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About Solving Some Functional Equations related to the Lagrange Inversion Theorem

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Abstract

Using the notion of the composita and the Lagrange inversion theorem, we present techniques for solving the functional equations $B(x) = H(xB(x)^r)$ and $A(F(x)) = xH(A(x))$, where $H(x)$, $B(x)$ and $F(x)$ are known generating functions and $A(x)$ is unknown, and r is a rational number. The first equation is a generalization of the Lagrange inversion formula for a rational power r in terms of compositae. For the second equation, the recurrent solution in terms of the compositae is obtained. Also we give some examples of the obtained results.

Keywords: Composita, Generating function, Lagrange inversion theorem, Functional equation

2010 MSC: 05A15, 30D05, 39B12

1. Introduction

The Lagrange inversion theorem is one of the important results in combinatorics and analysis. The Lagrange inversion formula is presented as follows:

Suppose $H(x) = \sum\limits_{n\geq 0} h(n)x^n$ with $h(0) \neq 0$ and let $A(x)$ be defined by

$$A(x) = xH(A(x)). \tag{1.1}$$

Then

$$n[x^n]A(x)^k = k[x^{n-k}]H(x)^n,$$

where $[x^n]A(x)^k$ is the coefficient of x^n in $A(x)^k$ and $[x^{n-k}]H(x)^n$ is the coefficient of x^{n-k} in $H(x)^n$.

Hence, this formula gives a solution for the equation $A(x) = xH(A(x))$ in terms of the coefficients of the generating function $A(x)$. Many other works related to the generating functions, especially for special numbers and polynomials, readers can find in [1, 2, 3, 4].

The Lagrange inversion formula and this type of functional equations are often useful in many areas of mathematics. Readers can start to study the Lagrange inversion theorem with the book titled "Enumerative Combinatorics, Volume 2" and written by Stanley [5].

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There are many generalizations and different forms of the Lagrange inversion theorem, that can be found in reviews made by Gessel [6] or Merlini, Sprugnoli, and Verri [7]. In [8] we generalized the equation (1.1) as follows:

$$B(x) = H(xB(x)^m),$$

where the generating function $xB(x)$ is substituted for $A(x)$ in the equation (1.1) and the power m is a natural number.

We also found that the coefficients of powers of the generating function $xB(x)$ (or a composita of $xB(x)$) are defined by

$$B_x^\Delta(n,k,m) = \frac{k}{i_{m-1}} H_x^\Delta(i_m, i_{m-1}),$$

where $i_m = (m+1)n - mk$, $H_x^\Delta(n,k)$ and $B_x^\Delta(n,k)$ are the compositae of the generating functions $xH(x)$ and $xB(x)$ respectively.

The notion of the composita of a generating function was introduced in [10, 9, 11]. The composita of a generating function $F(x)$ is the function of two variables defined by

$$F^\Delta(n,k) = \sum_{\pi_k \in C_n} f(\lambda_1) f(\lambda_2) \ldots f(\lambda_k),$$

where C_n is a set of all compositions of an integer n, π_k is the composition $\sum_{i=1}^k \lambda_i = n$ into k parts exactly.

Suppose $F(x) = \sum\limits_{n>0} f(n)x^n$ is a generating function where $f(0) = 0$. For this generating function we can write the following equation:

$$F(x)^k = \sum_{n>0} F(n,k)x^n.$$

The expression $F(n,k)$ is the composita, and it is denoted by $F^\Delta(n,k)$. In this case, the composita is an expression for the coefficients of powers of a generating function with the free term is equal to 0.

In this paper, we expand the results presented by the authors at the 2nd Mediterranean International Conference of Pure and Applied Mathematics and Related Areas (MICOPAM 2019) [12]. We study the application of the composita for solving the following generalized equations:

$$B(x) = H(xB(x)^r) \tag{1.2}$$

and

$$A(F(x)) = xH(A(x)), \tag{1.3}$$

where $H(x)$ and $B(x)$ are generating functions with the free term is not equal to 0, and r is a rational number.

