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Fractional Derivatives of Logarithmic Singular Functions and Applications to Special Functions

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Abstract

In 1974, Lavoie, Tremblay and Osler (Fundamental properties of fractional derivatives via Pochhammer integrals in 'Fractional calculus and its applications', Lecture Notes in Mathematics No.457, Springer-Verlag, (1974), 323-356) introduced a Pochhammer integral representation in the complex plane for the fractional derivative $D_z^\alpha z^p(\ln\ z)^\delta f(z)$ where $\delta = 0$ or 1. In the same vein, we present integral representations for the fractional derivative of functions with multiple branch-points (complex power, logarithm and their product) $D_{z-z_0}^\alpha U_{\delta,\theta;\,p,q}\,(z-z_0, w-z)\big|_{w=z}^{*}$ where $U_{\delta,\theta;\,p,q}\,(z-z_0, w-z) = f(z-z_0, w-z)(z-z_0)^p(w-z)^q\ [\ln(z-z_0)]^\delta\ [\ln(w-z)]^\theta$ for value δ, $\theta = 0$ or 1 using a Pochhammer contour integral enclosing the singularity points z_0, z and w. The symbol (*) indicates that $w \to z$ inside the Pochhammer contour used for the representation. The transformation formula for the fractional operator $D_{z-z_0}^\alpha U_{\delta,\theta;\,p+r,q}\,(z-z_0, w-z)\big|_{w=z}^{*} = \frac{\Gamma(1+p)}{\Gamma(-\alpha)} D_{z-z_0}^{-p-1} U_{\delta,\theta;r;\,q-\alpha-1}\,(w-z, z-z_0)\big|_{w=z}^{*}$ is derived. Some applications to special functions are given; in particular, a new form of the Leibniz rule is obtained. Another application includes many summation formulas involving the orthogonal polynomials and deduced from the Christoffel-Darboux identity for orthogonal polynomials.

Keywords: Fractional derivatives, Pochhammer contour, Transformation formulas, Special functions, Leibniz rules, Christoffel-Darboux formula, Logarithmic singular function

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1. Introduction

The fractional derivative of order α of the function $f(z)$ with respect to another function $g(z)$, $D_{g(z)}^\alpha f(z)$, is a generalization of the familiar derivative $d^n f(z)/(dg(z))^n$ to non-integral values of n. This concept has been introduced in many different ways, by generalizing the classical definitions of the nth derivative where the order n is replaced by an arbitrary α. We can find many surveys, various applications of 'fractional calculus' and discussions on several of these approaches in texts [24, 37, 38]. A large bibliography can be found in [47]. Fractional calculus has been successfully employed in many topics of mathematical analysis [38], in particular, to solve ordinary [19], partial [7, 36] and integral equations [6]. A large number of formulas from elementary calculus have been shown to be special cases of more general expressions involving fractional derivatives. These include Taylor's series [25, 28, 29], the Leibniz rule [10, 25, 26, 28, 31, 34], the chain rule and applications [27, 42], Lagrange's expansion [28] and others [30, 32, 48, 49].

This fractional operator has been intensively investigated in many directions [23, 25, 38]. To the best of the author's knowledge, the most widely-known representation for the fractional derivative is the Riemann-Liouville

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integral. The properties of this integral representation for a different class of functions have been extensively studied in the past; let us cite the important papers of M. Riesz [36] and of G.H. Hardy and J.E. Littlewood [16, 17, 18]. In the complex plane, the fractional derivative $D^{\alpha}_{g(z)}$ has several representations (formal series and integral). In particular, the Riemann-Liouville integral representation for $g(z) = z - z_0$ takes the following form:

$$D^{\alpha}_{z-z_0}\{(z-z_0)^p f(z)\} = \frac{1}{\Gamma(-\alpha)} \int_{z_0}^{z} f(\xi)(\xi - z_0)^p (z-\xi)^{-\alpha-1} d\xi \tag{1.1}$$

with two 'half-plane' restrictions $\Re(\alpha) < 0$ and $\Re(p) > -1$ and where the integration is done along a straight line from z_0 to z. Integrating by part m times, we obtain

$$D^{\alpha}_{z-z_0}\{(z-z_0)^p f(z)\} = \frac{d^m}{dz^m} D^{\alpha-m}_{z-z_0}\{(z-z_0)^p f(z)\}. \tag{1.2}$$

This allows us to modify the restriction $\Re(\alpha) < 0$ to $\Re(\alpha) < m$ [36]. The representation (1.1) is valid for $z \in R - z_0$ where R is simply the connected open region of analyticity of $f(z)$ containing the point z_0. One of the ways to obtain analytic continuation with respect to parameters of α and p is to use different kinds of loop contour integral representation for $D^{\alpha}_{z-z_0}\{(z-z_0)^p f(z)\}$. For instance, T.J. Osler [25, 26, 27, 28, 29, 30, 31, 32, 33] gave an important improvement for fractional calculus in the complex plane by using the Cauchy integral representation which uses a single loop contour starting and ending at z_0 after enclosing the branch point $\xi = z$.

$$D^{\alpha}_{z-z_0}\{(z-z_0)^p f(z)\} = \frac{\Gamma(1+\alpha)}{2\pi i} \int_{z_0}^{(z+)} f(\xi)(\xi - z_0)^p (\xi - z)^{-\alpha-1} d\xi. \tag{1.3}$$

Note that the Cauchy integral (1.3) was given as early as 1888 by Nekrasov [22]. The use of this contour has the effect of reducing the 'half-plane' restrictions $\Re(\alpha) < 0$ to $\alpha \neq -1, -2, -3, \ldots$.

In [20] (see also [43]), it is shown that a symmetric loop starting from z encircling the branch point z_0 changes the restriction $\Re(p) > -1$ to $p \neq 0, \pm 1, \pm 2, \ldots$ and that a double loop of the Pochhammer type drawn around the two points of branching z_0 and z makes it possible to reduce simultaneously both 'half-plane' restrictions $\Re(\alpha) < 0$ and $\Re(p) > -1$. The most important representations which have been proposed for this concept are reviewed in [21]. In particular, those representations which appear to be of greatest interest for use in exploring the special functions, are presented in detail.

Our interest centers on those functions whose fractional derivatives yield the classical "special functions of mathematical physics". If $\Theta(z)$ is a "special function" (such as an orthogonal polynomial (Laguerre, Hermite, Legendre, Jacobi, etc.), or a function (Bessel, Legendre, hypergeometric, Appell, etc.)), then it is usually possible to represent $\Theta(z)$ by means of fractional differentiation in the form $\Theta(z) = h(z)\, D^{\alpha}_{g(z)} F(z)$, where $g(z)$ is a univalent function, and $h(z)$ and $F(z)$ are functions which are of a more elementary nature than $\Theta(z)$ (See Section 2). For most representations of special functions $\Theta(z)$ with fractional derivatives, we find from experience that $g(z) = z - z_0$ and $F(z)$ must be an analytic function of the form $\Theta(z) = D^{\alpha}_{z-z_0}\, (z-z_0)^p (\ln(z-z_0))^{\delta} f(z)$ with $\delta = 0$ or 1, where $f(z)$ is analytic in a region R containing $z = z_0$. The notation used for the special functions is that of Erdelyi et al. [9]. Table 16.1 shows how we can represent the higher transcendental functions by taking fractional derivatives of more elementary functions. For an extensive table of fractional derivatives, see [9, vol. 2, pp.185-200].

From the set of fractional derivative representations studied in [21], the definition using a Pochhammer contour appears to be more efficient because it has the fewest restrictions on parameters α and p for $\Theta(z) = D^{\alpha}_{z-z_0}\, (z-z_0)^p (\ln(z-z_0))^{\delta} f(z)$ among those using Cauchy contours. In [43, 20], where the representation for fractional derivative using a Pochhammer contour is presumably presented for the first time, the analyticity of $\Theta(z) = D^{\alpha}_{z-z_0}\, (z-z_0)^p (\ln(z-z_0))^{\delta} f(z)$ is investigated in detail with reference to the four variables z, z_0, α, and p.

The primary purpose of this paper is to introduce the new Pochhammer contour integral representation for fractional derivatives which applies to logarithmic functions (Section 3)

$$D^{\alpha}_{z-z_0}\Omega(z) = D^{\alpha}_{z-z_0}\Big\{ f(z)(z-z_0)^p (w-z)^q\, [\ln(z-z_0)]^{\delta}\, [\ln(w-z)]^{\theta}\Big|^{*}_{w=z} \Big\} \tag{1.4}$$

for values $\delta, \theta = 0$ or 1. The use of the symbol (*) is to indicate the fact that $w = z$ is applied inside the integral used for the representation. To deduce the final version of the full representation for the fractional derivative of the branched function $\Omega(z)$, we must successively obtain the integral representations for each of the values of δ and θ. This approach was investigated for the first time by the author in [43]. Note that we can find a similar approach to Campos for the fractional derivative of analytic functions or functions with one or more branch points [4, 5].

In [20, 21], the power and usefulness of the Pochhammer representation of fractional derivatives was demonstrated by using it to prove basic theorems concerning the analytic behavior of $D^{\alpha}_{z-z_0}\{(z-z_0)^p\ \ln(z-z_0)\, f(z)\}$, $\delta = 0$ or $\mathbf{1}$, as functions of the three variables z, α and p. Without the use of the Pochhammer representation, the analyticity of these results would probably be quite lengthy and cumbersome to prove. Also, with the use of the Pochhammer contour representation, it becomes easier to deduce the analytic conditions for the complex functions considered in this paper.

Moreover, the Pochhammer representation defined in Section 3 also allows $f(z)$ to have an essential singularity at z_0. This is not the case for the Riemann-Liouville representation (1.1) and the Cauchy integral representation (1.3) used by Osler. In particular, if $f(z) = 1$ in (1.4), we obtain

$$\begin{aligned} &D^{\alpha}_{z-z_0}\{(z-z_0)^p(w-z)^q\,[\ln(z-z_0)]^{\delta}\,[\ln(w-z)]^{\theta}\}\Big|^{*}_{w=z} \\ &= \frac{\Gamma(1+p)\Gamma(-\alpha+q)}{\Gamma(-\alpha)\Gamma(1+p+q-\alpha)}(z-z_0)^{p+q-\alpha} \\ &\quad \{[\psi(1+p)-\psi(1+p+q-\alpha)+\ln(z-z_0)]^{\delta} \\ &\qquad [\psi(-\alpha+q)-\psi(1+p+q-\alpha)+\ln(z-z_0)]^{\theta} - \delta\theta\psi'(1+p+q-\alpha)\} \end{aligned} \tag{1.5}$$

where $\psi(z) = \Gamma(z)^{'}/\,\Gamma(z)$ is the Psi (or Digamma) function. These results appear presumably for the first time in [43].

In this paper, we do not intensively study the analytic properties of the functions of α, p, q, z_0, z, δ and θ , where in many cases, some singularities are removable. The detailed study of fractional derivative analyticity of these functions with respect to the set of variables and parameters will be the subject of a future paper. However, for each fractional representation, we give the more general conditions on the parameters to guarantee the existence of these functions.

Another observation can also be made about the Pochhammer representation. These representations present in the aggregate (integrand and contour) a perfect symmetry with respect to variables and parameters z_0 and z, p and $-\alpha-1$. This led to the surprising formula of transformation [43, 49]

$$D^{\alpha}_{z-z_0}\{(z-z_0)^p\ f(z)\} = \frac{\Gamma(1+p)}{\Gamma(-\alpha)}\ D^{-p-1}_{z-z_0}\left\{(z-z_0)^{-\alpha-1}f(w+z_0-z)\right\}\Big|_{w=z}. \tag{1.6}$$

The right-hand side of (1.6) also suggests considering the fractional derivative for the more general complex branched function

$$\begin{aligned} &\Lambda_{\,\delta,\theta;\,p,\,q}\,(z-z_0, w-z) = \\ &\qquad \left\{f(z-z_0, w-z)(z-z_0)^p(w-z)^q\,[\ln(z-z_0)]^{\delta}\,[\ln(w-z)]^{\theta}\right\}. \end{aligned} \tag{1.7}$$

One of the main results of this paper is presented in Theorem 4.1 . The proof is established by using the Pochhammer contour representation (3.47). We also give an extension to a more general class of functions. This theorem explicitly gives the full conditions of validity of the following transformation formula for fractional derivatives:

$$\begin{aligned} &D^{\alpha}_{z-z_0}f(z-z_0, w-z)(z-z_0)^{p+r}(w-z)^q\{\ln(z-z_0)\}^{\delta}\{\ln(w-z)\}^{\theta}\Big|^{*}_{w=z} \\ &= \frac{\Gamma(1+p)}{\Gamma(-\alpha)} \\ &D^{-p-1}_{z-z_0}f(w-z, z-z_0)(z-z_0)^{q-\alpha-1}(w-z)^{r}\{\ln(z-z_0)\}^{\theta}\{\ln(w-z)\}^{\delta}|^{*}_{w=z} \end{aligned} \tag{1.8}$$

(which is valid in view of (1.1) for only two 'half-plane' restrictions $\Re(\alpha) > 0$ and $\Re(p) > -1$).

In Section 5, we have chosen two examples of applications among many possibilities to illustrate the effectiveness of representations using Pochhammer outlines. The first application presents a new form of the Leibniz rule for the fractional derivative of the product of two functions is obtained and the analog formula((5.4) and (5.5)). In the second application uses the well-known Christoffel-Darboux formula ([1, Eq.(22.12.1)],[8, vol. 2, p.159, Eq.(10)])

$$\sum_{j=0}^{n} \frac{f_i(x) f_i(y)}{h_i} = \frac{k_n}{h_n k_{n+1}} \frac{f_{n+1}(x) f_n(y) - f_n(x) f_{n+1}(y)}{x - y} \tag{1.9}$$

(where $f_j(x)$ is the jth term of a set of orthogonal polynomials of squared norm h_j and k_j denotes the highest coefficient of $f_j(x)$) and (1.5) (with $z_0 = \delta = \theta = 0$) to obtain new summation formulas involving orthogonal polynomials are obtained ((5.13) and (5.14)).

Other more complicated examples using formulas ((1.4) with $\delta = \theta = 0$) and (1.6) can be found in [10, 11, 12, 13, 14, 40, 43, 49, 50].

Several of these results are obtained by using an associated operator noted ${}_{g(z)}O^{\alpha}_{\beta}$ and called the well-posed fractional operator, which was introduced by the author in [43]. This operator can significantly reduce the number of time-consuming calculations, as it was recently shown in [48]. A publication discussing the analyticity and various properties of this operator is in process of preparation.

No single representation for the fractional derivative is optimal for all applications. Nevertheless, the Pochhammer representation is often the most useful when trying to prove a general theorem on fractional differentiation. The experience of the author indicates that such proofs would be far less efficient and convincing if other representations are used. With these new representations of the fractional derivative, we can easily obtain results on special functions such as summation formulas, summation theorems, links between special functions, and so on.

