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On Sequences of Certain Contractive Mappings and Their Fixed Points

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Abstract

In 1988, M. Imdad, M.S. Khan and S. Sessa have introduced a general contractive condition of Reich-Hardy-Rogers type in a (complete) metric space (X, d). Let $(T_n)_n$ be a sequence of selfmaps of X satisfying this general condition. They proved that each selfmap T_n has a unique fixed point (say) z_n in X. Suppose that the sequence $(T_n)_n$ converges pointwise on X to a selfmap T. M. Imdad, M.S. Khan and S. Sessa sudied the convergence problem for the sequence $(z_n)_n$ in X. They established the convergence of this sequence under a regularity assumption, and they raised the question whether this regularity assumption is necessary or not ?

The aim of this paper is to answer positively to that question. Precisely, we prove that the main results of M. Imdad, M.S. Khan and S. Sessa are still valid without requiring the regularity assumption upon the sequence of fixed points $(z_n)_n$ of the sequence $(T_n)_n$.

Keywords: Complete metric space, fixed point, sequences of mappings, selfmaps of Reich-Hardy-Rogers type

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1. Introduction, Preliminaries and recalls

Let X be a non empty set and let $T : X \to X$ be a an arbitrary selfmapping of X. One of the fundamental problems to be considered for T is to study the set of fixed points by the map T. That is, the set $\mathrm{Fix}\,(T) := \{x \in X : Tx = x\}$. Of course, the presence of algebraic, topological, differential, partial orderings or other structures, will be of much help in that study. The uniform, metric or metric-like structures has developed a great background of results and methods to deal with the problem of finding, computing or approximating the set of fixed points for selfmappings, see for instance the book of Berinde [4].

It is well recognized that the first and most fundamental result in metric fixed point theory, is the Banach-Caccioppoli theorem.

Throughout this paper, (X, d) will be a complete metric space.

(A): A selfmapping $T : X \to X$, will be said a contraction on X, if there exists an $\alpha \in [0, 1)$, such that for each $x, y \in X$, we have

$$d(Tx, Ty) \leq \alpha\, d(x, y). \tag{1.1}$$

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Banach-Caccioppoli contraction principle (see [3] and [8]) says that each contraction T has a unique fixed point in X. This principle gives also a constructive method for computing and approximating this unique fixed point of T by using the Picard sequences.

This principle, is for ever a source of inspiration for further research in fixed point theory. In 2007, R. S. Palais (see [23]) has given a new simple proof of the Banach contraction principle together with a stopping rule for the sequence of successive approximations of the fixed points by Banach contractions.

Many generalizations for the Banach-Caccioppoli contraction principle were investigated through many directions.

One can say that the first direction of research is concerned with the introduction of new kinds of (more general) contractive conditions (see for instances [6], [18], [10], [33], [11], [29], [2], [30]. ...).

In connection with this research field, a fundamental work was presented by B.E. Rhoades [29] in 1977, aiming a comparison between a large number of general contractive conditions.

A second direction concerns other topological contexts defined for example by symmetrics, generalized distances (see [7]), pseudo-distances, two-metrics, b-metrics, partial metrics (see [17]), uniform structures, conic distances (see [13]) and many other kinds of new topological structures.

A third direction is concerned with finding common fixed points for set of maps under suitable conditions on these (multivalued or hybrid) sets of maps. In all cases, some properties (forms of commutativity) are required on these maps to ensure the existence of their common fixed points. Almost all these kinds of commutativity are generalizations of the concept of weak commutativity introduced by S. Sessa (see [31]).

A fourth direction of investigations in Fixed point theory concerns (the stability or) convergence problem for fixed points of a given sequence of selfmaps of a metric space. One can say that this line of research was initiated by F.F. Bonsall [5] and followed by several authors.

There are more other important research areas in Fixed point theory which are growing and are making great developements. Certainly, one can not quote all of them here.

In this paper we are mainly concerned with the convergence problem of fixed points of sequences of certain maps.

Precisely, let $(T_n)_n$ be a sequence of self-mappings of a metric space (X, d). Suppose that the sequence $(T_n)_n$ converges pointwise to a selfmap T of X. Suppose that each selfmap T_n has a fixed point (say) z_n. The natural questions are the following:

(a) Which conditions do ensure convergence of the sequence z_n ?