2. Solution for the functional equation $B(x) = H(xB(x)^r)$

Firstly, we give the formula for the composita of $xH(x)^r$.

Theorem 2.1. *The composita of $xH(x)^r$ is defined by*

$$H_x^\Delta(n,m,r) = \begin{cases} H^\Delta(1,1)^{mr}, & n = m \\ \sum\limits_{k=1}^{n-m} H_1^\Delta(n-m,k)\binom{mr}{k} H^\Delta(1,1)^{mr-k} & n > m, \end{cases} \tag{2.1}$$

where $H_1^\Delta(n,k)$ is the coefficients of the generating function $(H(x) - h(0))^k$ that are defined by the composita $H^\Delta(n,k)$ of the generating function $xH(x)$.

Proof. Suppose we have a generating function $H(x) = \sum\limits_{n\geq 0} h(n)x^n$ with $h(0) \neq 0$ and know the expression for the coefficients of powers of a generating function

$$H(x)^k = \sum_{n\geq 0} H(n,k)x^n.$$

Next we find the composition of the generating function $(H(x) - h(0))$. To do this, we apply the binomial theorem

$$\begin{aligned}(H(x) - h(0))^k &= \sum_{j=0}^{k} \binom{k}{j} H(x)^j (-1)^{k-j} h(0)^{k-j} \\ &= \sum_{n\geq k} \sum_{j=0}^{k} \binom{k}{j} H(n, j)(-1)^{k-j} h(0)^{k-j} x^n \\ &= \sum_{n\geq k} H_1^{\Delta}(n, k) x^n.\end{aligned}$$

According to [13], the relation between the coefficients of $H(x)^k$ and the composita $H^{\Delta}(n, k)$ of the generating function $xH(x)$ is the following:

$$H(n, j) = H^{\Delta}(n + j, j).$$

Since

$$h(0)^{k-j} H^{\Delta}(k - j, k - j),$$

we get the following expression for the composita $H_1^{\Delta}(n, k)$:

$$H_1^{\Delta}(n, k) = \sum_{j=0}^{k} \binom{k}{j} H^{\Delta}(n + j, j)(-1)^{k-j} H^{\Delta}(k - j, k - j). \tag{2.2}$$

By the formula (2.2), we introduce an operator $M(n, k, H^{\Delta})$ that transforms the composita of $xH(x)$ to the composita of $(H(x) - h(0))$.

Let consider $H(x)^r$ as the composition of $(x + h(0))^r$ and $(H(x) - h(0))$. The coefficients of the generating function $(x + h(0))^{mr}$ are equal to

$$\binom{mr}{n} h(0)^{mr-n} = \binom{mr}{n} H^{\Delta}(1, 1)^{mr-n}.$$

According to the formula for the composition of generating functions [10], we get

$$H(n, m, r) = \begin{cases} h(0)^{mr}, & n = 0 \\ \sum\limits_{k=1}^{n} H_1^{\Delta}(n, k) \binom{mr}{k} h(0)^{mr-k} & n > 0. \end{cases}$$

Then the desired formula for the composita of $xH(x)^r$ is

$$H_x^{\Delta}(n, m, r) = \begin{cases} H^{\Delta}(1, 1)^{mr}, & n = m \\ \sum\limits_{k=1}^{n-m} H_1^{\Delta}(n - m, k) \binom{mr}{k} H^{\Delta}(1, 1)^{mr-k} & n > m. \end{cases}$$

□

Example 2.2. Let us consider the generating function

$$H(x)^r = \left(\frac{e^x - 1}{x}\right)^r,$$

and find the composita of the generating function $xH(x)^r$.