2. Examples using the fractional derivative for some special functions

We briefly give some examples to show the efficiency of the new representations of the fractional derivative to obtain results on some special functions. For example, using (1.5) with $z_0 = \delta = \theta = 0$, it is easy to demonstrate that

$$\begin{aligned} &D_z^{\alpha} z^p (1 - wz + z^2)^{\beta}\Big|_{w=z}^{*} \\ &\quad = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} z^{p-\alpha} {}_3F_2\left[\begin{matrix} -\beta, 1+p, -\alpha \\ p/2 - \alpha/2 + 1/2, p/2 - \alpha/2 + 1 \end{matrix}\middle| \frac{z^2}{4}\right] \end{aligned} \tag{2.1}$$

where ${}_3F_2$ is a special case of the hypergeometric function ([8], Vol. 1, Eq(1),p. 182)

$${}_pF_q\left[\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q \end{matrix}\middle| z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!}. \tag{2.2}$$

We can also obtain new expressions for many special functions in terms of the fractional derivative. For example,

$$\begin{aligned} &F_3(\alpha, \alpha', \beta, \beta'; \gamma; xz, yz) \\ &\quad = \frac{\Gamma(\gamma)}{\Gamma(\gamma/2)} z^{1-\gamma} D_z^{-\gamma/2} z^{\gamma/2-1} {}_2F_1\left[\begin{matrix} \alpha, \beta \\ \gamma/2 \end{matrix}\middle| xz\right] {}_2F_1\left[\begin{matrix} \alpha', \beta' \\ \gamma/2 \end{matrix}\middle| y(w-z)\right]\Bigg|_{w=z}^{*} \end{aligned} \tag{2.3}$$

where $F_3(\alpha, \alpha', \beta, \beta'; \gamma; x, y)$ represents the third Appell function of two complex variables [8, Vol. 1, Eq. 8, p. 224]. Expressions such as (2.1) and (2.3) are useful because they offer the ability to derive new formulas often by specifying parameters. For example, if we put $z = 2$ in (2.1), we obtain after the following transformation:

$${}_3F_2\left[\begin{matrix} -\beta, 1+p, -\alpha \\ p/2 - \alpha/2 + 1/2, p/2 - \alpha/2 + 1 \end{matrix}\middle| 1\right] = {}_2F_1\left[\begin{matrix} -2\beta, 1+p \\ 1+p-\alpha \end{matrix}\middle| 2\right] \tag{2.4}$$

where $-\beta$ or $1 + p$ or both are negative or zero integers for the convergence. Successively putting $\alpha = -p - 1$ and $\alpha = -p$ in (2.4), we obtain, using the well-known Gauss theorem [35, p. 49]

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}\middle| 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{2.5}$$

the following three summation theorems

$$ {}_2F_1\left[\begin{matrix}-2n, 1+p\\ 1+\delta+2p\end{matrix}\middle| 2\right] = \frac{(1/2)_n}{(p+\delta+1/2)_n}\ (\delta = 0, 1); \qquad {}_2F_1\left[\begin{matrix}-2n-1, 1+p\\ 1+2p\end{matrix}\middle| 2\right] = 0. \tag{2.6} $$

Now, if we put $\alpha = \alpha' = \gamma/2$ in (2.3), we obtain with $z = 1$

$$ F_3(\gamma/2, \gamma/2, \beta, \beta'; \gamma; x, y) = (1-y)^{-\beta'} F_1(\gamma/2; \beta, \beta'; \gamma; x, y/(y-1)) \tag{2.7} $$

a link between the two Appell functions F_3 and F_1 ([3],[8, Vol.1, Eq.(6), p. 224.]). As another example, by putting $\beta' = \gamma/2$, $\alpha' = \gamma/2 - \alpha - \beta$ and $y = -x/(1-x)$ in (2.3), with $z = 1$, $\gamma \to 2\gamma$ and $2\gamma \neq 0, -1, -2, \ldots$, we obtain a presumably new reduction formula :

$$ F_3(\alpha, \gamma-\alpha-\beta, \beta, \gamma; 2\gamma; x, -x/(1-x)) = (1-x)^{\gamma}\, {}_2F_1\left[\begin{matrix}\alpha+\gamma, \beta+\gamma\\ 2\gamma\end{matrix}\middle| x\right]. \tag{2.8} $$

If we put $x = 1/2$ in (2.8) with $\gamma = 1/2 + \alpha/2 + \beta/2$, using the Gauss's second theorem [39, Eq. (III-6), p. 243]

$$ {}_2F_1\left[\begin{matrix}a, b\\ 1/2+a/2+b/2\end{matrix}\middle| \frac{1}{2}\right] = \frac{\Gamma(1/2)\Gamma(1/2+a/2+b/2)}{\Gamma(1/2+a/2)\Gamma(1/2+b/2)}, \tag{2.9} $$

we get

$$ \begin{aligned} &F_3(\alpha, 1/2-\alpha/2-\beta/2, \beta, 1/2-\alpha/2-\beta/2; 1+\alpha+\beta; 1/2, -1) \\ &\quad = \frac{2^{-1/2-\alpha/2-\beta/2}\,\sqrt{\pi}\,\Gamma(1+\alpha+\beta)}{\Gamma(3/4+3\alpha/4+\beta/4)\Gamma(3/4+\alpha/4+3\beta/4)}, \end{aligned} \tag{2.10} $$

Now, with $\beta = 1 - \alpha - 2\gamma$ and using Bailey's theorem [39, Eq. (III-7), p. 243]

$$ {}_2F_1\left[\begin{matrix}a, 1-a\\ b\end{matrix}\middle| 1\right] = \frac{\Gamma(b/2)\Gamma(1/2+b/2)}{\Gamma(b/2+a/2)\Gamma(1/2+b/2-a/2)}, \tag{2.11} $$

we obtain

$$ \begin{aligned} &F_3(\alpha, 3\gamma-1, 1-\alpha-2\gamma, \gamma/2; 2\gamma; 1/2, -1) \\ &\quad = \frac{2^{1-3\gamma}\,\sqrt{\pi}\,\Gamma(2\gamma)}{\Gamma(\gamma/2-\alpha/2+1/2)\Gamma(3\gamma/2+\alpha/2)}. \end{aligned} \tag{2.12} $$

Summation theorems (2.10) and (2.12) seem new. Note that most special functions can be represented in more than one way by fractional derivatives of elementary functions (see [21, Table 17.1, p, 261 and representations of Appell functions and confluent functions of Humbert, p. 260]. For example, we have the Psi function

$$ \psi(x) = -\gamma + \ln z - \Gamma(x) z^{1-x} D_z^{1-x} \ln z, \tag{2.13} $$

$$ = \Gamma(x-1) z^{1-x} D_z^{1-x}[(x-1)\{\ln w - \ln(w-z)\} - 1)] \ln z\big|_{w=z}, \tag{2.14} $$

the Jacobi polynomial [35, Eq.(1), p. 254],

$$ P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(1+\alpha+n)}{\Gamma(1+\alpha+\beta+n)} \frac{(1-z)^{-\alpha}}{2^n n!} D_{1-z}^{\beta+n} (1-z)^{\alpha+\beta+n} (1+z)^n, \tag{2.15} $$

$$ = \frac{(-1)^n \Gamma(1+\beta+n)}{\Gamma(1+\alpha+\beta+n)} \frac{(1+z)^{-\beta}}{2^n n!} D_{1+z}^{\alpha+n} (1+z)^{\alpha+\beta+n} (1-z)^n \tag{2.16} $$

and the third Appell function [8, Vol.1, Eq.(8), p. 224],

$$
\begin{aligned}
F_3(\alpha,\alpha',\beta,\beta';\gamma;x,y) \\
= \quad & \frac{\Gamma(\gamma-\alpha')\Gamma(\gamma-\beta')}{\Gamma(\alpha)\Gamma(\beta)} x^{1+\alpha'-\gamma} D_x^{\alpha+\alpha'-\gamma} x^{\alpha+\beta'-\gamma} D_x^{\beta+\beta'-\gamma} x^{\beta-1} \\
& \left\{(1-x)^{\alpha'+\beta'-\gamma}\, {}_2F_1\left[\begin{matrix}\alpha',\beta' \\ \gamma\end{matrix}\middle| x+y-xy\right]\right\}, \quad (2.17) \\
= \quad & \frac{\Gamma(\gamma-\alpha')\Gamma(\gamma)}{\Gamma(\alpha')\Gamma(\alpha)} y^{1+\alpha-\gamma} D_y^{\alpha+\alpha'-\gamma} y^{\alpha'-1} t^{1-\gamma} D_t^{\alpha-\gamma} t^{\alpha-1} \\
& \left\{(1-xt)^{-\beta}\,(1-y(w-t))^{-\beta'}\right\}\Big|_{w=t}. \quad (2.18)
\end{aligned}
$$

By using extensions of calculus formulas to the higher calculus of fractional derivatives, combined with fractional derivative representations for higher functions, it is possible to produce many interesting results involving the special functions. This observation has appeared previously [13, 15, 19, 25, 26, 27, 28].

It seems reasonable to assume that important properties of higher transcendental functions [12, 25, 26, 27, 28, 29, 30, 31, 41, 42, 44] and extensions of fractional operators [2, 44, 45, 46] could be derived from a knowledge of rules to manipulate fractional derivatives.

For instance, from (2.13) and (2.14), using (1.6) with $z_0 = 0$ and (1.1), it is easy to deduce the following integral representations of functions $\psi(x)$ and $\psi'(x)$:

$$
\begin{aligned}
\psi(x) &= -\gamma + \ln z - (x-1) z^{1-x} D_z^{-1} z^{x-2} \ln(w-z)\Big|_{w=z}, \\
&= -\gamma + \ln z - (x-1) z^{1-x} \int_0^z \xi^{x-2} \ln(z-\xi) d\xi, \qquad (\Re(x) \geq 2) \quad (2.19)
\end{aligned}
$$

$$
\begin{aligned}
\psi'(x) &= z^{1-x} D_z^{-1} z^{x-2} [(x-1)\{\ln w - \ln z\} - 1)] \ln(w-z)\Big|_{w=z} \\
&= z^{1-x} \int_0^z \xi^{x-2} [(x-1)\{\ln z - \ln \xi\} - 1)] \ln(z-\xi) d\xi, \qquad (\Re(x) \geq 2). \quad (2.20)
\end{aligned}
$$

Similarly, from (2.15), we can obtain the following relations for Jacobi polynomials:

$$\{\alpha - (1-z) d/dz\} P_n^{(\alpha,\beta)}(z) = (\alpha+n) P_n^{(\alpha-1,\beta+1)}(z) \quad (2.21)$$

and the application of (2.21) s times on itself gives the following interesting operational formula:

$$\{\alpha - s + 1 - (1-z) d/dz\}_s P_n^{(\alpha,\beta)}(z) = (\alpha+n-s+1)_s P_n^{(\alpha-s,\beta+s)}(z) \quad (2.22)$$

where

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1) & (n \in \mathbb{N}; \alpha \in \mathbb{C}) \\ 1 & (n = 0; \alpha \in \mathbb{C}\{0\}) \end{cases}. \quad (2.23)$$

From (2.17) and using the Euler transformation [35, Eq.(5), p. 60], if we put $\alpha + \alpha' = \beta + \beta' = \gamma$, we quickly find the reduction formula [8, Vol. 1,Eq. (4),p. 238],

$$F_3(\alpha, \gamma-\alpha, \beta, \gamma-\beta; \gamma; x, y) = (1-y)^{\alpha+\beta-\gamma} {}_2F_1\left[\begin{matrix}\alpha,\beta \\ \gamma\end{matrix}\middle| x+y-xy\right]. \quad (2.24)$$

Also, if we set $\gamma = \alpha + \alpha'$ in (2.18), we obtain after some calculations (with $t = 1$)

$$F_3(\alpha,\alpha',\beta,\beta';\alpha+\alpha';x,y) = (1-y)^{-\beta'} F_1(\alpha,\beta,\beta';\alpha+\alpha';x,y/(y-1)), \quad (2.25)$$

a special case but more general than (2.8).

These are three typical examples that demonstrate the effectiveness of fractional calculus in obtaining results in the field of special functions. Many others examples can be easily found [43, See Ch.3].

3. Representations of fractional derivatives via Pochhammer integrals

In this section, we briefly review some known representations for the fractional derivatives in the complex plane. As it was done in [21], we start with the analytic continuation of the Riemann-Liouville representation of $D^{\alpha}_{z-z_0}\{(z-z_0)^p f(z)\}$, which has the two 'half-plane' restrictions $\Re(\alpha) < 0$ and $\Re(p) > -1$, by the Cauchy integral representations (see also [20, 25, 26]). We gradually introduce representations of $D^{\alpha}_{z-z_0}\{(z-z_0)^p f(z)\}$ using known contour integrals to eliminate these restrictions step by step. Finally we get the representation using a Pochhammer contour that has the fewest restrictions on the parameters p, α amd z [20, 21].

In this paper, we introduce a new Pochhammer representation of fractional derivatives in the complex domain of (1.4) which contains multiple valued factors like general power and logarithmic functions. The symbol (*) indicates that the point $\xi = w$ is inside the Pochhammer contour integration in the complex plane ξ considered and w is in the neighborhood of the point $\xi = z$. The contour P is a four loop contour in the complex plane called 'Pochhammer contour' and is given by $P = C_1 \cup C_2 \cup C_3 \cup C_4$. The components of P are shown in Fig. 2, which also shows how the branch lines of the integrand (3.1) pass through the point $\xi = a$ without crossing P at any other point. In this integral, the point $\xi = w$ is inside the loops C_1 and C_3 plotted around the singular point $\xi = z$. The part of the branch line of the multiple valued factor $(w-\xi)^q$ between the points $\xi = w$ and $\xi = a$ is also totally inside the loops C_1 and C_3 (see Fig. 2), such that branch lines of $(z-\xi)^{-\alpha-1}$ and $(w-\xi)^q$ merge when $w \leftarrow z$. Again, this new form is suggested by the symmetry of the Pochhammer contour with regard to branch points z_0 and z (see Fig. 2). Of course, we rediscover all classical representations for the fractional derivative $D^{\alpha}_{z-z_0}\left\{(z-z_0)^p \{\ln(z-z_0)\}^{\delta} f(z)\right\}$ ($\delta = 0$ or 1) presented in the literature using the Pochhammer, Cauchy and Riemann coutour integrals in the complex plane.
We recall that our main interest centers on finding fractional derivatives with respect to $g(z)$ of order α of

$$f(g(z))[g(z)^p - g(w)]^q\{\ln[g(z)]\}^{\delta}\{\ln[g(w)-g(z)]\}^{\theta}\Big|^{*}_{w=z}$$

(see (3.47)), but to simplify, we treat the case $g(z) = z - z_0$. The results obtained can be easily generalized to $g(z)$.

In Theorem 3.1, we express the integral on the contour P as a sum of integrals on the components C_i, $i \in \{1,2,3,4\}$. We start by defining some notation.

Conventions and Notations 3.1 We introduce the following:

1- The region R is an open, simply connected set in the complex plane containing the point $z = g^{-1}(0)$;

2- The function $f(g(z))$ is an analytic function for $z \in R$;

3- The notation $\int_{C(a,b^+)} G(\xi)\, d\xi = \int_{C(a,b^+;G_1,G_2)} G(\xi)\, d\xi$ denotes integration on contours which start at $\xi = a$, where the integrand $G(\xi)$ takes the initial value $G(a) = G_1$ at the start of the integration process and the final value $G(a) = G_2$ after traversing the contour C (Fig. 1). We assume that the contour remains inside the region R and that the integrand $G(\xi)$ varies 'continuously' as we traverse the contour C

4- The integrand $G(\xi)$ will contain multiple valued factors such as $(\xi - z_0)^p$, $(z-\xi)^{\alpha}$, $\ln(\xi - z_0)$, $\ln(z-\xi)$, etc. The branch cut for these functions always passes through the beginning and ending point of the contour of integration, but never cuts the contour elsewhere. Unless otherwise stated, these functions denote the principal branch, which is that continuous range of the function for which $\arg(\xi - z_0)$ (or $\arg(z-\xi)$) is zero when $\xi - z_0$ (or $z - \xi$) is real and positive. In the event that the branch line is $\arg(\xi - z_0) = 0$ (or $\arg(z-\xi) = 0$), then we define the principal branch by $-2\pi < \arg(\xi - z_0)$ (or $\arg(z-\xi) \le 0$).

We begin by considering the following complex integral

$$I = \int_P F_{\delta,\theta}(\xi; z_0; z)d\xi \tag{3.1}$$

with the multiple valued function

$$F_{\delta,\theta}(\xi; z_0; z) = f(\xi)(\xi - z_0)^p (w-\xi)^q \left[\ln(\xi - z_0)\right]^{\delta} \left[\ln(w-\xi)\right]^{\theta} (z-\xi)^{-\alpha-1} \tag{3.2}$$

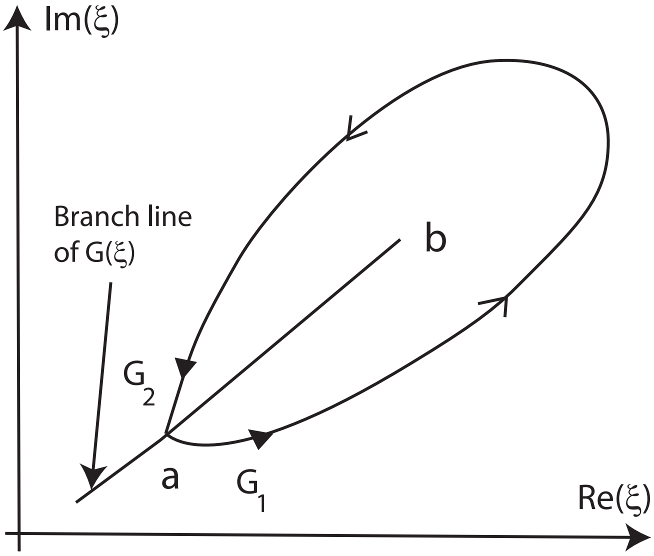

Figure 1. Single-loop contour

where the contour P is a four loop contour in the complex plane called 'Pochhammer contour' and is given by $P = C_1 \cup C_2 \cup C_3 \cup C_4$. The components of P are shown in Fig. 2, and also it shows how the branch lines of the integrand (3.1) pass through the point $\xi = a$ without crossing P at any other point. In this integral, the point $\xi = w$ is inside the loops C_1 and C_3 plotted around the singular point $\xi = z$. The part of the branch line of the multiple valued factor $(w - \xi)^q$ between the points $\xi = w$ and $\xi = a$ is also totally inside the loops C_1 and C_3 (see Fig. 2), such that branch lines of $(z - \xi)^{-\alpha-1}$ and $(w - \xi)^q$ merge when $w \leftarrow z$.