(b) When the sequence z_n has a limit (say) z. Is z a fixed point of the limit map T ?

We recall that one of the first results from this area of research is due to F.F. Bonsall [5]. This results concerns the Banach contractions. More precisely, Bonsall proved the following: " Let (X, d) be a complete metric space, and T and T_n ($n \in \mathbb{N}$) be a sequence of Banach contractions mapping X into itself having the same Lipschitz constant $k \in [0, 1)$, and with fixed points z and z_n respectively. Suppose that the sequence $(T_n)_{n\in\mathbb{N}}$ converges pointwise to T. Then the sequence $(z_n)_{n\in\mathbb{N}}$ converges to z in (X, d)"'. This result was generalized to several kinds of maps. Results in this direction are obtained by a number of authors. For instance see J. Achari ([1]), R.B. Fraser and S.B. Nadler ([12]), G.E. Hardy and T.D. Rogers ([14]), M. Imdad, M.S. Khan and S. Sessa ([15]), R.N. Mukherjee ([19]), N. Muresan ([20]), S.B. Nadler ([21] and [22]), S. Park ([24]), S. Reich ([26]), S.P. Singh ([32]. See also the references of the papers cited above.

Next, we make a brief recall on some convergence results known in the literature.

Let T, as before, be a selfmapping of a complete metric space (X, d).

E. Rakotch (see [25]) has introduced the following contractive condition:

(B): There exists a monotonically decreasing function $g : (0, +\infty) \to [0, 1)$ such that for each $x, y \in X$ with $x \neq y$,

$$d(Tx, Ty) \leq g(d(x, y))\, d(x, y). \tag{1.2}$$

S. Reich (see [26]) has considered the following contractive condition:

(C): There exist nonegative numbers a, b such that for each $x, y \in X$ with $x \neq y$,

$$d(Tx, Ty) \leq \ a\, d(x, y) + b\ (d(x, Tx) + d(y, Ty)), \tag{1.3}$$

where $a + 2b < 1$.

The inequality above contains the case of Kannan's mappings (see [16]).

S. Reich (see [27] and [28]) has also considered the following two contractive conditions which generalize (1.3).

(D): There exist monotonically deceasing functions $a, b : (0, +\infty) \to [0, 1)$ such that for each $x, y \in X$ with $x \neq y$,

$$d(Tx, Ty) \leq \ a(d(x, y))\, d(x, y) + b(d(x, y))\ (d(x, Tx) + d(y, Ty)), \tag{1.4}$$

where we have

$$a(t) + 2b(t) < 1, \quad \forall\, t \in (0, +\infty). \tag{1.5}$$

(E): There exist functions $a, b : (0, +\infty) \to [0, 1)$ such that (1.4) holds for each $x, y \in X$ with $x \neq y$, satisfying (1.5) and the following condition:

$$\limsup_{s \to t^+} (a(s) + 2b(s)) < 1. \tag{1.6}$$

In 1973, G.E. Hardy and T.D. Rogers (see [14]) have considered the following more general contractive condition:

(F): There exist monotonically deceasing functions $a, b, c : (0, +\infty) \to [0, 1)$ such that for each $x, y \in X$ with $x \neq y$,

$$\begin{aligned} d(Tx, Ty) \leq a(d(x, y))d(x, y) + b(d(x, y)) \left[d(x, Tx) + d(y, Ty)\right] \\ + c(d(x, y)) \left[d(x, Ty) + d(y, Tx)\right], \end{aligned} \tag{1.7}$$

where we have

$$a(t) + 2b(t) + 2c(t) < 1, \quad \forall\, t \in (0, +\infty). \tag{1.8}$$

M. Imdad, M.S. Khan and S. Sessa (see [15]) have introduced the following generalization of the condition (F).

(G): There exist functions $a, b, c : (0, +\infty) \to [0, 1)$ such that (1.7) holds for each $x, y \in X$ with $x \neq y$, satisfying (1.8) and the following condition:

$$\limsup_{s \to t^+} (a(s) + 2b(s) + 2c(s)) < 1, \quad \forall\, t \in (0, +\infty). \tag{1.9}$$

A discussion on the relationships between the conditions (A), (B), (C), (D), (E), (F) and (G) was made in [15]. The conclusion is that the condition (G) extends properly all the above conditions (A), (B), (C), (D), (E), and (F).