For that we represent the generating function as follows:

$$(xH(x)^r)^m = x^m (1 + H(x) - 1)^{mr} = x^m \left(1 + \frac{e^x - 1 - x}{x}\right)^{mr}.$$

The composita of the generating function $(e^x - 1)$ is

$$S_2(n,k)\frac{k!}{n!}, \tag{2.3}$$

where $S_2(n,k)$ are the Stirling numbers of the second kind [14].

The composita of $(-x)$ is equal to $\delta_{n,k}(-1)^k$. Then, according to the formula of the composita of a sum [10], the composita of $e^x - 1 - x$ is

$$\sum_{j=0}^{k}\binom{k}{j}\sum_{i=j}^{n-k+j} S_2(i,j)\frac{j!}{i!}\delta_{n-i,k-j}(-1)^{k-j},$$

where $\delta_{n-i,k-j}$ is the Kroneker symbol.

Since $\delta_{n-i,k-j}$ is not equal to 0 for $n - i = k - j$, then we have

$$\sum_{j=0}^{k}\binom{k}{j}S_2(n-k+j,j)\frac{j!}{(n-k+j)!}(-1)^{k-j}.$$

Thus, the composita of $\frac{e^x-1-x}{x}$ is

$$\sum_{j=0}^{k}\binom{k}{j}S_2(n+j,j)\frac{j!}{(n+j)!}(-1)^{k-j}.$$

Hence, the desired composita of $xH(x)^r$ is

$$H_x^{\Delta}(n,m,r) = \sum_{k=0}^{n-m}\binom{mr}{k}\sum_{j=0}^{k}\binom{k}{j}S_2(n+j,j)\frac{j!}{(n+j)!}(-1)^{k-j}.$$

For $r = -1$, we have the composita of the generating function of the Bernoulli numbers. It is represented in tabular form with parameter n in the rows and parameter m in the columns as follows:

$$\begin{array}{lllll}
1 & & & & \\
\frac{\alpha}{2} & 1 & & & \\
\frac{3\alpha^2+\alpha}{24} & \alpha & 1 & & \\
\frac{\alpha^3+\alpha^2}{48} & \frac{6\alpha^2+\alpha}{12} & \frac{3\alpha}{2} & 1 & \\
\frac{15\alpha^4+30\alpha^3+5\alpha^2-2\alpha}{5760} & \frac{2\alpha^3+\alpha^2}{12} & \frac{9\alpha^2+\alpha}{8} & 2\alpha & 1
\end{array}$$

By the formula (2.1), we introduce an operator $R(n,k,r,H^{\Delta})$ that transforms a formula for the coefficients of $(H(x) - h(0))^k$ to the composita of $xH(x)^r$.

Let us consider the equation (1.2) and represent it in the following way:

$$A(x) = xH(A(x))^r, \tag{2.4}$$

where $A(x) = xB(x)^r$.

The equation (2.4) is a type of the Lagrange inversion formula with a rational power r.

Theorem 2.3. *For the functional equation $B(x) = H(xB(x)^r)$, the solution in term of a composita is defined by*

$$B_x^{\Delta}(n,k,r) = R\left(n,k,\frac{1}{r},A^{\Delta}\right),$$

where $A^{\Delta}(n,k,r) = \frac{k}{n}H_x^{\Delta}(2n-k,n,r)$.

Proof. First, we show that the equation $B(x) = H(xB(x)^r)$ where $h(0) \neq 0$ and r is a rational number has a solution based on the Lagrange inversion theorem.