Theorem 3.1. *Let*

$$F_{\delta,\theta}(a; z_0; z) = f(a)(a - z_0)^p (z - a)^{q-\alpha-1} \left[\ln(a - z_0)\right]^{\delta} \left[\ln(z - a)\right]^{\theta} \tag{3.3}$$

where δ, $\theta = 0$ *or* 1, *denote the principal value of the integrand of (3.1) as defined in* Conventions and Notations 3.1, *when we begin to traverse P. Then using the notation adopted in (3.1) we have:*

$$\begin{aligned}
&\int_P F_{\delta,\theta}(\xi; z_0; z)\, d\xi = \\
&(1 - e^{2\pi i p}) \int_{C\left[a, z^+; F_{\delta,\theta}(a; z_0; z),\, F_{\delta,\theta}(a; z_0; z)\, e^{2\pi i(q-\alpha)}(1+2\pi i\theta/\ln(z-a))\right]} F_{\delta,\theta}(\xi; z_0; z)\, d\xi \\
&- 2\pi i\delta e^{2\pi i p} \int_{C\left[a, z^+; F_{0,\theta}(a; z_0; z),\, F_{0,\theta}(a; z_0; z)\, e^{2\pi i(q-\alpha)}(1+2\pi i\theta/\ln(z-a))\right]} F_{0,\theta}(\xi; z_0; z)\, d\xi \\
&\quad + (e^{2\pi i(p+q-\alpha)} - e^{2\pi i p}) \\
&\qquad \int_{C\left[a, z_0^+; F_{\delta,\theta}(a; z_0; z)\, e^{-2\pi i p},\, F_{\delta,\theta}(a; z_0; z)(1+2\pi i\delta/\ln(a-z_0))\right]} F_{\delta,\theta}(\xi; z_0; z)\, d\xi \\
&\quad + 2\pi i\theta e^{2\pi i(p+q-\alpha)} \\
&\qquad \int_{C\left[a, z_0^+; F_{\delta,0}(a; z_0; z)\, e^{-2\pi i p},\, F_{\delta,0}(a; z_0; z)(1+2\pi i\delta/\ln(a-z_0))\right]} F_{\delta,0}(\xi; z_0; z)\, d\xi.
\end{aligned} \tag{3.4}$$

Proof. Using the notation adopted in (3.1) we have:

$$\int_P = \int_{C_1\cup C_2\cup C_3\cup C_4} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

where

$$\begin{aligned}
C_1 &= C\left[a, z^+; F_{\delta,\theta}(a;z_0;z), F_{\delta,\theta}(a;z_0;z)e^{2\pi i(q-\alpha)}\left(1+\frac{2\pi i\theta}{\ln(z-a)}\right)\right]\\
C_2 &= C\left[\begin{array}{l} a, z_0^+;\\ F_{\delta,\theta}(a;z_0;z)e^{2\pi i(q-\alpha)}\left(1+\frac{2\pi i\theta}{\ln(z-a)}\right),\\ F_{\delta,\theta}(a;z_0;z)e^{2\pi i(p+q-\alpha)}\left(1+\frac{2\pi i\theta}{\ln(z-a)}\right)\left(1+\frac{2\pi i\delta}{\ln(a-z_0)}\right)\end{array}\right]\\
C_3 &= C\left[\begin{array}{l} a, z^-;\\ F_{\delta,\theta}(a;z_0;z)e^{2\pi i(p+q-\alpha)}\left(1+\frac{2\pi i\theta}{\ln(z-a)}\right)\left(1+\frac{2\pi i\delta}{\ln(a-z_0)}\right),\\ F_{\delta,\theta}(a;z_0;z)e^{2\pi p}\left(1+\frac{2\pi i\delta}{\ln(a-z_0)}\right)\end{array}\right]\\
C_4 &= C\left[a, z_0^-; F_{\delta,\theta}(a;z_0;z)e^{2\pi p}\left(1+\frac{2\pi i\delta}{\ln(a-z_0)}\right), F_{\delta,\theta}(a;z_0;z)\right].
\end{aligned}$$

Figure 2 shows these four components of P, and also shows how the two branch lines of the integrand of (3.1) both pass through the point $\xi = a$ without crossing P at any other point. We notice that the integrand of (3.1) returns to the initial value of $F_{\delta,\theta}(a;z_0;z)$ after having completely covered the four segments of the Pochhammer contour.

On C_2, we have

$$F_{\delta,\theta}(a;z_0;z)(1+2\pi i\ln(z-a)) = F_{\delta,\theta}(a;z_0;z) + 2\pi i\theta\, F_{\delta,0}(a;z_0;z)$$

therefore the integral becomes, with respect to *Conventions and Notations 3.1*,

$$\begin{aligned}
&\int_{C_2} F_{\delta,\theta}(\xi;z_0;z)\,d\xi =\\
&e^{2\pi i(p+q-\alpha)}\int_{C\left[a,z_0^+;F_{\delta,\theta}(a;z_0;z)\,e^{-2\pi ip},\,F_{\delta,\theta}(a;z_0;z)(1+2\pi i\delta/\ln(a-z_0))\right]} F_{\delta,\theta}(\xi;z_0;z)\,d\xi\\
&+2\pi i\theta e^{2\pi i(p+q-\alpha)}\int_{C\left[a,z_0^+;F_{\delta,0}(a;z_0;z)\,e^{-2\pi ip},\,F_{\delta,0}(a;z_0;z)(1+2\pi i\delta/\ln(a-z_0))\right]} F_{\delta,0}(\xi;z_0;z)\,d\xi.
\end{aligned}$$

Similarly, we can rewrite the integral over C_3 (and C_4) in terms of the integral over C_1 (and C_2). We obtain the following,

$$\begin{aligned}
&\int_{C_3} F_{\delta,\theta}(\xi;z_0;z)\,d\xi = -\int_{C_1} F_{\delta,\theta}(a;z_0;\xi)\,d\xi =\\
&-e^{2\pi ip}\int_{C\left[a,z^+;F_{\delta,\theta}(a;z_0;z),\,F_{\delta,\theta}(a;z_0;z)\,e^{2\pi i(q-\alpha)}(1+2\pi i\theta/\ln(z-a))\right]} F_{\delta,\theta}(\xi;z_0;z)\,d\xi\\
&-2\pi i\delta e^{2\pi ip}\int_{C\left[a,z^+;F_{0,\theta}(a;z_0;z),\,F_{0,\theta}(a;z_0;z)\,e^{2\pi i(q-\alpha)}(1+2\pi i\theta/\ln(z-a))\right]} F_{0,\theta}(\xi;z_0;z)\,d\xi
\end{aligned}$$

and

$$\begin{aligned}
&\int_{C_4} F_{\delta,\theta}(\xi;z_0;z)\,d\xi = -\int_{C_2} F_{\delta,\theta}(a;z_0;\xi)\,d\xi \qquad (3.5)\\
&= -e^{2\pi ip}\int_{C\left[a,z_0^+;F_{\delta,\theta}(a;z_0;z)\,e^{-2\pi ip},\,F_{\delta,\theta}(a;z_0;z)(1+2\pi i\delta/\ln(a-z_0))\right]} F_{\delta,\theta}(\xi;z_0;z)\,d\xi.
\end{aligned}$$

By grouping these integral expressions, we get (3.4). This completes the proof.

Next we examine various special cases of the contour integral (3.4) by deforming the Pochhammer contour integral in different ways. For each case, we try to identify the resulting contour integral with the fractional derivative. By this way, we deduce new Pochhammer integral representations for the fractional derivative of functions having multiple valued factors of types $(z-z_0)^p$, $(w-z)^q$, $\ln(z-z_0)$ or $\ln(w-z)$. Finally we obtain a full Pochhammer integral representation for fractional derivatives of (1.4).

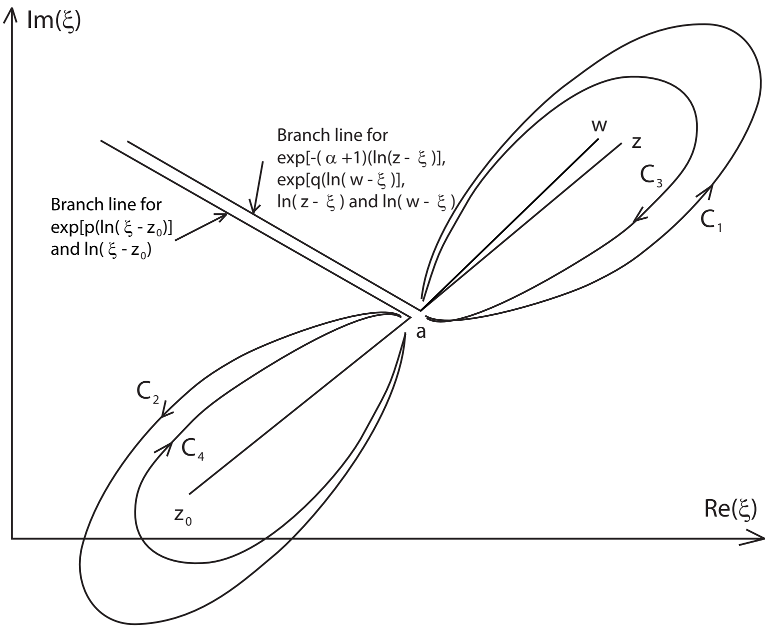

Figure 2. *The four components of the Pochhammer contour.*

3.1. Integral representation of $D^{\alpha}_{z-z_0}\{(z-z_0)^p(w-z)^q f(z)\}|^*_{w=z}$

The first case considered is obtained by putting $\delta = \theta = 0$ in (3.4) . The goal is now to express (3.4) in terms of the fractional derivative for these particular values of δ and θ. We have

$$\begin{aligned}
&\int_P F_{0,0}(\xi; z_0; z)\, d\xi \\
&= (1 - e^{2\pi i p}) \int_{C[a,\, z^+;\, F_{0,0}(a; z_0; z),\, F_{0,0}(a; z_0; z)\, e^{2\pi i(q-\alpha)}]} F_{0,0}(\xi; z_0; z)\, d\xi \\
&+ (e^{2\pi i(p+q-\alpha)} - e^{2\pi i p}) \int_{C[a,\, z_0^+;\, F_{0,0}(a; z_0; z)\, e^{-2\pi i p},\, F_{0,0}(a; z_0; z)]} F_{0,0}(\xi; z_0; z)\, d\xi \\
&= (1 - e^{2\pi i p}) \int_{C[a,\, z^+;\, F_{0,0}(a; z_0; z),\, F_{0,0}(a; z_0; z)\, e^{2\pi i(q-\alpha)}]} F_{0,0}(\xi; z_0; z)\, d\xi \\
&+ (1 - e^{2\pi i(\alpha - q)}) \int_{C[a,\, z_0^+;\, F_{0,0}(a; z_0; z)\, e^{2\pi i(q-\alpha)},\, F_{0,0}(a; z_0; z)\, e^{2\pi i(p+q-\alpha)}]} F_{0,0}(\xi; z_0; z)\, d\xi. \qquad (3.6)
\end{aligned}$$

Now we can deform $C(a, z^+)$ into the union of three contours $C_1 \cup C_2 \cup C_3$ as shown in Fig. 3, where C_1 is a straight line segment from a almost to z, C_2 is a small circle centered at $\xi = z$, C_3 is C_1 traversed in opposite directions. Similarly, the contour $C(a, z_0^+)$ can be deformed into $C_4 \cup C_5 \cup C_6$. The path C_4 is a straight line segment from a to almost z_0, C_5 is a small circle centered at $\xi = z_0$. C_6 is a straight line segment from z_0 almost to a. With the two 'half-plane' restrictions $\Re(\alpha - q) < 0$ and $\Re(p) < -1$, integrals over C_2 and C_5 approach zero as both radii of contours tend to zero.

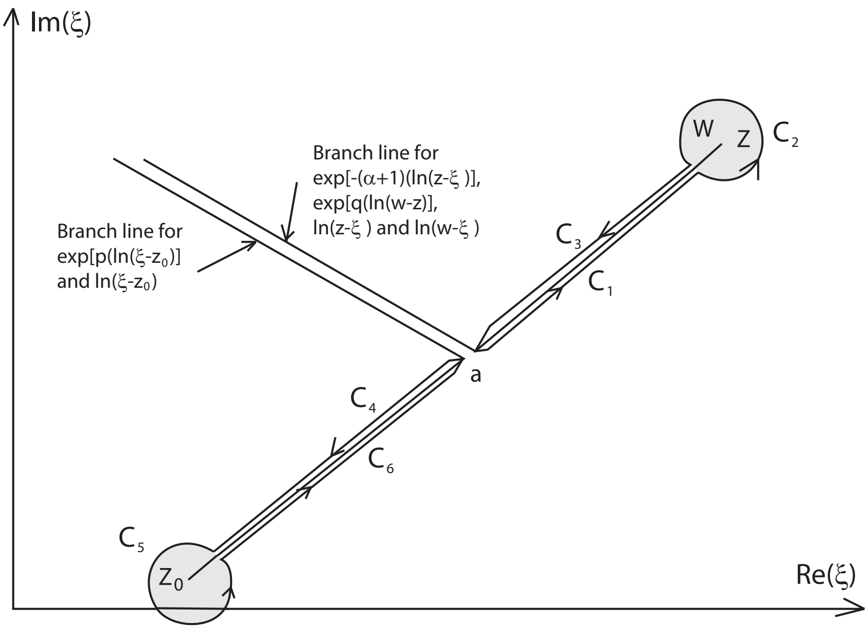

Figure 3. *The contour used in integral* (3.7).

Thus we get

$$\int_P F_{0,0}(\xi;z_0;z)\,d\xi = (1-e^{2\pi ip})[\int_a^z F_{0,0}(\xi;\,z_0;\,z)d\xi + e^{2\pi i(q-\alpha)}\int_z^a F_{0,0}(\xi,\,z_0;\,z)d\xi]$$
$$+\ (1-e^{2\pi i(\alpha-q)})[e^{2\pi i(q-\alpha)}\int_a^{z_0} F_{0,0}(\xi;\,z_0;\,z)d\xi + e^{2\pi i(p+q-\alpha)}\int_{z_0}^a F_{0,0}(\xi;\,z_0;\,z)d\xi]$$
$$= \int_a^z F_{0,0}(\xi)d\xi - e^{2\pi ip}\int_a^z F_{0,0}(\xi)d\xi - e^{2\pi i(q-\alpha)}\int_a^z F_{0,0}(\xi)d\xi$$
$$+\ e^{2\pi i(p+q-\alpha)}\int_a^z F_{0,0}(\xi)d\xi - e^{2\pi i(q-\alpha)}\int_{z_0}^a F_{0,0}(\xi)d\xi + \int_{z_0}^a F_{0,0}(\xi)d\xi$$
$$+\ \int_{z_0}^z F_{0,0}(\xi)d\xi - e^{2\pi ip}\int_{z_0}^z F_{0,0}(\xi)d\xi - e^{2\pi i(q-\alpha)}\int_{z_0}^z F_{0,0}(\xi)d\xi$$
$$+\ e^{2\pi i(p+q-\alpha)}\int_{z_0}^z F_{0,0}(\xi)d\xi$$
$$= (1-e^{2\pi ip})(1-e^{2\pi i(q-\alpha)})\int_{z_0}^z f(\xi)(\xi-z_0)^p(z-\xi)^{-(\alpha-q)-1}d\xi$$
$$= (1-e^{2\pi ip})e^{2\pi i(p+q-\alpha)}\int_{z_0}^a F_{0,0}(\xi)d\xi - e^{2\pi ip}\int_{z_0}^a F_{0,0}(\xi)d\xi$$
$$=\ (1-e^{2\pi i(q-\alpha)})\int_{z_0}^z f(\xi)(\xi-z_0)^p(w-\xi)^q(z-\xi)^{-\alpha-1}d\xi\Big|_{w=z}^{*}. \tag{3.7}$$

Recall that the symbol (*) indicates that $w \to z$ inside the Pochhammer contour. For the integral (3.7), it means $w = z$ before evaluating. Using the Riemann-Liouville integral representation (1.1), we recognize the fractional derivative with $q = 0$ on the right-hand side of the last equality (3.7). Since $(1-e^{2\pi ip}) = -2ie^{i\pi p}\sin p\pi$ and $\Gamma(-\alpha)(1-e^{-2\pi i\alpha}) = -2\pi ie^{-i\pi\alpha}/\Gamma(1+\alpha)$, we rediscover from (3.7) the classical Pochhammer representation for fractional derivative [21, Eq.(13.6), p.256]

$$D^{\alpha}_{z-z_0}\{(z-z_0)^p f(z)\} = \frac{e^{-i\pi(p-\alpha-1)}\Gamma(1+\alpha)}{4\pi\sin\pi p}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{-\alpha-1}d\xi. \tag{3.8}$$

3.1.1. *Integral representation of* $D^{\alpha}_{z-z_0}\ \{(z-z_0)^p(w-z)^q f(z)\}|^*_{w=z}$

With the adopted *Conventions and Notations 3.1*, we have for p, α and $\alpha-q$ not integers and $z \in R-\{z_0\}$

$$\begin{aligned}&D^{\alpha}_{z-z_0}\{(z-z_0)^p(w-z)^q f(z)\}\Big|^*_{w=z}\\ &= \frac{e^{-i\pi(p-\alpha-1)}\Gamma(1+\alpha)}{4\pi\sin\pi p}\left\{\frac{e^{-i\pi q}\sin\alpha\pi}{\sin(\alpha-q)\pi}\right\}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}d\xi\end{aligned} \tag{3.9}$$

where the symbol (*) indicates that w is in the neighborhood of the point $\xi = z$. The point w is inside the loop $C(a,\ z^+)$ (see Fig. 2) which merges with the singular point z when $w \to z$ in the right-hand integral of (3.9).