It was observed in [15], that if a selfmapping T of a complete metric space (X, d) satisfies (1,7), (1.8) and (1.9), then T has a unique fixed point in X.

Thus according to [15], the following fixed point theorem can be proved.

Theorem 1.1. *Let T be a selfmapping of a complete metric space (X, d) satisfying the conditions (1,7), (1,8) and (1,9).*

Then T has a unique fixed point z in X.

In the sequel of this paper, $\mathbb{N}$ will be the set of all positive integers.

The following result was proved in [14] and [29].

Theorem 1.2. *([14], [29]) Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of selfmappings of a complete metric space (X, d) satisfying the condition (F) with the same functions a, b, c with fixed points z_n. Suppose that a selfmapping T of X can be defined pointwise by $T(x) := \lim_{n \to +\infty} T_n(x)$, for any x in X.*

Then T has a unique fixed point z in X and $z = \lim_{n \to +\infty} z_n$.

Theorem 1.2 extends the result obtained by Bonsall for mappings $(T_n)_{n\in\mathbb{N}}$ satisfying the condition (A). Theorem 1.2 generalizes Theorem 6 of [26] and Theorem 4 of [27] for mappings $(T_n)_{n\in\mathbb{N}}$ satisfying the conditions (C) and (D) respectively.

Theorem 1.2 extends also other results due to Chatterjea [9] and S.P. Singh [32].

To generalize Theorem 1.2, M. Imdad, M.S. Khan and S. Sessa have established in [15] the following result:

Theorem 1.3. *(Theorem 2, [15]). Let $\{T_n\}$ be a sequence of selfmappings of a complete metric space (X, d) satisfying the conditions (1,7), (1,8) and (1,9) with the same functions a, b, c with fixed points z_n.*
We suppose that :
(i) a mapping T of X into itself can be defined pointwise by $T(x) := \lim_{n\to+\infty} T_n(x)$, for all x in X, and
(ii) the sequence $\{z_n\}$ is regular.
Then T has a unique fixed point z in X and $z = \lim_{n\to+\infty} z_n$.

In the previous theorem, the sequence $\{z_n\}$ is regular means that it possesses a limit.

In Remark 1 of [15], it was raised the question whether the assumption (ii) in Theorem 1.3 is necessary or not ?
The aim of this paper is to answer to this question.
Precisely, we prove that the conclusions of Theorem 1.3 are still valid without need of the assumption (ii). So, condition (ii) can be removed.

The remainder of this paper is organized as follows:
In section two, we establish the main result of this paper (see Theorem 2.1). In Remark 2.2, we discuss relationships between our main result and other earlier results. We give a list of some of these results that can be obtained as direct consequences of Theorem 2.1.
In section three, we study an illustrative example to support our main result.
In section four, we have ended this paper by some concluding remarks concerning this work.

2. The result

The main result of this paper reads as follows.

Theorem 2.1. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of selfmappings of a complete metric space (X, d) satisfying the conditions (1,7), (1,8) and (1,9) with the same functions a, b, c with fixed points z_n. We suppose that a mapping of X into itself can be defined pointwise by $T(x) := \lim_{n\to+\infty} T_n(x)$, for all x in X. Then the map T has a unique fixed point z in X and the sequence $\{z_n\}$ converges to z in X.*

Proof. Since the metric $d : X \times X \to [0, +\infty)$ is continuous in the product metric space $X \times X$, then the limit mapping will also satisfy the conditions (1,7), (1,8) and (1,9) with the same functions a, b, c. These condition imply that T has a unique fixed point (say) z.
We claim that the sequence $\{z_n\}$ converges to the point z.
To get a contradiction, suppose that the contrary holds. Then, there exist an $\epsilon > 0$ and a subsequence $\{z_{\varphi(n)}\}$ of $\{z_n\}$ satisfying

$$d(z_{\varphi(n)}, z) \geq \epsilon, \quad \forall n \in \mathbb{N}. \tag{2.1}$$