Getting the r-th power of both parts of the equation and multiplying by x, we get

$$xB(x)^r = xH(xB(x)^r)^r.$$

Substituting $A(x)$ for $xB(x)^r$, we get the following equation:

$$A(x) = xH(A(x))^r. \tag{2.5}$$

Next, using the composita of $xH(x)$ and the operator R, we find the composita of $xH(x)^r$

$$H_x^\Delta(n,k,r) = R(n,k,r,H^\Delta).$$

Solving the equation (2.5) with respect to the composita $A^\Delta(n,k,r)$ of the generating function $A(x)$ [9], we get

$$A^\Delta(n,k,r) = \frac{k}{n} H_x^\Delta(2n-k,n,r).$$

Then, solving the equation $A(x) = xB(x)^r$ with respect to $B(x)$, we get

$$B(x) = \left(\frac{A(x)}{x}\right)^{\frac{1}{r}}$$

or

$$B(x) = \left(1 + \frac{A(x)}{x} - 1\right)^{\frac{1}{r}}.$$

Hence, the composita of $xB(x)$ is

$$B^\Delta(n,k) = R\left(n,k,\frac{1}{r},A^\Delta\right),$$

where $A^\Delta(n,k,r)$ is the composita of the generating function $A(x)$ and $A^\Delta(n,k,r) = \frac{k}{n} H_x^\Delta(2n-k,n,r)$. □

Example 2.4. Applying the obtained results for the equation

$$B(x) = \frac{1}{1 - xB(x)^r},$$

we get that the composita of $xB(x)$ is equal to

$$B^\Delta(n,m,r) = \sum_{k=0}^{n-m} \binom{\frac{m}{r}}{k} \sum_{j=0}^{n-m} \frac{j(-1)^{j-k}}{n-m+j} \binom{k}{j} \binom{r(n-m+j)+n-m-1}{n-m}$$

and the expression for the coefficients of $B(x)$ is equal to

$$b(n,r) = \sum_{k=0}^{n} \binom{\frac{1}{r}}{k} \sum_{j=0}^{n} \frac{j(-1)^{j-k}}{n+j} \binom{k}{j} \binom{r(n+j)+n-1}{n}.$$

Example 2.5. Suppose we have the following equation:

$$B(x) = e^{xB(x)^r}.$$

To find the composita of the generating function $xB(x)$, we get the r-th power of both parts of the equation

$$B(x)^r = \left(e^{xB(x)^r}\right)^r.$$

Replacing $B(x)^r$ by $A(x)$, we get

$$A(x) = e^{xrA(x)}.$$

Since the composita of $H(x) = xe^{rx}$ is

$$H^{\Delta}(n,k,r) = \frac{k^{n-k}}{(n-k)!} r^{n-k},$$

then we get

$$A^{\Delta}(n,k,r) = \frac{k}{n} H^{\Delta}(n,k,r) = \frac{k}{n} \frac{n^{n-k}}{(n-k)!} r^{n-k}.$$

Hence, the desired composita of $xB(x)$ is

$$B^{\Delta}(n,m,r) = \frac{r^{n-m}}{(n-m)!} \sum_{k=0}^{n-m} \binom{\frac{m}{r}}{k} \sum_{j=0}^{k} j(-1)^{j-k} \binom{k}{j} (n-m+j)^{n-m-1}.$$

Using the obtained result, we get the following identities for the composita of $xB(x)$:

$$B^{\Delta}(n,m,r+1) = \frac{m}{n} B^{\Delta}(2n-m,n,r),$$

that is

$$\frac{(\alpha+1)^{n-m}}{(n-m)!} \sum_{k=0}^{n-m} \binom{\frac{m}{\alpha+1}}{k} \sum_{j=0}^{k} j(-1)^{j-k} \binom{k}{j} (n-m+j)^{n-m-1} =$$

$$= \frac{m\alpha^{n-m}}{n(n-m)!} \sum_{k=0}^{n-m} \binom{\frac{n}{\alpha}}{k} \sum_{j=0}^{k} j(-1)^{j-k} \binom{k}{j} (n-m+j)^{n-m-1}$$

or in the general case

$$B^{\Delta}(n,m,r+\beta) = B^{\Delta}((\beta+1)n-\beta m, \beta n-(\beta-1)m, r) \frac{m}{\beta n-(\beta-1)m},$$

where $\beta \in \mathbb{N}$.