3.1.2. *Analytic continuation of* $D^{\alpha}_{z-z_0}\{(z-z_0)^p(w-z)^q f(z)\}\Big|^*_{w=z}$

Returning to the right hand side of (3.7) where $\Re(\alpha-q) < 0$ and $\Re(p) > -1$, it is easy to show that

$$\begin{aligned}&\int_{z_0}^{z} f(\xi)(\xi-z_0)^p(w-\xi)^q(z-\xi)^{-\alpha-1}d\xi\Big|^*_{w=z}\\ &= \Gamma(-\alpha)\, D^{\alpha}_{z-z_0}\ \{(z-z_0)^p(w-z)^q f(z)\}|^*_{w=z}\\ &= \left[(w-z_0)^q\int_{z_0}^{z} f(\xi)\sum_{n=0}^{\infty}\frac{(-q)_n}{n!}\frac{(\xi-z_0)^{p+n}}{(w-z_0)^n}(z-\xi)^{-\alpha-1}d\xi\right]\Bigg|_{w=z}\\ &\qquad \left(with\ \left|\frac{\xi-z_0}{w-z_0}\right| < 1\right)\\ &= \sum_{n=0}^{\infty}\frac{(-q)_n}{n!}(z-z_0)^{q-n}\int_{z_0}^{z} f(\xi)(\xi-z_0)^{p+n}(z-\xi)^{-\alpha-1}d\xi.\end{aligned} \tag{3.10}$$

If $f(\xi) = \sum c_k(\xi-z_0)^k$ ($f(\xi)$ is analytic at $\xi - z_0$), we can integrate term by term in (3.10) , using (1.1), to obtain:

$$\begin{aligned}&\int_{z_0}^{z} f(\xi)(\xi-z_0)^p(w-\xi)^q(z-\xi)^{-\alpha-1}d\xi\Big|_{w=z}\\ &= \sum c_k\frac{\Gamma(-\alpha)\Gamma(1+p+k)}{\Gamma(1+p-\alpha+k)}F(-q,\ 1+p+k;\ 1+p-\alpha+k;\ 1)(z-z_0)^{p+q-\alpha+k}.\end{aligned} \tag{3.11}$$

Using the Gauss summation theorem (2.5) and the basic formula ((1.5) with $\delta = \theta = q = 0$) $D^{\alpha}_{z-z_0}(z-z_0)^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha}(z-z_0)^{p-\alpha}$, after elementary simplifications, we can conclude that

$$\begin{aligned}&\Gamma(-\alpha)D^{\alpha}_{z-z_0}\{(z-z_0)^p(w-z)^q f(z)\}|^*_{w=z}\\ &= \int_{z_0}^{z} f(\xi)(\xi-z_0)^p(w-\xi)^q(z-\xi)^{-\alpha-1}d\xi\Big|_{w=z} \quad (\ w = z \text{ before the integration})\\ &= \Gamma(-\alpha+q)D^{\alpha-q}_{z-z_0}(z-z_0)^p f(z) \qquad \text{if } \Re(\alpha-q) < 0.\end{aligned} \tag{3.12}$$

From (3.12) with $\Re(\alpha-q) > 0$, we have

$$D^{\alpha}_{z-z_0}\{(z-z_0)^p(w-z)^q f(z)\}|^*_{w=z} = \frac{\Gamma(-\alpha+q)}{\Gamma(-\alpha)}D^{\alpha-q}_{z-z_0}(z-z_0)^p f(z). \tag{3.13}$$

Also, the right-hand side can be used to obtain an analytic continuation of the left-hand side for all values of parameters α and q except $q-\alpha=0,-1,-2,\dots$ with $\alpha\neq 0,1,2,\dots$. This is justified in (3.9) by using a Pochhammer contour integral plotted around z_0 and z. It is interesting to note that we can write (3.13) in the following form

$$D^{\alpha}_{z-z_0}(z-z_0)^p\frac{(w-z)^q}{\Gamma(q)}f(z)\}\Big|^*_{w=z} = D^{-q}_{z-z_0}(z-z_0)^p\frac{(w-z)^{-\alpha}}{\Gamma(-\alpha)}f(z)\}\Big|^*_{w=z} \tag{3.14}$$

showing that we can interchange the roles of q and $-\alpha$.

Remark 3.2. If q equals zero or a positive (or a negative) integer and α not an integer, we replace the correction factor $\dfrac{e^{-\pi q}\sin\alpha\pi}{\sin(\alpha-q)\pi}$ by 1. This is sufficient to be able to conclude that the Pochhammer representation of the fractional derivative $D^{\alpha}_{z-z_0}\{(z-z_0)^p f(z)\}$ remains valid if the function $f(z)$ has an isolated and essential singularity at $\xi=z$. We also notice that if w is outside the loop $C(a, z^+)$, it is easy to prove that Representation (3.9) remains valid only for $q=0,1,2,\dots$, or with $\Re(-\alpha+q)>0$ if $q=-1,-2,\dots$.

Remark 3.3. If q and α are integers, then

$$\begin{aligned} &D^{\alpha}_{z-z_0}\{(z-z_0)^p(w-z)^q f(z)\}\Big|^*_{w=z}\\ &\quad=\begin{cases}\dfrac{(-1)^q\alpha!}{(\alpha-q)!}D^{\alpha-q}_{z-z_0}(z-z_0)^p f(z) & \alpha\geq q\geq 0\\ 0 & \text{q}\geq\alpha\geq 0.\end{cases}\end{aligned}$$

We remember that we have to put $w=z$ after the operation, as shown with the notation $|^*_{w=z}$ on the left-hand side of (3.13). For example, if $\alpha=2$ and $q=1$, we can verify that

$$\begin{aligned} &D^{2}_{z-z_0}\{(z-z_0)^p(w-z)f(z)\}\Big|^*_{w=z} = D^{2}_{z-z_0}\{(z-z_0)^p(w-z)f(z)\}\Big|_{w=z}\\ &=\frac{-2!}{1!}D_{z-z_0}(z-z_0)^p f(z) = -2p(z-z_0)^{p-1}f(z)-2(z-z_0)^p f^{'}(z).\end{aligned}$$

Remark 3.4. We must pay special attention to the cases $\alpha=0,1,2,\dots$ and $q=-1,-2,-3,\dots$. Recall that we must put $w=z$ in the integrand before integrating (and not after) on the left side of the equation(3.13). Also $\alpha-q>0$, and we must use the analytic continuation formula from (3.13) to obtain the correct significance and value of $D^{\alpha}_{z-z_0}\{(z-z_0)^p(w-z)^q f(z)\}\big|^*_{w=z}$. For example, if $\alpha=2$ and $q=-1$, we have

$$D^{2}_{z-z_0}\{(z-z_0)^p(w-z)^{-1}f(z)\}\Big|^*_{w=z} = -\frac{\Gamma(3)e^{-i\pi p}}{4\pi\sin\pi p}\int_p f(\xi)(\xi-z_0)^p(z-\xi)^{-4}d\xi$$

which is equal to

$$\begin{aligned} &-\frac{1}{3}(z-z_0)^{p-3}[(z-z_0)^3 f^{'''}(z)+3p(z-z_0)^2 f^{''}(z)\\ &+3p(p-1)(z-z_0)f^{'}(z)+p(p-1)(p-2)f(z)].\end{aligned}$$

Note that if we use (3.8) on the right hand side of the identity (3.13), we obtain

$$\begin{aligned} &D^{2}_{z-z_0}\{(z-z_0)^p(w-z)^{-1}f(z)\}\Big|^*_{w=z}\\ &=\lim_{\alpha\to 2}\frac{e^{-i\pi(p-\alpha-2)}}{4\sin\pi p\ \sin(-1-\alpha)\pi\,\Gamma(-\alpha)}\int_p f(\xi)(\xi-z_0)^p(z-\xi)^{-\alpha-2}d\xi\\ &=\lim_{\alpha\to 2}\frac{-e^{-i\pi(p-\alpha-2)}\Gamma(1+\alpha)}{4\pi\sin\pi p}\int_p f(\xi)(\xi-z_0)^p(z-\xi)^{-\alpha-2}d\xi,\end{aligned} \tag{3.15}$$

which is in agreement with the Pochhammer representation integral (3.9).

3.2. Integral representation of $D^{\alpha}_{z-z_0}\left\{(z-z_0)^p(w-z)^q(\ln(z-z_0)^{\delta}f(z)\right|^{*}_{w=z}$

Now returning to (3.4), if $a \to z_0$ with the 'half-plane' restriction $\Re(p) > -1$ and if we set $\theta = 0$, the integral over the contour $C[a,\, z_0]$ becomes zero. The integral expression (3.4) takes the following form:

$$\begin{aligned}&\int_P F_{\delta,0}(\xi;\, z_0;\, z)d\xi = \\ &= (1-e^{2\pi ip})\int_{C[z_0,\,z^+;\,F_{\delta,0}(z_0;z_0;z),\,F_{\delta,0}(z_0;z_0;z)e^{2\pi i(q-\alpha)}]} F_{\delta,0}(\xi;\, z_0;\, z)d\xi \\ &\quad - 2\pi i\delta e^{2\pi ip}\int_{C[z_0,\,z^+;\,F_{0,0}(z_0;z_0;z),\,F_{0,0}(z_0;z_0;z)e^{2\pi i(q-\alpha)}]} F_{0,0}(\xi;\, z_0;\, z)d\xi.\end{aligned} \tag{3.16}$$

Putting $\delta = 0$ in (3.16), we can write

$$\begin{aligned}&\int_{C[z_0,\,z^+;\,F_{0,0}(z_0;z_0;z),\,F_{0,0}(z_0;z_0;z)e^{2\pi i(q-\alpha)}]} f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}d\xi \\ &= \frac{1}{(1-e^{2\pi ip})}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}d\xi.\end{aligned} \tag{3.17}$$

Now we deform the contour of integration $C[z_0,\, z^+]$ into the union of three contours $C_1 \cup C_2 \cup C_3$ as shown in Fig. 4. The path C_1 is a straight line segment from z_0 to almost z, C_2 is a small circle centered at $\xi = z$, C_3 is C_1 traversed in the opposite direction. With the additional 'half-plane' restriction $\Re(\alpha - q) < 0$, the integral over C_2 approaches zero as the radius of the contour tends to zero. Also we must make sure to use the correct branch of the function $(z-\xi)^{q-\alpha-1}$. In accordance with *Conventions and Notations 3.1*, the left-hand side of (3.17) can be written in the following form:

$$\begin{aligned}&\int_{C[z_0,\,z^+;\,F_{0,0}(z_0;z_0;z),\,F_{0,0}(z_0;z_0;z)e^{2\pi i(q-\alpha)}]} f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}d\xi \\ &= (1-e^{2\pi i(q-\alpha)})\int_{z_0}^{z} f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}d\xi \\ &= (1-e^{2\pi i(q-\alpha)})\int_{z_0}^{z} f(\xi)(\xi-z_0)^p(w-\xi)^q(z-\xi)^{-\alpha-1}d\xi\Big|_{w=z} \\ &= (1-e^{2\pi i(q-\alpha)})\Gamma(-\alpha+q)D^{\alpha-q}_{z-z_0}(z-z_0)^pf(z) \\ &= (1-e^{2\pi i(q-\alpha)})\Gamma(-\alpha)D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^qf(z)\Big|_{w=z}.\end{aligned} \tag{3.18}$$

Isolating the fractional derivative expression in the right hand side of (3.18), we obtain:

$$\begin{aligned}&D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^qf(z)\Big|_{w=z} \\ &= \frac{1}{\Gamma(-\alpha)(1-e^{-2\pi i\alpha})}\left\{\frac{(1-e^{-2\pi i\alpha})}{(1-e^{2\pi i(q-\alpha)})}\right\}\int_{z_0}^{(z^+)} f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}d\xi.\end{aligned} \tag{3.19}$$

Now using (3.19) and from (3.17), we finally deduce the following representation:

3.2.1. ***First form of Cauchy integral representation of***

$$D^{\alpha}_{z-z_0}\left\{(z-z_0)^p(w-z)^qf(z)\right\}\Big|^{*}_{w=z}$$

With the adopted *Conventions and Notations 3.1*, we have for $\Re(p) > -1$, α and $\alpha - q$ are not integers and $z \in R - \{z_0\}$

$$\begin{aligned}&D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^qf(z) \\ &= \frac{1}{\Gamma(-\alpha)(1-e^{-2\pi i\alpha})}\left\{\frac{(1-e^{-2\pi i\alpha})}{(1-e^{2\pi i(q-\alpha)})}\right\}\int_{z_0}^{(z^+)} f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}d\xi\end{aligned} \tag{3.20}$$

where the symbol (*) indicates that w is in the neighborhood of the point $\xi = z$. The point w is inside the loop $C(z_0, z^+)$ (see Fig. 4) which merges with the singular point z when $w \to z$ in the right-hand integral of (3.20). In (3.20), it is easy to show that the singularities at $\alpha = -1, -2, \ldots, -N$ and $\alpha - q = -1, -2, \ldots, -M$ (by integrating by parts M times in (3.20)), N and M arbitrary positive numbers, can be removed. The special case $q = 0$ is called

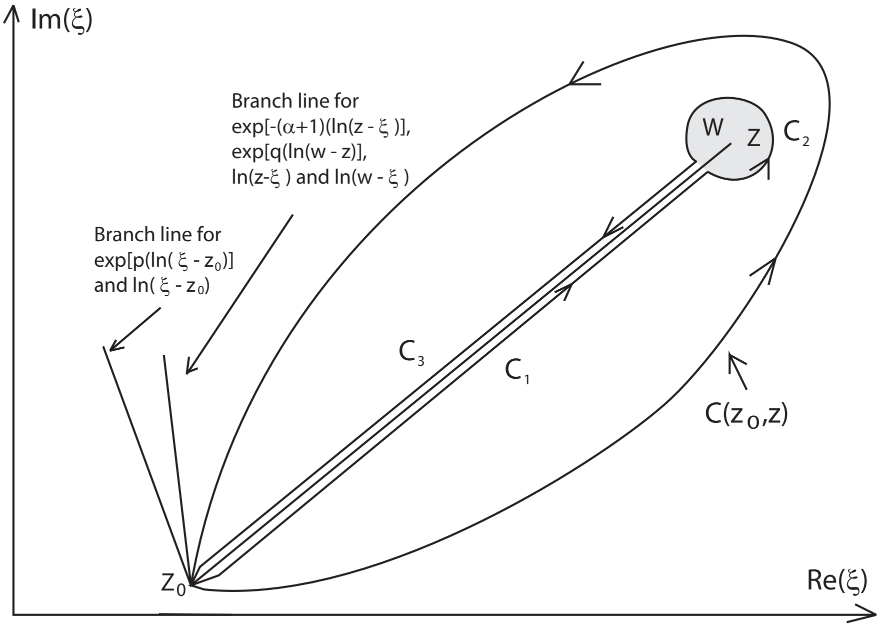

Figure 4. *The contour used in integral* (3.18).

the first Cauchy type of representation for the fractional derivative ([20, Eq. (2.6), p. 327], [21, Eq. (11.3), p. 252]), largely used by Osler [27, 28, 29, 30]:

$$D^{\alpha}_{z-z_0}(z-z_0)^p f(z) = \frac{\Gamma(1+\alpha)e^{i\pi(\alpha+1)}}{2\pi i}\int_{z_0}^{(z^+)} f(\xi)(\xi-z_0)^p(z-\xi)^{-\alpha-1}d\xi \tag{3.21}$$

which is valid for α not a negative integer, $\Re(p) > -1$ and $z \in R - \{z_0\}$. However, note that (3.21) is defined for all values of α. The apparent singularities at $\alpha = -1, -2, -3, \ldots$ due to factor $\Gamma(1+\alpha)$ are removable since in this case the integrand is analytic, and thus the integral is zero. If both α and q are integers in (3.20) then $\dfrac{e^{-i\pi q}\sin\alpha\pi}{\sin(\alpha-q)\pi}$ will be replaced by 1, as was the case for the Pochhammer contour (see Section 3.1.2).