Here, $\varphi : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. To simplify notations, we set $y_n := z_{\varphi(n)}$ for all $n \in \mathbb{N}$. Then (2.1) says that

$$d(y_n, z) \geq \epsilon, \quad \forall n \in \mathbb{N}. \tag{2.2}$$

We consider

$$\eta := \inf_{n\in\mathbb{N}} d(y_n, z). \tag{2.3}$$

(2.2) implies that $\eta \geq \epsilon > 0$. Thus, for all n, we have $y_n \neq z$ and then we can apply the inequality (1.7). By this inequality and the triangular property of the metric d, for all $n \in \mathbb{N}$, we get the following successive inequalities:

$$\begin{aligned} d(y_n, z) &= d(T_{\varphi(n)}y_n, Tz) \leq d(T_{\varphi(n)}y_n, T_{\varphi(n)}z) + d(T_{\varphi(n)}z, Tz) \\ &\leq d(T_{\varphi(n)}z, z) + a_n\, d(y_n, z) + b_n\, d(z, T_{\varphi(n)}z) \\ &+ c_n \left[d(y_n, z) + d(z, T_{\varphi(n)}z) + d(z, T_{\varphi(n)}y_n)\right] \\ &= (a_n + 2c_n)\, d(y_n, z) + (1 + b_n + c_n)\, d(T_{\varphi(n)}z, z), \qquad (2.4) \end{aligned}$$

where we have used the notations: $a_n := a(d(y_n, z))$, $b_n := b(d(y_n, z))$ and $c_n := a(d(y_n, z))$, for all $n \in \mathbb{N}$.

Thus, we get

$$d(y_n, z) \leq \frac{2d(T_{\varphi(n)}z, z)}{1 - (a_n + 2c_n)}, \forall n \in \mathbb{N}. \tag{2.5}$$

By definition of η, there exists a subsequence $\{y_{\psi(n)}\}$ of the sequence $\{y_n\}$ (where $\psi : \mathbb{N} \to \mathbb{N}$ is a strictly increasing map) such that

$$\eta \leq d(y_{\psi(n)}, z) < \eta + \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{2.6}$$

From (2.6), we see that

$$\lim_{n \to +\infty} d(y_{\psi(n)}, z) = \eta > 0. \tag{2.7}$$

By setting $\theta := \varphi \circ \psi$ we obtain a strictly increasing map from $\mathbb{N}$ into itself. Therefore $\{y_{\psi(n)}\} = \{z_{\theta(n)}\}$ is a subsequence of the initial sequence $\{z_n\}$.

As in [15], We follow ideas of S. Reich [28] and obtain, from the assumptions (1.8) and (1.9) about the functions a, b, c, the existence of two functions $h, k : (0, +\infty) \to (0, +\infty)$ satisfying the following property:

$$\forall t > 0,\ \forall s \in (0, +\infty) :\ t \leq s < t + h(t) \Longrightarrow a(s) + 2b(s) + 2c(s) \leq k(t)\ < 1. \tag{2.8}$$

By (2.6) (or (2.7)), there exists a positive integer n_1 such that

$$\forall n \in \mathbb{N},\ n \geq n_1 \Longrightarrow \eta \leq d(y_{\psi(n)}, z) < \eta + h(\eta). \tag{2.9}$$

By (2.8), it follows that

$$a(d(y_{\psi(n)}, z)) + 2c(d(y_{\psi(n)}, z)) \leq k(\eta)\ < 1,\ \forall n \geq n_1. \tag{2.10}$$

(2.10) gives the following inequality:

$$\frac{1}{1 - a(d(y_{\psi(n)}, z)) + 2c(d(y_{\psi(n)}, z))} \leq \frac{1}{1 - k(\eta)},\ \forall n \geq n_1. \tag{2.11}$$

(2.5) and (2.11) will imply the following inequality:

$$d(y_{\psi(n)}, z) \leq \frac{2d(T_{\theta(n)}z, z)}{1 - k(\eta)}, \forall n \in \mathbb{N}. \tag{2.12}$$

