5.2), we get

$$\frac{2e^{tx\log a}}{\lambda\left(\frac{1}{\lambda}e^{t\log a}+1\right)} = \sum_{n=0}^{\infty} y_{9,n}(x,\lambda;a)\frac{t^n}{n!}.$$

Combining the above function with (1.2), after some elementary calculations, we get the following relation between the polynomials $\mathcal{E}_n(x;\lambda)$ and $y_{9,n}(x,\lambda;a)$:

Theorem 5.1.

$$y_{9,n}(x,\lambda;a)=\frac{(\log a)^n}{\lambda}\mathcal{E}_n\left(x;\frac{1}{\lambda}\right). \tag{5.3}$$

Here, we note that due to the relation in equation (5.3), all the constraints given in [76]-[77], [78] also apply to the generating functions given in equation (5.1) and equation (5.2).

Here, by using these generating functions, we give not only fundamental properties of these polynomials and numbers, but also new identities and formulas including these numbers and polynomials, the generalized Eulerian type numbers, the Euler numbers, the Fubini numbers, the Stirling numbers, the Dobinski numbers, the numbers $\mathrm{Y}_n(\lambda;a)$.

By using (5.1) and (5.2), we have

$$y_{9,n}(0,\lambda;a)=y_{9,n}(\lambda;a).$$

From the equations (5.1) and (5.2), we have

$$\sum_{n=0}^{\infty}y_{9,n}(x,\lambda;a)\frac{t^n}{n!}=\sum_{n=0}^{\infty}(x\log a)^n\frac{t^n}{n!}\sum_{n=0}^{\infty}y_{9,n}(\lambda;a)\frac{t^n}{n!}.$$

Using the Cauchy product in the above equation yields

$$\sum_{n=0}^{\infty}y_{9,n}(x,\lambda;a)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(x\log a)^{n-j}y_{9,j}(\lambda;a)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, computation formula for the polynomials $y_{9,n}(x,\lambda;a)$ is given by the following theorem:

Theorem 5.2. *Let $n\in\mathbb{N}_0$. Then we have*

$$y_{9,n}(x,\lambda;a)=\sum_{j=0}^{n}\binom{n}{j}(x\log a)^{n-j}y_{9,j}(\lambda;a). \tag{5.4}$$

Setting $a=e$ and $\lambda=-2$ in (5.1), yields the following equation:

$$\frac{2}{e^t-2}=\sum_{n=0}^{\infty}y_{9,n}(-2;e)\frac{t^n}{n!}.$$

Combing the above equation with (1.10), we have

$$\sum_{n=0}^{\infty}y_{9,n}(-2;e)\frac{t^n}{n!}=-2\sum_{n=0}^{\infty}w_g(n)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have following corollary:

Corollary 5.3. *Let $n\in\mathbb{N}_0$. Then we have*

$$y_{9,n}(-2;e)=-2w_g(n). \tag{5.5}$$

By aid of (1.11) and (5.5), we also arrive at the following corollary:

Corollary 5.4. *Let $n\in\mathbb{N}_0$. Then we have*

$$y_{9,n}(-2;e)=-2\sum_{j=0}^{n}j!S_2(n,j).$$

Substituting $a^t = e^{e^t-1}$ and $\lambda = 0$ into (5.1), we have

$$\frac{2}{e^{e^t-1}} = \sum_{m=0}^{\infty} y_{9,m}(\lambda; e) \frac{(e^t-1)^m}{m!}.$$

Combining the above equation with (1.7) and (1.12), we obtain

$$2\sum_{n=0}^{\infty} D(n)\frac{t^n}{n!} = \sum_{m=0}^{\infty} y_{9,m}(\lambda; e) \sum_{n=0}^{\infty} S_2(n,m)\frac{t^n}{n!}.$$