Similarly, $D^{\alpha}_{z-z_0}\{(z-z_0)^p(w-z)^q f(z)\}\Big|^{*}_{w=z} = \dfrac{\Gamma(-\alpha+q)}{\Gamma(-\alpha)}D^{\alpha-q}_{z-z_0}(z-z_0)^p f(z)$ will be used as an analytic continuation if $q - \alpha = 0, -1, -2, \ldots$ with $\alpha \neq 0, 1, 2, \ldots$. In the case where both α and q are positive integers or zero, then $D^{\alpha}_{z-z_0}\{(z-z_0)^p(w-z)^q f(z)\}\Big|^{*}_{w=z}$ will be interpreted as $\dfrac{(-1)^q\alpha!}{(\alpha-q)!}D^{\alpha-q}_{z-z_0}(z-z_0)^p f(z)$ and is zero if $q \geq \alpha$.

Moreover, if we return to (3.16) with $\delta = 1$, we can write

$$\begin{aligned}&\int_{C[z_0,z^+;F_{1,0}(z_0;z_0;z),F_{1,0}(z_0;z_0;z)e^{2\pi i(q-\alpha)}]} F_{1,0}(\xi;z_0;z)d\xi\\ &= \frac{1}{(1-e^{2\pi ip})}\int_P F_{1,0}(\xi;z_0;z)d\xi\\ &\quad+\frac{2\pi i e^{2\pi ip}}{(1-e^{2\pi ip})}\int_{C[z_0,z^+;F_{0,0}(z_0;z_0;z),F_{0,0}(z_0;z_0;z)e^{2\pi i(q-\alpha)}]} F_{0,0}(\xi;z_0;z)d\xi.\end{aligned} \tag{3.22}$$

Multiplying both sides of (3.22) by $\dfrac{\Gamma(1+\alpha)e^{i\pi(\alpha+1)}}{2\pi i}\left\{\dfrac{e^{-i\pi q}\sin\alpha\pi}{\sin(\alpha-q)\pi}\right\}$ and identifying each term obtained with the representation for fractional derivatives (3.20) previously deduced, we obtain the new Pochhammer integral representation.

3.2.2. *Integral representation of*

$$D^{\alpha}_{z-z_0}\left\{(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}f(z)\right\}\Big|^{*}_{w=z}$$

With the adopted *Conventions and Notations 3.1*, we have for p, both α and $\alpha-q$ are not integers and $z\in R-\{z_0\}$

$$\begin{aligned} &D^{\alpha}_{z-z_0}\{(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}f(z)\Big|^{*}_{w=z} \\ &=\frac{\Gamma(1+\alpha)e^{-i\pi(p-\alpha-1)}}{4\pi\sin\pi p}\left\{\frac{e^{-i\pi q}\sin\alpha\pi}{\sin(\alpha-q)\pi}\right\} \\ &\quad\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\{\ln(\xi-z_0)\}^{\delta}d\xi \\ &\qquad-\frac{\delta\pi\, e^{-i\pi(q-\alpha)}}{4\Gamma(-\alpha)\sin^2\pi p\,\sin(\alpha-q)\pi}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}d\xi \end{aligned}\tag{3.23}$$

where the symbol (*) indicates that w is in the neighborhood of the point $\xi=z$. The point w is inside the loop $C(z,z^+)$ (see Fig. 3) which merges with the singular point z when $w\to z$ in the right-hand integral of (3.23). If we put $\delta=q=0$ in (3.23), we obtain once again the classical Pochhammer representation (3.8) already defined by Lavoie-Osler-Tremblay ([21, Eq. (13.6), p. 256], [20], Eq. (3.1), p. 337).

Remark 3.5. If we put $q=0$, we rediscover the Pochhammer representation for fractional derivatives

$$\begin{aligned} &D^{\alpha}_{z-z_0}(z-z_0)^p\{\ln(z-z_0)\}^{\delta}f(z) \\ &=\frac{\Gamma(1+\alpha)e^{-i\pi(p-\alpha-1)}}{4\pi\sin\pi p}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{-\alpha-1}\{\ln(\xi-z_0)\}^{\delta}d\xi \\ &\quad-\delta\frac{\Gamma(1+\alpha)e^{i\pi(\alpha+1)}}{4\sin^2\pi p}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{-\alpha-1}d\xi \end{aligned}\tag{3.24}$$

largely used by Lavoie-Osler-Tremblay [20, 21] which is valid for α not a negative integer, p not an integer and $z\in R-\{z_0\}$. Like for (3.20), it is easy to prove that the singularities at $\alpha=-1,-2,-3,\ldots$ and $\alpha-q=-1,-2,-3,\ldots$ in (3.23) are apparent. This is also the case for singularities at $p=0,1,2,3,\ldots$ in (3.23) and (3.24) which can be removed ([20], Th. 3.2, p. 340). However the singularities $p=-1,-2,-3,\ldots$ are not removable except if we consider the function $(z-z_0)^p/[\Gamma(1+p)]^{1+\delta}$ ([20], Observation 3.1, p. 342). For both α and q integers, the factor $\dfrac{e^{-i\pi q}\sin\alpha\pi}{\sin(\alpha-q)\pi}$ will be replaced by 1. In addition, the transformation formula

$$\begin{aligned} &D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}f(z)\Big|^{*}_{w=z} \\ &=\frac{\Gamma(-\alpha+q)}{\Gamma(-\alpha)}D^{\alpha-q}_{z-z_0}(z-z_0)^p\{\ln(z-z_0)\}^{\delta}f(z) \end{aligned}\tag{3.25}$$

can be used as an analytic continuation for the left-hand side except for $q-\alpha=0,-1,-2,\ldots$ with $\alpha\neq 0,1,2,\ldots$.

Remark 3.6. As in the previous Section 3.1, we must pay special attention when α and q are both integers such that $q<0\leq\alpha$. In this case, we have $\alpha-q>0$ and we must put $w=z$ before differentiating. Also with (3.25) we have

$$\begin{aligned} &D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}f(z)\Big|^{*}_{w=z} \\ &=(-1)^q\frac{\alpha!}{(\alpha-q)!}D^{\alpha-q}_{z-z_0}(z-z_0)^p\{\ln(z-z_0)\}^{\delta}f(z). \end{aligned}\tag{3.26}$$

For example, if we put $\alpha = 1$, $q = -1$ and $\delta = 1$ in (3.26), we obtain

$$D^1_{z-z_0}(z-z_0)^p(w-z)^{-1}\ln(z-z_0)f(z)\Big|^{*}_{w=z} = \frac{-1}{2}D^2_{z-z_0}(z-z_0)^p\ln(z-z_0)f(z) \tag{3.27}$$
$$= \frac{-1}{2}\{[p(p-1)\ln(z-z_0)+2p-1]f(z)+2(z-z_0)[p\ln(z-z_0)+1]f'(z)$$
$$+(z-z_0)^2\ln(z-z_0)f''(z)\}(z-z_0)^{p-2}.$$

Note that if we use (3.24) on the right hand side of the identity (3.25), using the well-known $\Gamma(\alpha)\Gamma(1-\alpha) = \pi/sin(\pi\alpha)$, we obtain

$$D^\alpha_{z-z_0}(z-z_0)^p\{\ln(z-z_0)\}^\delta f(z) \tag{3.28}$$
$$= \frac{e^{-i\pi(p+q-\alpha-1)}}{4\Gamma(-\alpha)\sin\pi p\sin\pi(q-\alpha)}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{-\alpha-1}\{\ln(\xi-z_0)\}^\delta d\xi$$
$$-\delta\frac{\pi e^{i\pi(\alpha-q+1)}}{4\Gamma(-\alpha)\sin^2\pi p\sin\pi(q-\alpha)}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{-\alpha-1}d\xi$$

which, after some elementary transformations, is in agreement with the Pochhammer representation integral (3.23).

More explicitly, for $\alpha = 1$, $q = -1$ and $\delta = 1$, (3.23) becomes

$$D_{z-z_0}\{(z-z_0)^p(w-z)^{-1}\{\ln(z-z_0)\}f(z)\Big|^{*}_{w=z} \tag{3.29}$$
$$= \frac{\Gamma(2)e^{-i\pi p}}{4\pi\sin\pi p}\Big\{\frac{e^{i\pi}\sin\alpha\pi}{\sin(\alpha+1)\pi}\Big\}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{-3}\{\ln(\xi-z_0)\}d\xi$$
$$-\frac{\Gamma(2)}{4\sin^2\pi p}\Big\{\frac{e^{i\pi}\sin\alpha\pi}{\sin(\alpha-q)\pi}\Big\}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{-3}d\xi$$
$$= -\frac{e^{-i\pi p}}{4\pi\sin\pi p}\frac{\partial}{\partial p}\Big[\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{-3}d\xi\Big] + \frac{1}{4\sin^2\pi p}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{-3}d\xi$$
$$+\frac{1}{4\sin^2\pi p}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{-3}d\xi$$
$$= -\frac{1}{2}\Big(\frac{e^{-i\pi p}}{\sin\pi p}\frac{\partial}{\partial p}\Big[e^{i\pi p}\sin\pi p\,D^2_{z-z_0}\{(z-z_0)^p f(z)\Big] + \frac{e^{i\pi p}\pi}{\sin\pi p}D^2_{z-z_0}\{(z-z_0)^p f(z)\Big)$$
$$= -\frac{1}{2}\frac{\partial}{\partial p}\Big(D^2_{z-z_0}\{(z-z_0)^p f(z)\Big)$$
$$= -\frac{1}{2}\frac{\partial}{\partial p}\Big(p(p-1)(z-z_0)^{p-2}f(z)+2p(z-z_0)^{p-1}f'(z)+(z-z_0)^p f''(z)\Big)$$
$$= -\frac{1}{2}\frac{\partial}{\partial p}\Big(p(p-1)(z-z_0)^{p-2}f(z)\Big) - \frac{\partial}{\partial p}\Big(p(z-z_0)^{p-1}f'(z)\Big)$$
$$-\frac{1}{2}\frac{\partial}{\partial p}\Big((z-z_0)^p f''(z)\Big)$$

which is equal to the right-side of (3.27).

3.3. Pochhammer integral representation of

$$D^\alpha_{z-z_0}(z-z_0)^p(w-z)^q\{\ln(w-z)\}^\theta f(z)\Big|^{*}_{w=z}$$

In the previous section, we considered $\theta = 0$ in (3.4). Now we start with $\delta = 0$, the 'half-plane' restriction $\Re(q-\alpha) > 0$ and $a \to z$. Under these conditions, the integral over the contours $C[a,z]$ becomes zero and the integral expression (3.4) reduces to the following form:

$$\int_P F_{0,\theta}(\xi;\, z_0;\, z)d\xi = \tag{3.30}$$
$$= (e^{2\pi i(p+q-\alpha)} - e^{2\pi ip}) \int_{C[z,\, z_0^+;\, F_{0,\theta}(z;z_0;z)e^{-2\pi ip}, F_{0,\theta}(z;z_0;z)]} F_{0,\theta}(\xi;\, z_0;\, z)d\xi$$
$$+ 2\pi i\theta e^{2\pi i(p+q-\alpha)} \int_{C[z,\, z_0^+;\, F_{0,0}(z;z_0;z)e^{-2\pi ip}, F_{0,0}(z;z_0;z)]} F_{0,0}(\xi;\, z_0;\, z)d\xi.$$

Now, if we set $\theta = 0$ in (3.30), this equation is reduced to the following equality

$$\int_{C[z,\, z_0^+;\, F_{0,0}(z;z_0;z)e^{-2\pi ip}, F_{0,0}(z;z_0;z)]} f(\xi)(\xi - z_0)^p (z-\xi)^{q-\alpha-1} d\xi \tag{3.31}$$
$$= \frac{1}{(e^{2\pi i(p+q-\alpha)} - e^{2\pi ip})} \int_P f(\xi)(\xi - z_0)^p (z-\xi)^{q-\alpha-1} d\xi.$$

Now we can deform the contour of integration $C[z,\, z_0^+]$ into $C_4 \cup C_5 \cup C_6$ as shown in Fig. 5. The path C_4 is a straight line segment from z to almost z_0, C_5 is a small circle centered at $\xi = z_0$, C_6 is a straight line segment from z_0 to almost z. If we add the 'half-plane' restriction $\Re(p) > -1$, the complex integral over C_5 approaches zero as the radius of the contour tends to zero. Choosing the correct branch of the function $(\xi - z_0)^p$ with respect to the *Conventions and Notations 3.1*, the left-hand side of (3.31) can be written in the following form

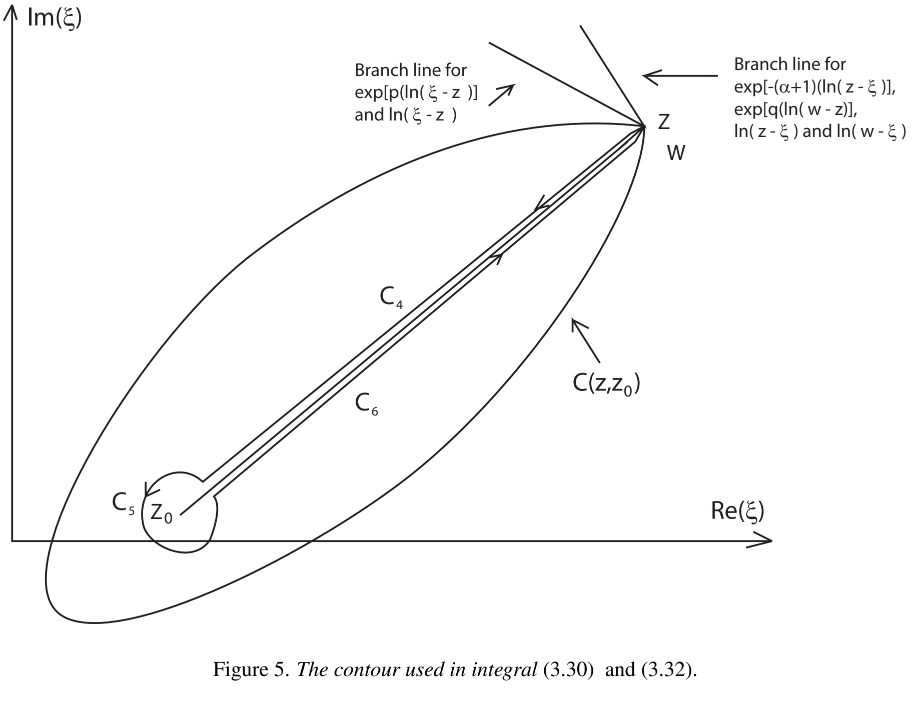

Figure 5. *The contour used in integral* (3.30) and (3.32).

$$\int_{C[z,\, z_0^+;\, F_{0,0}(z;z_0;z)e^{-2\pi ip}, F_{0,0}(z;z_0;z)]} f(\xi)(\xi - z_0)^p (z-\xi)^{q-\alpha-1} d\xi \tag{3.32}$$
$$= (1 - e^{-2\pi ip}) \int_{z_0}^{z} f(\xi)(\xi - z_0)^p (z-\xi)^{q-\alpha-1} d\xi$$
$$= (1 - e^{-2\pi ip})\Gamma(-\alpha + q) D_{z-z_0}^{\alpha-q} (z - z_0)^p f(z)$$
$$= (1 - e^{-2\pi ip})\Gamma(-\alpha) D_{z-z_0}^{\alpha} (z - z_0)^p (w - z)^q f(z)\Big|_{w=z}.$$

Also, the fractional derivative term on the right-hand side of (3.32) can be isolated and we can write

$$D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q f(z)\Big|_{w=z} = \frac{1}{(1-e^{-2\pi i p})\Gamma(-\alpha)} \int_{C[z,\,z_0^+;\,F_{0,0}(z;z_0;z)e^{-2\pi i p},F_{0,0}(z;z_0;z)]} f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}d\xi. \tag{3.33}$$

We finally deduce the following representation:

3.3.1. *Second form of Cauchy representation of*

$$D^{\alpha}_{z-z_0}\left\{(z-z_0)^p(w-z)^q f(z)\right\}\Big|^{*}_{w=z}$$

With the adopted*Conventions and Notations 2.1*, we have for $\Re(\alpha-q)<0$, p not an integer and $z\in R-\{z_0\}$

$$D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q f(z)\Big|_{w=z} = \frac{e^{i\pi p}}{2i\sin\pi p\Gamma(-\alpha)}\int_z^{(z_0^+)} f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}d\xi. \tag{3.34}$$

The point w is outside the loop $C(z,\,z_0^+)$ which merges with the singular point z when $w\to z$ on the right hand side of (3.33).