On the other hand, we know that the sequence $\{T_{\theta(n)}z\}$ converges to $Tz = z$. Hence, there exists another positive integer (say) n_2 such that

$$\forall n \in \mathbb{N},\ n \geq n_2 \ \Longrightarrow d(T_{\theta(n)}z, z) < \frac{\eta(1 - k(\eta))}{4}. \tag{2.13}$$

By taking $p := \max\{n_1, n_2\}$, from (2.12) and (2.13), we obtain the following inequalities:

$$\forall n \in \mathbb{N},\ n \geq p \ \Longrightarrow 0 < \eta \leq d(y_{\psi(n)}, z) \leq \frac{2d(T_{\theta(n)}z, z)}{1 - k(\eta)} \leq \frac{\eta}{2},$$

which is a contradiction. We conclude that the whole sequence $\{z_n\}$ converges to the point z which is, as we know, a fixed point of T. In fact, we have $\text{Fix}(T) = \{z\}$. This completes the proof. □

Remark 2.2. Since the condition (G) extends properly all the conditions (A)–(F), Theorem 2.1 extends properly Theorem 1.2. In particular our main result unifies and gives a common proper extensions to the following theorems: Theorem 6 of [26], and Theorem 4 of [27] for mappings $(T_n)_{n \in \mathbb{N}}$ satisfying the conditions (C) and (D) respectively. It extends also other results due to Chatterjea [9] and S.P. Singh [32].

3. Example

In this section, we furnich an example illustrating the result of this paper.

Let $X : [0,1] \cup \mathbb{N} \setminus \{1\}$, where $\mathbb{N}$ is the set of positive integers. We define a distance d on X by setting:

$$d(x,y) := \begin{cases} |x-y|, & \text{if } x,y \in [0,1], \\ x+y, & \text{if one of } x,y \in \mathbb{N} \setminus \{1\}. \end{cases}$$

It is easy to check that d is a metric on X. In fact, D.W. Boyd and J.S.W. Wong introduced (in [6]) this metric space (X,d) and they proved that (X,d) is a complete metric space.

For all integer $n \in \mathbb{N}$, we denote T_n the map defined for all $x \in X$, by setting

$$T_n(x) := \begin{cases} \frac{x}{2} - \frac{nx^2}{4n+2} + \frac{1}{4}, & \text{if } x \in [0,1], \\ x-1, & \text{if } x \in \mathbb{N} \setminus \{1\}. \end{cases}$$

Then we have the following properties:

(1) For all integer $n \in \mathbb{N}$, we have $T_n([0,1]) \subset [0,1]$ and $T_n(X) \subset X$. Thus, each T_n is a selfmap of X.

(2) The sequence T_n converges pointwise to the selfmap T of X, given by:

$$T(x) := \begin{cases} \frac{x}{2} - \frac{x^2}{4} + \frac{1}{4}, & \text{if } x \in [0,1], \\ x-1, & \text{if } x \in \mathbb{N} \setminus \{1\}. \end{cases}$$

(3) For all integer $n \in \mathbb{N}$, and for all $x,y \in [0,1]$ with $x \neq y$, we have the following inequalities:

$$\begin{aligned} d(T_nx, T_nx) &= \tfrac{1}{2}|x-y|\Big(1 - \tfrac{n}{2n+1}(x+y)\Big) \\ &\leq \tfrac{1}{2}|x-y|\Big(1 - \tfrac{1}{3}(x+y)\Big) \\ &\leq \tfrac{1}{2}|x-y|\Big(1 - \tfrac{1}{3}|x-y|\Big) \\ &= a_1(d(x,y))d(x,y), \end{aligned}$$

whre $a_1(t) := \frac{1}{2} - \frac{t}{6}$, for all $t \in (0,1]$.

(4) For all integer $n \in \mathbb{N}$, and for all $x,y \in \mathbb{N} \setminus \{1\}$ with $x > y$, we have the following inequalities:

$$\begin{aligned} d(T_nx, T_nx) &= d(x-1, y-1) \\ &= x-1+y-1 \\ &\leq x+y-1 \\ &= d(x,y)(1 - \tfrac{1}{d(x,y)}) \\ &= a_2(d(x,y))d(x,y), \end{aligned}$$

whre $a_2(t) := 1 - \frac{1}{t}$, for all $t > 1$.