Thus,

$$2\sum_{n=0}^{\infty} D(n)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} y_{9,m}(\lambda; e) S_2(n,m)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 5.5. *Let $n \in \mathbb{N}_0$. Then we have*

$$D(n) = \frac{1}{2}\sum_{m=0}^{n} y_{9,m}(\lambda; e) S_2(n,m).$$

By replacing λ by $-\lambda$ in (5.1), we get

$$\frac{2}{a^t-\lambda} = \sum_{n=0}^{\infty} y_{9,n}(-\lambda; a)\frac{t^n}{n!}.$$

Combing the above equation with (1.18), we have

$$2\sum_{n=0}^{\infty} \mathrm{Y}_n(\lambda; a)\frac{t^n}{n!} = \sum_{n=0}^{\infty} y_{9,n}(-\lambda; a)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have following corollary:

Corollary 5.6. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathrm{Y}_n(\lambda; a) = \frac{y_{9,n}(-\lambda; a)}{2}.$$

Setting $a = 1$, $x = 0$ and $\lambda = 1$ in (1.5), yields the following equation:

$$\frac{1-u}{b^t-u} = \sum_{n=0}^{\infty} \mathcal{H}_n(u; 1, b, c; 1)\frac{t^n}{n!}.$$

Combining the above equation with (5.1), we get

$$\sum_{n=0}^{\infty} \mathcal{H}_n(u; 1, b, c; 1)\frac{t^n}{n!} = \frac{1-u}{2}\sum_{n=0}^{\infty} y_{9,n}(-u; b)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following result:

Corollary 5.7. *Let $n \in \mathbb{N}_0$. Then we have*

$$y_{9,n}(-u; b) = \frac{2}{1-u}\mathcal{H}_n(u; 1, b, c; 1). \tag{5.6}$$

Substituting $b = e$ into (5.6), we get

$$y_{9,n}(-u; e) = \frac{2}{1-u}H_n(u).$$

When $u = -1$ in the above equation, we arrive at the following corollary:

Corollary 5.8. *Let $n \in \mathbb{N}_0$. Then we have*

$$y_{9,n}(1; e) = E_n.$$

5.1. Interpolation function for the numbers $y_{9,n}(\lambda; a)$

Here, using the following derivative operator in equation (4.11) to the generating function for the numbers $y_{9,n}(\lambda; a)$, we define unification of zeta type function which is interpolates the numbers $y_{9,n}(\lambda; a)$ at negative integers.

By using (5.1), and assuming that $|\frac{a^t}{\lambda}| < 1$, we get

$$\frac{2}{\lambda}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a^t}{\lambda}\right)^n = \sum_{n=0}^{\infty} y_{9,n}(\lambda; a)\frac{t^n}{n!}. \tag{5.7}$$

By applying the following derivative operator in equation (4.11) to both sides of the equation (5.7), we get

$$y_{9,k}(\lambda; a) = \frac{2(\log a)^k}{\lambda}\sum_{n=1}^{\infty}\frac{(-1)^n n^k}{\lambda^n}, \tag{5.8}$$

where $|\frac{1}{\lambda}| < 1$.

With the help of analytic continuation technique applied to Lerch type zeta function, using (5.8), we arrive at the following definition of the interpolation function for the number $y_{9,k}(\lambda; a)$:

Definition 5.9. Let $a \geq 1$. For

$$\lambda \in \mathbb{C}\ (|\frac{1}{\lambda}| < 1) \text{ and } s \in \mathbb{C},$$

a unification of zeta type function $\mathcal{Z}_3(s; a, \lambda)$ is defined by

$$\mathcal{Z}_3(s; a, \lambda) = \frac{2}{(\log a)^s}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s\lambda^{n+1}} \tag{5.9}$$

where $\mathrm{Re}(s) > 1$.

Note that the function $\mathcal{Z}_3(s; a, \lambda)$ has the following property: This function is analytic continuation, except $s = 1$ and $\lambda = 1$ in whole complex plane. Combining (5.9) with (5.8) at negative integer, the following Theorem show that the $\mathcal{Z}_3(s; a, \lambda)$ interpolates the numbers $y_{9,k}(\lambda; a)$ at negative integer.