Remark 3.7. At the beginning of the contour around z_0, we have
$(z-z_0)^p = e^{p(\ln|z-z_0|+i\arg(z-z_0)-2\pi i)}$, and at the end we have
$(z-z_0)^p = e^{p(\ln|z-z_0|+i\arg(z-z_0))}$.

Remark 3.8. We note the missing symbol (*) in (3.33) and (3.34). We can omit it because the contour $C(z,\,z_0^+)$ and $C(z,\,g^{-1}(0)^+)$ (see Fig. 2.4) passes through the point $\xi=z$. For this reason, we must also have the 'half-plane' restriction $\Re(\alpha-q)<0$. Note also that the contour of integration in (3.34) does not pass through $\xi=z_0$, and thus the integral gives no restriction on p. Restrictions on p come from the constant $e^{i\pi p}/2i\sin\pi p$. We can however show that the singularities at $p=0,1,2,\ldots,M$ (with M an arbitrary positive number) can be removed. On the other hand, the singularities at $p=-1,-2,-3,\ldots$ are not removable except if we consider the function $(z-z_0)^p/\Gamma(1+p)$ instead of $(z-z_0)^p$.

The special case $q=0$ is called the second Cauchy representation for the fractional derivative, introduced for the first time by Lavoie-Osler-Tremblay ([21, Eq.(12.1), p. 252], [20, Eq. (2.7), p.327])

$$D^{\alpha}_{z-z_0}(z-z_0)^p f(z) = \frac{e^{i\pi p}}{2i\sin\pi p\Gamma(-\alpha)}\int_z^{(z_0^+)} f(\xi)(\xi-z_0)^p(z-\xi)^{-\alpha-1}d\xi \tag{3.35}$$

which is valid for $\Re(\alpha)<0$, p not an integer and $z\in R-\{z_0\}$. More generally, returning to (3.30) with $\theta=1$, we can write

$$\begin{aligned}&\int_{C[z,\,z_0^+;\,F_{0,1}(z;z_0;z)e^{-2\pi i p},F_{0,1}(z;z_0;z)]} f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-q}\ln(z-\xi)d\xi \\ &= \frac{1}{(e^{2\pi i(p+q-\alpha)}-e^{2\pi i p})}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\ln(z-\xi)d\xi \\ &\quad -\frac{2\pi i e^{2\pi i(p+q-\alpha)}}{(e^{2\pi i(p+q-\alpha)}-e^{2\pi i p})}\int_{C[z,\,z_0^+;\,F_{0,0}(z;z_0;z)e^{-2\pi i p},F_{0,0}(z;z_0;z)]} f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}d\xi.\end{aligned} \tag{3.36}$$

Multiplying both sides of (3.36) by $\dfrac{e^{i\pi p}}{2i\sin\pi p\Gamma(-\alpha)}$, using (3.33) and (3.35), we obtain

$$\begin{aligned}&D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q\ln(w-z)f(z)\Big|_{w=z}\\&=\frac{e^{i\pi p}}{2i\Gamma(-\alpha)(e^{2\pi i(p+q-\alpha)}-e^{2\pi ip})\sin\pi p}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\ln(z-\xi)d\xi\\&\quad-\frac{2\pi ie^{2\pi i(p+q-\alpha)}}{(e^{2\pi i(p+q-\alpha)}-e^{2\pi ip})}D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^qf(z)\Big|_{w=z}.\end{aligned}\tag{3.37}$$

By using the Pochhammer representation integral (3.20) on the right hand side of (3.37) we obtain

$$\begin{aligned}&D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q\ln(w-z)f(z)\Big|_{w=z}\\&=\frac{e^{i\pi p}}{2i\Gamma(-\alpha)(e^{2\pi i(p+q-\alpha)}-e^{2\pi ip})\sin\pi p}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\ln(z-\xi)d\xi\\&\quad-\frac{2\pi ie^{2\pi i(p+q-\alpha)}}{(e^{2\pi i(p+q-\alpha)}-e^{2\pi ip})}\frac{e^{-i\pi(p-\alpha-1)}\Gamma(1+\alpha)}{4\pi\sin\pi p}\left\{\frac{e^{-i\pi q}\sin\alpha\pi}{\sin(\alpha-q)\pi}\right\}\\&\qquad\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-q}d\xi.\end{aligned}\tag{3.38}$$

After making simplifications, we obtain from (3.38) the new Pochhammer representation for

$$D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q\{\ln(w-z)\}^{\theta}f(z)\Big|^{*}_{w=z}.$$

3.3.2. *Pochhammer integral representation of*

$$D^{\alpha}_{z-z_0}\left\{(z-z_0)^p(w-z)^q\{\ln(w-z)\}^{\theta}f(z)\right\}\Big|^{*}_{w=z}$$

With the adopted *Conventions and Notations 3.1* for $p\neq 0,\pm1,\pm2,\ldots$ and $q-\alpha\neq 0,\pm1,\pm2,\ldots$ and $z\in R-\{z_0\}$ we have

$$\begin{aligned}&D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q\{\ln(w-z)\}^{\theta}f(z)\Big|^{*}_{w=z}\\&=\frac{e^{i\pi(\alpha-p-1)}\Gamma(1+\alpha)}{4\pi\sin\pi p}\left\{\frac{e^{-i\pi q}\sin\alpha\pi}{\sin(\alpha-q)\pi}\right\}\int_p f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\{\ln(z-\xi)\}^{\theta}d\xi\\&\qquad+\frac{\pi\theta e^{-i\pi p}}{4\Gamma(-\alpha)\sin\pi p\ \sin^2(q-\alpha)\pi}\int_p f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\,d\xi\end{aligned}\tag{3.39}$$

where the symbol (*) indicates that w is in the neighborhood of the point $\xi=z$. The point w is inside the loop $C(a,z^{+})$ (see Fig. 3) which merges with the singular point z when $w\to z$ in the right-hand integral of (3.39). If we put $\theta=q=0$, we obtain once again the classical Pochhammer representation (3.9) already defined by Lavoie-Osler-Tremblay ([20, Eq. (2.9), p. 331], [21, Eq. (13.7), p. 256]). Again, we can prove that singularities at $p=0,1,2,\ldots$ can be removed. If $\theta=0$, we find (3.9) and the singularities at $q-\alpha=0,-1,-2,\ldots$ with $\alpha=0,1,2,\ \ldots$ can be removed. If $\theta=1$, we have

$$\begin{aligned}&D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q\{\ln(w-z)\}f(z)\Big|^{*}_{w=z}\\&\qquad=\frac{\Gamma(-\alpha+q)}{\Gamma(-\alpha)}D^{\alpha-q}_{z-z_0}(z-z_0)^p\{\ln(w-z)\}f(z)\Big|^{*}_{w=z}\end{aligned}\tag{3.40}$$

which can be used as an analytic continuation for the left-hand side except for the case $q-\alpha=0,-1,-2,\ldots$ with $\alpha\neq 0,1,2,\ldots$. For $\alpha=0,\pm1,\pm2,\pm3,\ldots$, the representation is valid if $\alpha-q\neq 0,1,2,\ldots$, we will use (3.40) as an analytic continuation. If both α and $q-\alpha$ positive integers or zero then $D^{\alpha}_{z-z_0}\{(z-z_0)^p(w-z)^q\ln(w-z)f(z)\}\Big|^*_{w=z}=0$. Finally, we can show that function

$$D^{\alpha}_{z-z_0}\Big\{\frac{(z-z_0)^p}{\Gamma(1+p)}\frac{(w-z)^q}{\Gamma(q-\alpha)^{1+\theta}}\{\ln(w-z)\}^{\theta}f(z)\Big\}\Big|^*_{w=z}$$

is an entire function of p and $q-\alpha$.

3.4. Pochhammer integral representation of

$$D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}\{\ln(w-z)\}^{\theta}f(z)\Big|^*_{w=z}$$

Now consider the general case obtained from (3.4) with both $\delta\neq 0$ and $\theta\neq 0$. Again, in (3.4), if we let $a\to z_0$ with $\Re(p)>-1$ (see Fig. 4), integrals over $C[a,\,z_0^+]$ vanish and we can write

$$\begin{aligned}&\int_{C\left[z_0,\,z^+;\,F_{\delta,\theta}(z_0;\,z_0;\,z),\,F_{\delta,\theta}(z_0;\,z_0;\,z)\,e^{2\pi i(q-\alpha)}(1+2\pi i\theta/\ln(z-z_0))\right]}F_{\delta,\theta}(\xi;z_0;z)\,d\xi \qquad (3.41)\\&=\frac{1}{(1-e^{2\pi ip})}\int_p F_{\delta,\theta}(\xi;z_0;z)\,d\xi+\frac{2\pi i\delta e^{2\pi ip}}{(1-e^{2\pi ip})}\\&\int_{C\left[z_0,\,z^+;\,F_{0,\theta}(z_0;\,z_0;\,z),\,F_{0,\theta}(z_0;\,z_0;\,z)\,e^{2\pi i(q-\alpha)}(1+2\pi i\theta/\ln(z-z_0))\right]}F_{0,\theta}(\xi;z_0;z)\,d\xi.\end{aligned}$$

Now we can deform the contour of integration $C[z_0,\,z^+]$ into $C_1\cup C_2\cup C_3$ as shown in Fig. 3. The contour C_1 is a straight line segment from z_0 to almost z, C_2 is a small circle centered at $\xi=z$, C_3 is C_1 traversed in opposite direction. With the additional 'half-plane' restriction $\Re(\alpha-q)<0$, the integral over C_2 approaches zero as the radius of contour tends to zero. Again we must make sure that the correct branches of function $(z-\xi)^{q-\alpha-1}$ and $\ln(z-\xi)$ are used. In accordance with *Conventions and Notations 3.1*, the left-hand side of (3.41) can be written in the following form

$$\begin{aligned}&\int_{C\left[z_0,\,z^+;\,F_{\delta,\theta}(z_0;\,z_0;\,z),\,F_{\delta,\theta}(z_0;\,z_0;\,z)\,e^{2\pi i(q-\alpha)}(1+2\pi i\theta/\ln(z-z_0))\right]}F_{\delta,\theta}(\xi;z_0;z)\,d\xi \qquad (3.42)\\&=\int_{z_0}^{z}f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\{\ln(\xi-z_0)\}^{\delta}\{\ln(z-\xi)\}^{\theta}d\xi\\&\quad-e^{2\pi i(q-\alpha)}\left[\int_{z_0}^{z}f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\{\ln(\xi-z_0)\}^{\delta}\{\ln(z-\xi)\}^{\theta}d\xi\right.\\&\qquad\left.+2\pi i\theta\int_{z_0}^{z}f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\{\ln(\xi-z_0)\}^{\delta}\,d\xi\right]\\&=(1-e^{2\pi i(q-\alpha)})\int_{z_0}^{z}f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\{\ln(\xi-z_0)\}^{\delta}\{\ln(z-\xi)\}^{\theta}d\xi\\&\quad-2\pi i\theta e^{2\pi i(q-\alpha)}\int_{z_0}^{z}f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\{\ln(\xi-z_0)\}^{\delta}d\xi.\end{aligned}$$

From the Riemann-Liouville representation (1.1), we can identify the fractional derivatives term on the left-hand side of (3.42). Thus we have

$$\begin{aligned}&\int_{C[z_0,z^+;F_{\delta,\theta}(z_0;z_0;z),F_{\delta,\theta}(z_0;z_0;z)e^{2\pi i(q-\alpha)}(1+2\pi i\theta\ln(z-a))]}F_{\delta,\theta}(\xi;z_0;z)d\xi \qquad (3.43)\\&=(1-e^{2\pi i(q-\alpha)})\\&\quad\Gamma(-\alpha)D^{\alpha}_{z-z_0}f(z)(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}\{\ln(w-z)\}^{\theta}\Big|^*_{w=z}\\&\qquad-2\pi i\theta\Gamma(-\alpha)e^{2\pi i(q-\alpha)}D^{\alpha}_{z-z_0}f(z)(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}\Big|^*_{w=z}.\end{aligned}$$

We now return to (3.41) with (3.42) and we can write

$$
\begin{aligned}
&(1-e^{2\pi i(q-\alpha)})\Gamma(-\alpha)D^{\alpha}_{z-z_0}f(z)(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}\{\ln(w-z)\}^{\theta}\Big|^{*}_{w=z} \\
&\quad -2\pi i\theta\,\Gamma(-\alpha)e^{2\pi i(q-\alpha)}D^{\alpha}_{z-z_0}f(z)(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}\Big|^{*}_{w=z} \\
&=\frac{1}{(1-e^{2\pi ip})}\int_P F_{\delta,\theta}(\xi;\,z_0;\,z)d\xi \\
&+\frac{2\pi i\delta e^{2\pi ip}}{(1-e^{2\pi ip})}\Big[(1-e^{2\pi i(q-\alpha)})\Gamma(-\alpha)D^{\alpha}_{z-z_0}f(z)(z-z_0)^p(w-z)^q\{\ln(w-z)\}^{\theta}\Big|^{*}_{w=z} \\
&\quad -2\pi i\theta\,\Gamma(-\alpha)e^{2\pi i(q-\alpha)}D^{\alpha}_{z-z_0}f(z)(z-z_0)^p(w-z)^q\Big|^{*}_{w=z}\Big]
\end{aligned}
\tag{3.44}
$$

which can be rewritten in the following form

$$
\begin{aligned}
&D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}\{\ln(w-z)\}^{\theta}f(z)\Big|^{*}_{w=z} \\
&=\frac{e^{i\pi(\alpha-p+1)}\Gamma(1+\alpha)}{4\pi\sin\pi p}\Big\{\frac{e^{-i\pi q}\sin\alpha\pi}{\sin(\alpha-q)\pi}\Big\} \\
&\quad\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\{\ln(\xi-z_0)\}^{\delta}\{\ln(z-\xi)\}^{\theta}d\xi \\
&\quad-\frac{\pi\theta e^{i\pi(q-\alpha)}}{\sin(q-\alpha)\pi}D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}f(z)\Big|^{*}_{w=z} \\
&\quad-\frac{\pi\delta e^{i\pi p}}{\sin\pi p}D^{\alpha}_{z-z_0}f(z)(z-z_0)^p(w-z)^q\{\ln(w-z)\}^{\theta}f(z)\Big|^{*}_{w=z} \\
&\quad+\frac{\pi^2\delta\theta e^{i\pi(p+q-\alpha)}}{\sin\pi p\ \sin(\alpha-q)\pi}D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q f(z)\Big|^{*}_{w=z}.
\end{aligned}
\tag{3.45}
$$

If we use the new Pochhammer integrals representations (3.23), (3.39) and (3.9) in (3.45), we obtain the next full Pochhammer representation for the general case of the fractional derivative.

3.4.1. ***Pochhammer integral representation of***

$$D^{\alpha}_{z-z_0}(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}\{\ln(w-z)\}^{\theta}f(z)\Big|^{*}_{w=z}$$

With the adopted *Conventions and Notations 3.1* we have for p, both α and $\alpha-q$ are not integers $z\in R-\{z_0\}$

$$
\begin{aligned}
&D^{\alpha}_{z-z_0}f(z)(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}\{\ln(w-z)\}^{\theta}\Big|^{*}_{w=z} \\
&=\frac{e^{i\pi(\alpha-p+1)}\Gamma(1+\alpha)}{4\pi\sin\pi p}\left\{\frac{e^{-i\pi q}\sin\alpha\pi}{\sin(\alpha-q)\pi}\right\} \\
&\quad\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\{\ln(\xi-z_0)\}^{\delta}\{\ln(z-\xi)\}^{\theta}d\xi \\
&+\theta\frac{\pi e^{-i\pi p}}{4\Gamma(-\alpha)\sin\pi p\ \sin^2(\alpha-q)\pi}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\{\ln(\xi-z_0)\}^{\delta}d\xi \\
&-\delta\frac{\pi e^{-i\pi(q-\alpha)}}{4\Gamma(-\alpha)\sin^2\pi p\ \sin(\alpha-q)\pi}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}\{\ln(z-\xi)\}^{\theta}d\xi \\
&-\delta\theta\frac{\pi^2}{4\Gamma(-\alpha)\sin^2\pi p\ \sin^2(\alpha-q)\pi}\int_P f(\xi)(\xi-z_0)^p(z-\xi)^{q-\alpha-1}d\xi
\end{aligned}
\tag{3.46}
$$

where the symbol (*) indicates that w is in the neighborhood of the point $\xi=z$. The point w is inside the loop $C(a,\,z^+)$ (see Fig .2) which merges with the singular point z when $w\to z$ in the right hand integral of (3.46). Restrictions and extensions of (3.47) are the same as the ones in equation (3.46). For each choice $\delta=0/1$, $\theta=0/1$, we find the set of previous representations of the fractional operator.