3. Solution for the functional equation $A(F(x)) = xH(A(x))$

In this section, we consider another functional equation $A(F(x)) = xH(A(x))$ and find the recurrent solution that will hold in the following theorem:

Theorem 3.1. *Suppose we have a generating function $F(x) = \sum_{n>0} f(n)x^n$ with $f(1) \neq 0$ and its composita $F^{\Delta}(n,k)$, a generating function $H(x) = \sum_{n\geq 0} h(n)x^n$ with $h(0) \neq 0$ and the composita $H^{\Delta}(n,k)$ of the generating function $xH(x)$. Then, for the composita of the generating function defined by the functional equation $A(F(x)) = xH(A(x))$, there hold the following recurrent formula:*

$$A^{\Delta}(n,m) = \begin{cases} \left(\frac{h(0)}{f(1)}\right)^n, & n = m \\ \frac{\sum_{k=1}^{n-m} A^{\Delta}(n-m,k)H^{\Delta}(k+m,m) - F^{\Delta}(n,k+m-1)A^{\Delta}(k+m-1,m)}{F^{\Delta}(n,n)}, & n > m. \end{cases} \quad (3.1)$$

Proof. We rewrite the equation (1.3) as

$$(A(F(x)))^m = (xG(A(x)))^m. \quad (3.2)$$

Next we consider the equation (3.2) in terms of compositae.

According to the formula for the composita of the composition of generating functions [10], the left part has the following composita:

$$L^{\Delta}(n,m) = \sum_{k=m}^{n} F^{\Delta}(n,k)A^{\Delta}(k,m)$$

or

$$L^{\Delta}(n,m) = F^{\Delta}(n,n)A^{\Delta}(n,m) + \sum_{k=1}^{n-m} F^{\Delta}(n,k+m-1)A^{\Delta}(k+m-1,m).$$

For the right part, there hold

$$R^{\Delta}(n,m) = \sum_{k=1}^{n-m} A^{\Delta}(n-m,k)H^{\Delta}(k+m,m).$$

Equating both parts, we get

$$F^{\Delta}(n,n)A^{\Delta}(n,m) =$$

$$\sum_{k=1}^{n-m} A^{\Delta}(n-m,k)H^{\Delta}(k+m,m) - F^{\Delta}(n,k+m-1)A^{\Delta}(k+m-1,m).$$

Since the index of $A(n,m)$ in the left part is less then in the right part and

$$A^{\Delta}(n,n) = \left(\frac{h(0)}{f(1)}\right)^{n},$$

then we get the desired formula for $A^{\Delta}(n,m)$. □

Example 3.2. Let us consider the functional equation

$$A(e^x - 1) = xe^{A(x)}.$$

The composita of the generating function $xH(x) = xe^x$ is

$$H_x^{\Delta}(n,k) = \frac{k^{n-k}}{(n-k)!}.$$

Applying the formulas (2.3) and (3.1), we get the following recurrent formula:

$$A^{\Delta}(n,m) = \begin{cases} 1, & n = m \\ \sum\limits_{k=1}^{n-m} \left(\frac{m^k A^{\Delta}(n-m,k)}{k!} - \frac{(m+k-1)!A^{\Delta}(m+k-1,m)S(n,m+k-1)}{n!}\right), & n > m. \end{cases}$$

This formula calculates the values of the sequence A201777 in [15].

4. Conclusion

Using the properties of a composita and the Lagrange inversion theorem, we have obtained an explicit formula for solving the equation $B(x) = H(xB(x)^r)$, where $H(x)$ and $B(x)$ are generating functions with the free term not equal to 0 and r is a rational number. We have considered the functional equation $A(F(x)) = xH(A(x))$ and found a recurrent solution with respect to $A(x)$ in terms of the composita. Those results can contribute to the development of methods for solving functional and iterative equations related to the Lagrange inversion theorem.

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