Theorem 5.10. *Let $\lambda \in \mathbb{C}\left(|\frac{1}{\lambda}| < 1\right)$ and $k \in \mathbb{N}$. Then we have*

$$\mathcal{Z}_3(-k; a, \lambda) = y_{9,k}(\lambda; a).$$

Corollary 5.11. *Let $\lambda \in \mathbb{C}\left(|\frac{1}{\lambda}| < 1\right)$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^k}{\lambda^n} = \frac{\lambda y_{9,k}(\lambda; a)}{2(\log a)^k}.$$

Remark 5.12. In [75] and [47], Srivastava et al. studied and investigated many properties of the following unification of the Riemann-type zeta functions: For $\beta \in \mathbb{C}$ ($|\beta| < 1$),

$$\zeta_\beta(s; v, c, d) = \left(-\frac{1}{2}\right)^{v-1}\sum_{n=1}^{\infty}\frac{\beta^{dn}}{c^{d(n+1)}n^s}, \tag{5.10}$$

where $v \in \mathbb{N}_0$ and c and d are positive real numbers. Substituting $s = -m$ into the above equation, we have

$$\zeta_\beta(-m; v, c, d) = \frac{(-1)^v m!}{(m+v)!}y_{m+v,\beta}(v, c, d),$$

where the numbers $y_{m+v,\beta}(v,c,d)$ which is a unification of the Bernoulli, Euler and Genocchi numbers defined by Ozden[46] by means of the following generating function:

$$\frac{2^{v-1}t^v}{\beta^b e^{tx} - a^b} = \sum_{m=m}^{\infty} y_{m,\beta}(v,c,d)\frac{t^m}{m!}.$$

In [34], Kim et al. applied derivative operator $\frac{d^k}{dt^k}$ to the λ-Bernoulli numbers, the constructed interpolation of these numbers.

By combining (5.9) with (5.10), we get the following result:

Corollary 5.13. *Under the conditions given above equations (5.9) and (5.10), then we have*

$$\mathcal{Z}_3(s;a,\lambda) = \frac{2}{\lambda(\log a)^s}\zeta_{-\frac{1}{\lambda}}(s;1,1,1)$$

Remark 5.14. Recently many authors have studeied on the unification of the Bernoulli, Euler and Genocchi numbers $y_{m,\beta}(v,c,d)$ with their interplation function (*cf.* [4], [15], [22], [39], [40], [42], [45], [46], [64], [74]-[81]).

Remark 5.15. Most of the reductions to other zeta-type functions and their applications have not been discussed here. Relationships between zeta type functions have already been given in detail in the relevant studies, given the list of the references [1]–[81].

6. A new family of combinatorial sums and numbers arising from (4.13)

In this section, new numbers including combinatorial sums are defined, and some open problems are raised involving these new numbers.

Let's define the following new combinatorial numbers that arise from the equation (4.13):

$$y(n,\lambda) = \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)\lambda^{j+1}(\lambda-1)^{n+1-j}} \tag{6.1}$$

Substituting $\lambda = 2$ into (6.1), we get

$$y(n) := y(n,2) = \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)2^{j+1}}. \tag{6.2}$$

Open problems:

1. *One of the first questions that comes to mind what is generating function for the numbers* $y(n)$ and the numbers $y(n,\lambda)$.
2. *Some of the other questions are what are the special families of numbers the numbers* $y(n)$ *are related to.*
3. *What are the combinational applications of the numbers* $y(n)$*.*
4. *Can we find a special arithmetic function representing this family of numbers?*

We can partially solve the second question as follows:

By (1.14), Kucukoglu and Simsek [37] also gave the following novel finite combinatorial sum in terms of the numbers $\mathfrak{D}_n(\lambda)$:

$$\sum_{k=0}^{n-1} \frac{1}{k+1}\left(\frac{\lambda-1}{\lambda}\right)^k = \frac{(-1)^{n+1}\lambda\mathfrak{D}_n(\lambda)}{n!}\left(\frac{\lambda-1}{\lambda^2}\right)^n + \frac{\lambda\log\lambda}{\lambda-1}, \tag{6.3}$$

(*cf.* [37]).