3.4.2. ***Extension of the Pochhammer integral representation to*** $D^{\alpha}_{g(z)}$

For the derivative of order α with respect to an arbitrary function $g(z)$, we have

$$D^{\alpha}_{g(z)} f(z)[g(z)^p - g(w)]^q \{\ln[g(z)]\}^{\delta}\{\ln[g(w) - g(z)]\}^{\theta}\Big|^*_{w=z} \tag{3.47}$$
$$= \frac{e^{i\pi(\alpha-p+1)}\Gamma(1+\alpha)}{4\pi\sin\pi p}\left\{\frac{e^{-i\pi q}\sin\alpha\pi}{\sin(\alpha-q)\pi}\right\}$$
$$\int_P f(\zeta)[g(\zeta)]^p[g(z)-g(\zeta)]^{q-\alpha-1}\{\ln[g(\zeta)\}^{\delta}\{\ln[g(z)-g(\zeta)]\}^{\theta} g(\zeta)d\zeta$$
$$+\,\theta\frac{\pi e^{-i\pi p}}{4\Gamma(-\alpha)\sin\pi p\ \sin^2(\alpha-q)\pi}\int_P f(\zeta)[g(\zeta)]^p[g(z)-g(\zeta)]^{q-\alpha-1}\{\ln[g(\zeta)]^{\delta} g(\zeta)d\zeta$$
$$-\,\delta\frac{\pi e^{-i\pi(q-\alpha)}}{4\Gamma(-\alpha)\sin^2\pi p\ \sin(\alpha-q)\pi}$$
$$\int_P f(\zeta)[g(\zeta)]^P[g(z)-g(\zeta)]^{q-\alpha-1}\{\ln[g(z)-g(\zeta)]\}^{\theta} g(\zeta)d\zeta$$
$$-\,\delta\theta\frac{\pi^2}{4\Gamma(-\alpha)\sin^2\pi p\ \sin^2(\alpha-q)\pi}\int_P f(\zeta)[g(\zeta)]^p[g(z)-g(\zeta)]^{q-\alpha-1} g(\zeta)d\zeta$$

obtained by a simple change of variables $\xi = g(\zeta)$ in (3.46) with $z_0 = 0$, and after replacing $f[g(\zeta)]$ by $f(\zeta)$. The Pochhammer contour P is plotted with respect to the point $g^{-1}(0)$ and z. Once again, if both α and q are integers, the factor $\dfrac{e^{-i\pi q}\sin\alpha\pi}{\sin(\alpha-q)\pi}$ will be replaced by 1. The singularities at $p = 0, 1, 2, \ldots, M$ (an arbitrary positive number) can be removed. Except for $q - \alpha = 0, -1, -2, \ldots$, with $\alpha \neq 0, 1, 2, \ldots$, we will use the analytic continuation

$$D^{\alpha}_{z-z_0}\Big\{(z-z_0)^p(w-z)^q\{\ln(z-z_0)\}^{\delta}\{\ln(w-z)\}^{\theta} f(z)\Big\}\Big|^*_{w=z} \tag{3.48}$$
$$= \frac{\Gamma(-\alpha+q)}{\Gamma(-\alpha)} D^{\alpha-q}_{z-z_0}\Big\{(z-z_0)^p\{\ln(z-z_0)\}^{\delta}\{\ln(w-z)\}^{\theta} f(z)\Big\}\Big|^*_{w=z}.$$

Moreover, we can show that the function

$$D^{\alpha}_{z-z_0}\Big\{\frac{(z-z_0)^p}{\Gamma(1+p)^{1+\delta}}\frac{(w-z)^q}{\Gamma(q-\alpha)^{1+\theta}}\{\ln(z-z_0)\}^{\delta}\{\ln(w-z)\}^{\theta} f(z)\Big\}\Big|^*_{w=z}$$

is an entire function of p and $q - \alpha$.

4. A transformation formula for fractional derivatives.

In this section, we deduce a transformation formula obtained by noticing the symmetry with respect to the pair of parameters $\alpha + 1$ and p and the pairs of functions $\{(\xi - z_0), \ln(\xi - z_0)\}$ and $\{(z - \xi), \ln(z - \xi)\}$ which appear in the integrand of Pochhammer integral representations, and the four-loop contour drawn around the branch points z_0 and z.

Theorem 4.1. Let $f(z - z_0, w - z)$ be an analytic function on a simply connected open set R containing the points z_0 and w. Also assume that $f(0, w - z_0) \neq 0$ and $f(z - z_0, 0) \neq 0$. If p, α, $\alpha - q$ and $p + r$ are not integers and $z \in R - \{z_0\}$, then the transformation formula

$$D^{\alpha}_{z-z_0}\Big\{f(z-z_0,\, w-z)(z-z_0)^{p+r}(w-z)^q\{\ln(z-z_0)\}^{\delta}\{\ln(w-z)\}^{\theta}\Big\}\Big|^*_{w=z} \tag{4.1}$$
$$= \frac{\Gamma(1+p)}{\Gamma(-\alpha)}$$
$$D^{-p-1}_{z-z_0}\Big\{f(w-z,\, z-z_0)(z-z_0)^{q-\alpha-1}(w-z)^r\{\ln(z-z_0)\}^{\theta}\{\ln(w-z)\}^{\delta}\Big\}\Big|^*_{w=z}$$

is valid for $z \in R - \{z_0\}$ where $\delta, \theta = 0$ or 1.

Proof: From equation (3.46), if we put

$$\begin{aligned}\mathbf{I} &= D^{\alpha}_{z-z_0} \Lambda_{\delta,\theta,p+r,q}(z-z_0;\, w-z) \\ &= D^{\alpha}_{z-z_0} \Big\{ f(z-z_0,\, w-z)(z-z_0)^{p+r}(w-z)^{q}\{\ln(z-z_0)\}^{\delta}\{\ln(w-z)\}^{\theta}\Big\}\Big|^{*}_{w=z},\end{aligned} \tag{4.2}$$

we have

$$\begin{aligned}\mathbf{I} = &\frac{e^{i\pi(\alpha-p-r+1)}\Gamma(1+\alpha)}{4\pi \sin \pi(p+r)} \left\{ \frac{e^{-i\pi q} \sin \alpha\pi}{\sin(\alpha-q)\pi} \right\} \\ &\int_P f(\xi - z_0,\, z-\xi)\ \Xi_{\delta,\theta}(p+r,\, q-\alpha-1;\, \xi-z_0,\, z-\xi) d\xi \\ &+ \theta \frac{\pi e^{-i\pi(p+r)}}{4\Gamma(-\alpha) \sin \pi(p+r) \sin^2(\alpha-q)\pi} \\ &\int_P f(\xi - z_0,\, z-\xi)\ \Xi_{\delta,0}(p+r,\, q-\alpha-1;\, \xi-z_0,\, z-\xi) d\xi \\ &- \delta \frac{\pi\ e^{-i\pi(q-\alpha)}}{4\Gamma(-\alpha) \sin^2 \pi(p+r)\ \sin(\alpha-q)\pi} \\ &\int_P f(\xi - z_0,\, z-\xi)\ \Xi_{0,\theta}(p+r,\, q-\alpha-1;\, \xi-z_0,\, z-\xi) d\xi \\ &- \delta\theta \frac{\pi^2}{4\Gamma(-\alpha) \sin^2 \pi(p+r) \sin^2(\alpha-q)\pi} \\ &\int_P f(\xi - z_0,\, z-\xi)\ \Xi_{0,0}(p+r,\, q-\alpha-1;\, \xi-z_0,\, z-\xi) d\xi\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}&\Xi_{\delta,\theta}(p+r,\, q-\alpha-1;\, \xi-z_0,\, z-\xi) \\ &= (\xi-z_0)^{p+r}(z-\xi)^{q-\alpha-1}\{\ln(\xi-z_0)\}^{\delta}\{\ln(z-\xi)\}^{\theta}.\end{aligned} \tag{4.4}$$

The Pochhammer contour is a four loop contour in the complex plane and is given by $P = C_1 \cup C_2 \cup C_3 \cup C_4$. The Fig. 2.1 shows the components of P. Note that we can always choose $C_3 = -C_1$ and $C_4 = -C_2$ which yields $P = C_1 \cup C_2 \cup (-C_1) \cup (-C_2)$, where the minus signs indicate that contours are crossing in the opposite direction with a arbitrary in R; also the values of integrals in (4.3) remain unchanged. Now, with the substitution $\xi = z + z_0 - \zeta$ in (4.3), being in the ξ-plane, the variable of integration ξ starts at $\xi = a$, encloses $\xi = z$ once in the positive sense, and returns to $\xi = z + z_0 - a$, the variable of integration ζ in the ζ-plane starts at $\zeta = z + z_0 - a$, encloses $\zeta = z_0$ once in the positive sense, and returns to $\zeta = z + z_0 - a$. In other words, the new contour in the complex plane ζ which is the image of C_1 is equivalent to C_2 in the ξ-plane except for the starting point which is $\zeta = z + z_0 - a$ instead of $\xi = a$.

Similarly, with the same substitution $\xi = z + z_0 - \zeta$ in (4.3), the image of C_2, of the ζ-plane is equivalent to C_1 in the ξ-plane except again for the starting point which is $\zeta = z + z_0 - a$ instead of $\xi = a$. Globally the image of the Pochhammer contour $P = C_1 \cup C_2 \cup C_3 \cup C_4$ of the ξ-plane in the ζ-plan is equivalent to $P = C_2 \cup C_1 \cup (-C_2) \cup (-C_1) = -\{C_1 \cup C_2 \cup -(C_1) \cup (-C_2)$ which is $-P$ except again the starting point which is $\zeta = z + z_0 - a$ instead of $\xi = a$. Using the fact that $\Xi_{\delta,\theta}(p+r,\, q-\alpha-1;\, \xi-z_0, z-\xi) = \Xi_{\delta,\theta}(q-\alpha-1,\, p+r;\, z-\zeta,\, \zeta-z_0)$ with $\xi = z + z_0 - \zeta$, this substitution in (4.3) gives

$$\mathbf{I} = \frac{e^{i\pi(\alpha-p-r+1)}\Gamma(1+\alpha)}{4\pi\sin\pi(p+r)}\Big\{\frac{e^{-i\pi q}\sin\alpha\pi}{\sin(\alpha-q)\pi}\Big\} \tag{4.5}$$
$$\int_P f(z-\zeta,\,\zeta-z_0)\,\Xi_{\theta,\delta}(q-\alpha-1,\,p+r;\,z-\zeta,\,\zeta-z_0)(-d\zeta)$$
$$+\,\theta\frac{\pi e^{-i\pi(p+r)}}{4\Gamma(-\alpha)\sin\pi(p+r)\sin^2(\alpha-q)\pi}$$
$$\int_P f(z-\zeta,\,\zeta-z_0)\,\Xi_{0,\delta}(q-\alpha-1,\,p+r;\,z-\zeta,\,\zeta-z_0)(-d\zeta)$$
$$-\,\delta\frac{\pi\, e^{-i\pi(q-\alpha)}}{4\Gamma(-\alpha)\sin^2\pi(p+r)\ \sin(\alpha-q)\pi}$$
$$\int_P f(z-\zeta,\,\zeta-z_0)\,\Xi_{\theta,0}(q-\alpha-1,\,p+r;\,z-\zeta,\,\zeta-z_0)(-d\zeta)$$
$$-\,\delta\theta\frac{\pi^2}{4\Gamma(-\alpha)\sin^2\pi(p+r)\sin^2(\alpha-q)\pi}$$
$$\int_P f(z-\zeta,\,\zeta-z_0)\,\Xi_{0,0}(q-\alpha-1,\,p+r;\,z-\zeta,\,\zeta-z_0)(-d\zeta).$$

If we replace P in the ζ-plan by $-P$ in the ξ-plane with the starting point $\xi = z+z_0-a$ instead of $\xi = a$, and if we take $\Gamma(1+p)/\Gamma(-\alpha)$ as a factor, after elementary calculations we can rewrite (4.5) in the following form

$$\mathbf{I} = \frac{\Gamma(1+p)}{\Gamma(-\alpha)}\Big[\frac{e^{i\pi([-1-p]-[q-\alpha+1]+1)}\Gamma(1+[-1-p])}{4\pi\sin\pi(q-\alpha)}\Big\{\frac{e^{-i\pi r}\sin(-1-p)\pi}{\sin(-1-p-r)\pi}\Big\} \tag{4.6}$$
$$\int_P f(z-\xi,\,\xi-z_0)\,\Xi_{\theta,\delta}(q-\alpha-1,\,p+r;\,z-\xi,\,\xi-z_0)d\xi$$
$$+\,\delta\frac{\pi e^{-i\pi(q-\alpha-1)}}{4\Gamma(-[-1-p])\sin\pi(q-\alpha-1)\sin^2([-1-p]-r)\pi}$$
$$\int_P f(z-\xi,\,\xi-z_0)\,\Xi_{\theta,0}(q-\alpha-1,\,p+r;\,z-\xi,\,\xi-z_0)d\xi$$
$$-\,\theta\frac{\Gamma(1+[-1-p])e^{i\pi([-1-p]+1)}}{4\sin^2\pi(q-\alpha-1)}\Big\{\frac{e^{-i\pi r}\sin(-1-p)\pi}{\sin(-1-p-r)\pi}\Big\}$$
$$\int_P f(z-\xi,\,\xi-z_0)\,\Xi_{0,\delta}(q-\alpha-1,\,p+r;\,z-\xi,\,\xi-z_0)d\xi$$
$$-\,\theta\delta\frac{\pi^2}{4\Gamma(-[-1-p])\sin^2\pi(q-\alpha-1)\sin^2(-1-p-r)\pi}$$
$$\int_P f(z-\xi,\,\xi-z_0)\,\Xi_{0,0}(q-\alpha-1,\,p+r;\,z-\xi,\,\xi-z_0)d\xi\Big].$$

Identifying the right-hand side of (4.6) with (4.4) and (3.46), we obtain

$$\mathbf{I} = \frac{\Gamma(1+p)}{\Gamma(-\alpha)} \tag{4.7}$$
$$D^{-p-1}_{z-z_0}\Big\{f(w-z,\,z-z_0)(z-z_0)^{q-\alpha-1}(w-z)^r\{\ln(z-z_0)\}^\theta\{\ln(w-z)\}^\delta\Big\}\Big|^*_{w=z}.$$

The conditions of parameters in (4.1) are the same as the ones given in (3.46). The Theorem is proved. With the change of variable $\xi - z_0 = g(\zeta)$, we get the generalized version of the formula to the operator $D^{\alpha}_{g(z)}$:

$$D^{\alpha}_{g(z)}\Big\{f(z, g^{-1}[g(w) - g(z)])\,[g(z)]^{p+r}[g(w) - g(z)]^{q} \quad \{\ln[g(z)]\}^{\delta}\{\ln[g(w) = g(z)]\}^{\theta}\Big\}\Big|^{*}_{w=z} = \frac{\Gamma(1+p)}{\Gamma(-\alpha)} D^{-p-1}_{g(z)}\Big\{f[g^{-1}[g(w) - g(z)], z]\,[g(z)]^{q-\alpha-1}[g(w) - g(z)]^{r} \quad \{\ln[g(z)]\}^{\theta}\{\ln[g(w) - g(z)]\}^{\delta}\Big\}\Big|^{*}_{w=z}. \tag{4.8}$$

Now we will examine possible extensions of formula (4.1) with respect to the set of parameters α, p, q, r, δ and θ. If we use the analytic continuation formula (3.48) on both sides of the transformation formula (4.1), replacing α by $\alpha + q$ and p by $p - r$, we can write

$$D^{\alpha}_{z-z_0}\Big\{f(z - z_0,\, w - z)\frac{(z - z_0)^{p}}{\Gamma(1+p)}\{\ln(z - z_0)\}^{\delta}\{\ln(w - z)\}^{\theta}\Big\}\Big|^{*}_{w=z} = D^{-p-1}_{z-z_0}\Big\{f(w - z,\, z - z_0)\frac{(z - z_0)^{-\alpha-1}}{\Gamma(-\alpha)}\{\ln(z - z_0)\}^{\theta}\{\ln(w - z)\}^{\delta}\Big\}\Big|^{*}_{w=z} \tag{4.9}$$

which is symmetric with respect to α and $-1 - p$. We can obtain directly (4.9) from (4.1) by putting $q = r = 0$. Note that we can also obtain (4.9) with the analytic continuation formula (3.48) on both sides of the transformation formula (4.1). We must pay special attention when $-\alpha$ and $1 + p$ are negative integers in (4.9), taking into account the fact that z_0 and z are branch-points of logarithm functions. Also, we cannot simultaneously have $\delta = \theta = \mathbf{1}$. Moreover, we can give an interpretation of the left-hand side of (4.9) when p = -1, -2, -3, ... and $\delta = 0$ by the right-hand side of (4.9) if $\alpha \neq 0, 1, 2, \ldots$ Inversely, the right-hand side of (4.9) can be defined with $\alpha = 0, 1, 2, \ldots$ and $\theta = 0$ by using the left-hand side of (4.9) if $p \neq$ -1, -2, -3, ...