(5) For all integer $n \in \mathbb{N}$, let $x = 2$ and let $y \in [0,1]$. Then we have the following inequalities:

$$\begin{aligned} d(T_n2, T_ny) &= d(1, T_ny) \\ &= 1 - T_ny \\ &\leq 1 \leq 1 + y = y + 2 - 1 = (y+2)(1 - \tfrac{1}{y+2}) \\ &= d(2,y)(1 - \tfrac{1}{d(2,y)}) \\ &= a_2(d(2,y))d(2,y), \end{aligned}$$

whre $a_2(t) := 1 - \frac{1}{t}$, for all $t > 1$.

(6) For all integer $n \in \mathbb{N}$, let $x \in \mathbb{N} \setminus \{1,2\}$ and let $y \in [0,1]$ with $x \neq y$. Then we have the following inequalities:

$$\begin{aligned} d(T_n2, T_ny) &= d(x-1, T_ny) \\ &= x - 1 + T_ny \\ &\leq x - 1 + y + \tfrac{1}{4} \\ &= x + y - \tfrac{3}{4} = (x+y)(1 - \tfrac{3}{4(x+y)}) \\ &= d(x,y)(1 - \tfrac{1}{4d(x,y)}) \\ &= a_3(d(x,y))d(x,y), \end{aligned}$$

whre $a_3(t) := 1 - \frac{3}{4t}$, for all $t > 1$.

(7) Let us define $b(t) = c(t) = 0$ for all $t > 0$ and define the function a on $(0, +\infty)$ by setting:

$$a(t) := \begin{cases} \frac{1}{2} - \frac{t}{6}, & \text{for all } t \in (0, 1]. \quad \text{if } x, y \in [0, 1], \\ 1 - \frac{3}{4t}, & \text{for all } t > 1. \end{cases}$$

Then each T_n satisfies the condition (G) with these functions a, b and, c.

Therefore each T_n has a unique fixed point z_n in X.

By virtue of our result, the sequence $(z_n)_n$ converges to the unique fixed point z of the limit map T.

(8) All these can be checked by computations. Indeed, each fixed point z_n is given by

$$z_n = \frac{2n+1}{2n}\left(\sqrt{\frac{4n+1}{2n+1}} - 1\right). \tag{3.1}$$

By taking the limit in (3.1), we get

$$\lim_{n\to\infty} z_n = \sqrt{2} - 1.$$

(9) It easy to see that $z = \sqrt{2} - 1$ is the unique fixed point of the limit map T.

(10) Conclusion: All conditions and all conclusions are realized in this example.

4. Concluding remarks

M. Imdad, M.S. Khan and S. Sessa [15] introduced the general contractive condition (G) (see the introduction of this paper) which is a Reich-Hardy-Rogers type in a metric space (X, d).

Let $(T_n)_n$ be a sequence of selfmaps of a complete metric space (X, d) satisfying condition (G). M. Imdad, M.S. Khan and S. Sessa (in [15]) proved that each selfmap T_n has a unique fixed point (say) z_n in X. Suppose that the sequence $(T_n)_n$ converges pointwise on X to a map T, M. Imdad, M.S. Khan and S. Sessa sudied the convergence problem for the sequence $(z_n)_n$ in X. Precisely, in Theorem 1.2, they established the convergence of this sequence by assuming a regularity condition. They ended their paper [15] by raising the question whether this regularity assumption is necessary or not ?

In this paper we answer positively to that question. Precisely, we prove in our main result (see Theorem 2.1) that the the conclusions of the main result of M. Imdad, M.S. Khan and S. Sessa are still valid without requiring the regularity assumption upon the sequence of fixed points $(z_n)_n$ of the sequence $(T_n)_n$.

Our result extends and unifies several convergence results, published previously, using one of the conditions (A), (B), (C), (D), (E) or (F) (see the introduction of this work).

We provide also an illustrative example to support our main result (see section three).

We point out that the example studied here, shows also, that condition (G) extends properly condition (F). This fact was previously established in [15].

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