Combining (6.1) and (6.3) shows that there exist a relationship between the numbers $y(n, \lambda)$ and λ-Apostol-Daehee numbers. In general response to the second question, is there any other relationships of the numbers $y(n, \lambda)$ with other well-known numbers and functions?

Even substituting $\lambda = 2$ into (6.3), and replacing n by $n + 1$, we get

$$\sum_{k=0}^{n} \frac{1}{(k+1)\,2^k} = \frac{(-1)^n\, \mathfrak{D}_{n+1}(2)}{2^{2n-1}\,(n+1)!} + 2\log 2 \tag{6.4}$$

which, by (6.2), yields that there exist a relationships between the numbers $y(n)$ and the λ-Apostol-Daehee numbers $\mathfrak{D}_n(\lambda)$ as in the following form:

$$y\,(n) = \frac{\mathfrak{D}_{n+1}(2)}{2^{2n}\,(n+1)!} + (-1)^n \log 2.$$

Recently, reciprocals of binomial coefficients, combinatorial sums have been studied in many different areas (*cf.* [5], [16], [60], [62], [80]).

With the help of the beta function and the gamma function, Sury et al. [80, Eq. (3)] gave the following combinatorial sum:

$$\frac{1}{n+1} \sum_{j=0}^{n} \frac{\lambda^j}{\binom{n}{j}} = \sum_{j=0}^{n} \frac{\lambda^{n+1} + \lambda^{n-j}}{(j+1)(1+\lambda)^{n+1-j}}.$$

The left-hand side of the above type sum has been recently studied by many mathematicians such as Mansour [44], Simsek [60], [61] and [62], and Sury et al. [80]; and also the references cited therein.

Substituting $\lambda = 1$ into the above equation, we obtain

$$y(n, -1) = \frac{1}{2\,(n+1)} \sum_{j=0}^{n} \frac{1}{\binom{n}{j}}. \tag{6.5}$$

The left hand side is the analogue of our alternating combinatorial sum in the equation (6.2).

Remark 6.1. In [60], [61] and [62], the author showed that the finite sums, containing reciprocals of binomial coefficients, are also related to the Beta-type polynomials and the Bernstein basis functions. The readers may refer to the aforementioned papers in order to see these relationships.

7. Conclusion

In this paper, two new special combinatorial numbers and polynomials are constructed with the aid of p-adic integrals including Volkenborn and fermionic integrals. By these generating functions, fundamental properties of these combinatorial numbers and polynomials are investigated, and we derive various new identities and formulas containing the newly introduced combinatorial numbers and polynomials, Bernoulli numbers and polynomials, Euler numbers and polynomials, Apostol-Bernoulli numbers and polynomials, Apostol-Euler numbers and polynomials, Stirling numbers of the second kind, Daehee numbers, Changhee numbers, the generalized Eulerian type numbers, Eulerian polynomials, Fubini numbers, Dobinski numbers. Additionally, we introduced a presumably new zeta-type function, which interpolates a special case of one of the newly introduced combinatorial numbers at negative integers. As a conclusion, the results of this paper have the potential to affect many researchers not only in combinatorics, but also in other relevant fields.

For future studies:

It is planned to investigate more properties and relations of the newly introduced combinatorial numbers and polynomials.

Examining the fundamental properties of the new zeta-type functions defined in this study and their relations with other fields are among our future projects. It includes our other main projects in solving the obvious problems that arise in defining these functions.

As a result, we believe that the results obtained in this article have the potential to be used in many different areas.

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