Each side of (4.9) can be seen as a convolution integral operator of appropriate functions. Note that it is possible to make a link between the complex function $\frac{(z-z_0)^{-\alpha-1}}{\Gamma(-\alpha)}$ and the distribution function $I^{-\alpha}_{+} = \frac{x_{+}^{-\alpha-1}}{\Gamma(-\alpha)}$ [38, p. 145-160] which gives as a particular case $I^{n}_{+} = \delta^{(n)}(x)$, $\delta(z)$ being the Dirac distribution function. In the next section we will give some applications of the new transformation (4.1) and (4.8) with the new Pochhammer representations for fractional derivatives of singular functions proposed in section 3.

5. Some applications

Transformations (4.1) and (4.8) can be applied in many subjects and formulas involving fractional derivatives. For example, if we use the transformation formula (4.8) with $\delta = \theta = 0$ and $g(z) = 1 - z$, from (2.15), we obtain

$$P^{(\alpha,\beta)}_{n}(z) = \frac{\Gamma(1+\alpha+n)}{\Gamma(-\beta+n)}\frac{(1-z)^{-\alpha}}{2^{n}n!}D^{-\alpha-\beta-n-1}_{1-z}(1-z)^{-\beta-n-1}(2+w-z)^{n}\Big|^{*}_{w=z} \tag{5.1}$$

which eventually gives the following for the Jacobi polynomial [35, Eq. (2), p. 254]:

$$P^{(\alpha,\beta)}_{n}(z) = \frac{(1+\alpha)_{n}}{n!}\Big(\frac{1+z}{2}\Big)^{n}{}_{2}F_{1}\left[\begin{matrix}-n, -\beta-n\\ 1+\alpha\end{matrix}\middle|\frac{z-1}{z+1}\right]. \tag{5.2}$$

In this section we examine two particular applications: the Leibniz rule and the Christoffel-Darboux formula (1.9). For the first, we can find many papers in the literature treating the Leibniz rule and its generalizations in the form of an infinite series and an integral analogue [26, 28, 30, 31]. We will limit our example to the now-classic series form (5.3)(see [26, Eq.(1.1), p.658]. We recall that the proof of (5.3) without using the Pochhammer representations for fractional derivatives requires the following three 'half-plane' restrictions $\Re(P+Q) > -1$, $\Re(P) > -1$ and $\Re(Q) > -1$ even if only the first one is necessary. In [20], it is shown how the introduction of the Pochhammer representation can improve the Leibniz rule as an infinite series (5.3) (or as an integral analogue version) given in ([26, Th.1, p.664)],[30, Th.4.1, p.907]). In this section, we show how the Pochhammer representation with its symmetric properties with respect of the parameters and variables can allow the use of the transformation (4.1) to obtain new forms of the Leibniz rule.

For the second application, it is the first time that we use the fractional derivative and the Christoffel-Darboux formula (1.9) together in an application. Recall that the Christoffel-Darboux formula involves the set of orthogonal polynomials $\{f_n(x)\}_{n=0}^{\infty}$ on the interval $[a, b]$ for the weight function $w(x)$. The transformation formula (4.1) with $\delta = \theta = 0$ and $z_0 = 0$ applied on the Christoffel-Darboux formula slightly modified allows us to deduce many summation formulas involving the classical orthogonal polynomials and other special functions.

5.1. The Leibniz Rule.

If we use the transformation formula (4.1) with $\delta = \theta = 0$ in the well-known Leibniz rule [26]

$$\begin{aligned} &D^{\alpha}_{z-z_0}(z-z_0)^{\mu+\nu}u(z)v(z) \\ &\quad = a\sum_{-\infty}^{\infty}\binom{\alpha}{\gamma+an}D^{\alpha-\gamma-an}_{z-z_0}(z-z_0)^{\mu}u(z)\cdot D^{\gamma+an}_{z-z_0}(z-z_0)^{\nu}v(z) \end{aligned} \tag{5.3}$$

(with $0 < a \leq 1$), we obtain after elementary substitutions the following new form of the Leibniz rule

$$\begin{aligned} &\binom{\mu+\nu}{\mu}D^{-\mu-\nu-1}_{z-z_0}\left\{\frac{(z-z_0)^{-\alpha-1}}{\Gamma(-\alpha)}u(z)v(z)\right\} = a\sum_{-\infty}^{\infty}\binom{\alpha}{\gamma+an} \\ &D^{-\mu-1}_{z-z_0}\left\{\frac{(z-z_0)^{-\alpha+\gamma+an-1}}{\Gamma(-\alpha+\gamma+an)}u(z)\right\}\cdot D^{-\nu-1}_{z-z_0}\left\{\frac{(z-z_0)^{-\gamma-an-1}}{\Gamma(-\gamma-an)}v(z)\right\}. \end{aligned} \tag{5.4}$$

If $a \to 0$ in (5.4), we obtain the interesting integral analogue

$$\begin{aligned} &\binom{\mu+\nu}{\mu}D^{-\mu-\nu-1}_{z-z_0}\left\{\frac{(z-z_0)^{-\alpha-1}}{\Gamma(-\alpha)}u(z)v(z)\right\} = \int_{-\infty}^{\infty}\binom{\alpha}{\gamma+\omega} \\ &D^{-\mu-1}_{z-z_0}\left\{\frac{(z-z_0)^{-\alpha+\gamma+\omega-1}}{\Gamma(-\alpha+\gamma+\omega)}u(z)\right\}\cdot D^{-\nu-1}_{z-z_0}\left\{\frac{(z-z_0)^{-\gamma-\omega-1}}{\Gamma(-\gamma-\omega)}v(z)\right\}d\omega. \end{aligned} \tag{5.5}$$

If R (a simply connected region where z_0 is an interior point) is the region of analyticity of $f(z)$, Osler ([26, 30]) showed that formula (5.3) is valid inside the region $\aleph \subset R$ defined by the set of all z such that the closed disk $|\xi - z| \leq |z - z_0|$ contains only points ξ in R. Recalling that formula (4.1) is obtained by the change of variable $\xi = z + z_0 - \zeta$, we see that the image of $\aleph$ becomes the set of all z such that the closed disk $|\zeta - z_0| \leq |z - z_0|$ contains only points ζ in R. Therefore we have $\aleph \equiv R$ (the region of convergence of (5.3)and (5.4)), α, γ, μ and ν are all complex numbers except for α which is a negative integer and $\Re(\mu+\nu) > -1$. In addition to eliminating the restrictions $\Re(\mu) > -1$ and $\Re(\nu) > -1$ initially imposed on (5.3), the Pochhammer representation for the fractional derivative presents another important advantage, that of allowing the function $f(z)$ to have an essential singularity at z_0. Moreover, due to the use of a Pochhammer contour, we can show that the singularities $-\alpha+\gamma+an = 0, -1, -2, \ldots$ and $-\gamma-an = 0, -1, -2, \ldots$ are apparent.

We conclude this application by assigning specific values to $u(z)$, $v(z)$, α, γ, μ and ν in (5.4). For example, putting $z_0 = 0$, $u(z) = (1-z)^{-A}$, $v(z) = 1$, $\alpha = B$, $\mu = B - D$, $\nu = C + D$, replacing γ by $\gamma + D$, in computing the fractional derivatives in (5.4), and using the fact that the gaussian function can be written in terms of fractional derivatives like the following ([9],Vol.2,p.191-212),

$${}_2F_1\left[\begin{matrix}a, b\\ c\end{matrix}\middle| z\right] = \frac{\Gamma(c)}{\Gamma(b)}z^{1-c}D^{b-c}_{z}z^{b-1}(1-z)^{-a} \tag{5.6}$$

we obtain

$$\begin{aligned} &\binom{B+C}{D+C}\,{}_2F_1\left[\begin{matrix}A+E, -B\\ 1+C\end{matrix}\middle| z\right] = a\sum_{n=-\infty}^{n=+\infty}\binom{B}{D+\gamma+an}\binom{C}{C-\gamma-an} \\ &{}_2F_1\left[\begin{matrix}A, D-B+\gamma+an\\ 1+\gamma+an\end{matrix}\middle| z\right]\,{}_2F_1\left[\begin{matrix}E, -D-\gamma-an\\ 1+C-\gamma-an\end{matrix}\middle| z\right]. \end{aligned} \tag{5.7}$$

If $A = E = 0$ in (4.9), we obtain the well-known Dougall formula ([8], vol. 1, Eq. (1), p. 7). Let us note that we can find other forms of the Leibniz rule in [10] where the transformation formula (4.1) could be applied.

5.2. The Christoffel-Darboux Formula.

Before using the transformation (4.1), we first modify the Christoffel-Darboux formula (1.9). Replacing x by w, y by z and putting $z_0 = 0$, multiplying both sides by $z^p(w-z)^q$, we obtain

$$\sum_{m=0}^{n} \frac{1}{h_m} \frac{f_m(w) z^p f_m(z)}{(w-z)^{-q}} = \frac{k_n}{k_{n+1} h_n} \frac{f_{n+1}(w) z^p f_n(z) - f_n(w) z^p f_{n+1}(z)}{(w-z)^{-q+1}}. \tag{5.8}$$

Now applyng the operator D_z^α on each side of (5.14) where $w = z$ before the operation, we have

$$\sum_{m=0}^{n} \frac{f_m(z)}{h_m} D_z^\alpha z^p (w-z)^q f_m(z)\Big|_{w=z}^{*} = \frac{k_n}{k_{n+1} h_n} \tag{5.9}$$
$$\left[f_{n+1}(z) D_z^\alpha z^p (w-z)^{q-1} f(z)\Big|_{w=z}^{*} - f_n(z) D_z^\alpha z^p (w-z)^{q-1} f_{n+1}(z)\Big|_{w=z}^{*} \right].$$

Now using (3.13) and the transformation formula (4.1) with $\delta = \theta = 0$ and z_0, we get

$$\begin{aligned} D_z^\alpha z^p (w-z)^{q-1} f_n(z)\Big|_{w=z}^{*} &= \frac{\Gamma(-\alpha+q-1)}{\Gamma(-\alpha)} D_z^{\alpha-q+1} z^p f_n(z) \\ &= \frac{\Gamma(1+p)}{\Gamma(-\alpha)} D_z^{-p-1} z^{-\alpha+q-2} f_n(w-z)\Big|_{w=z}^{*} \\ &= \frac{\Gamma(1+p)}{\Gamma(-\alpha)} \sum_{k=0}^{n} \frac{f_n^{(k)}(z)}{k!} (-1)^k D_z^{-p-1} z^{-\alpha+q+k-2} \\ &= \frac{\Gamma(1+p)\Gamma(-\alpha+q-1) z^{-\alpha+p+q-1}}{\Gamma(-\alpha)\Gamma(-\alpha+p+q)} \sum_{k=0}^{n} \frac{f_n^{(k)}(z)}{k!} \frac{(-\alpha+q-1)_k}{(-\alpha+p+q)_k} (-z)^k. \end{aligned} \tag{5.10}$$

Using (5.10) in (5.9) and simplifying we can write

$$\begin{aligned} &\frac{(-\alpha+q-1)}{(-\alpha+p+q)} z \sum_{m=0}^{n} \frac{f_m(z)}{h_m} \sum_{k=0}^{m} \frac{f_m^{(k)}(z)}{k!} \frac{(-\alpha+q)_k}{(-\alpha+p+q+1)_k} (-z)^k \\ &= \frac{k_n}{k_{n+1} h_n} \Bigg[f_{n+1}(z) \sum_{k=0}^{n} \frac{f_n^{(k)}(z)}{k!} \frac{(-\alpha+q-1)_k}{(-\alpha+p+q)_k} (-z)^k \\ &\qquad - f_n(z) \sum_{k=0}^{n+1} \frac{f_{n+1}^{(k)}(z)}{k!} \frac{(-\alpha+q-1)_k}{(-\alpha+p+q)_k} (-z)^k \Bigg]. \end{aligned} \tag{5.11}$$

The first term ($k = 0$) on the right-hand side of (5.11) equals zero. After sliding the index of summation, and putting $a = \alpha + q$, $b = -\alpha + p + q + 1$, if we define the associated polynomial $\omega_n(a;\, b;\, z)$ of the orthogonal polynomial $f_n(z)$ by

$$\omega_n(a;\, b;\, z) = \sum_{k=0}^{n-1} \frac{f_n^{(k+1)}(z)}{(k+1)!} \frac{(a)_k}{(b)_k} (-z)^k, \tag{5.12}$$

we obtain from (5.11) the following summation formula

$$\begin{aligned} &\sum_{m=0}^{n} \frac{f_m^2(z)}{h_m} - \frac{az}{b} \sum_{m=0}^{n} \frac{f_m(z)}{h_m} \omega_n(a+1;\, b+1;\, z) \\ &= \frac{k_n}{k_{n+1} h_n} [f_{n+1}(z)\omega_n(a;\, b;\, z) - f_n(z)\omega_{n+1}(a;\, b;\, z)] \end{aligned} \tag{5.13}$$

where $\{f_n(z)\}_{n=0}^{\infty}$ is a system of orthogonal polynomials on the interval $[a,\, b]$ for the weight function $w(x)$. Recall that we have by definition,

$$h_n = \int_a^b w(z)[f_n(x)]^2 dx \,\text{and}\, f_n(x) = k_n x^n + k_n x^{n-1} + \dots.$$

The special case $a = 0$ in (5.13) gives the well-known formula [8, Vol. 2,Eq. (11), p. 159],

$$\sum_{m=0}^{n} \frac{f_m^2(z)}{h_m} = \frac{k_n}{k_{n+1} h_n} [f_{n+1}(z) f_n(z) - f_n(z) f_{n+1}(z)]. \tag{5.14}$$

As an example, if we consider the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$, then [1, Tables 22.2 and 22.3, p. 774-775],

$$h_n = \frac{2^{\alpha+\beta+1}\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{(2n+\alpha+\beta+1)n!\Gamma(1+\alpha+\beta+n)}, k_n = \frac{1}{2^n}\binom{2n+\alpha+\beta}{n},$$

and with some calculations we get the following explicit summation formula

$$\begin{aligned} &\sum_{m=0}^{n} \frac{(2m+\alpha+\beta+1)(1+\alpha+\beta)_m}{(1+\beta)_m} P_m^{(\alpha,\beta)}(z) {}_3F_2\left[\begin{matrix} -m, 1+\alpha+\beta+m \\ 1+\alpha, 1+b \end{matrix}\middle| \frac{1-z}{2}\right] \\ &= \frac{2b(1+\alpha+\beta)(2+\alpha+\beta)_n}{(1-z)(b-a)(2+\alpha+\beta+2n)(1+\beta)_n} \\ &\qquad \{(1+\alpha+n) P_n^{(\alpha,\beta)}(z)\ {}_3F_2\left[\begin{matrix} -n-1, 2+\alpha+\beta+n, a \\ 1+\alpha, b \end{matrix}\middle| z\right] \\ &\qquad\qquad - (1+n) P_{n+1}^{(\alpha,\beta)}(z)\ {}_3F_2\left[\begin{matrix} -n, 1+\alpha+\beta+n, a \\ 1+\alpha, b \end{matrix}\middle| z\right]. \end{aligned} \tag{5.15}$$

By the same technique, it is possible to deduce similar summation formulas for the other classical orthogonal polynomials.

Furthermore, this can be used as an efficient tool to obtain new associated forms of the known generalized Leibniz rules for the fractional derivatives. There are many other potential areas where the new Pochhammer integral representations and transformation formulas can be used to obtain new extensions of results involving fractional derivatives. Among them, we can examine, a generalized Taylor series, the chain rule, the derivative of composite a function, and so on.

6. Conclusion.

In this paper, we show that Pochhammer contours are efficient tools for representing the fractional derivative of analytic functions (exponential, trigonometric and hyperbolic functions, etc.) and functions with a single branch-point (complex powers, logarithmic functions, and their products). These representations of increasing complexity are presumably new. They are obtained progressively to finally obtain the most general definition (3.47) of the fractional derivative that is known in the literature, that of the product of an analytic function (or having an essential singularity at z_0) and four other functions with a branch-point (two complex powers and two logarithmic functions). From this new representation, we have been able to deduce a generalization of an important transformation formula for the fractional derivative already published in [49]. In addition, the examples presented in sections 2 and 5 definitely show the usefulness of these new representations for discovering new formulas involving classical functions of mathematical physics, particularly a new form of the Leibniz formula for the fractional derivative of a product of functions. Our last application makes it possible to obtain new summation formulas arising from the classical Christoffel-Darboux formula associated with orthogonal polynomials. This article clearly shows that these new transformations are powerful tools to investigate and obtain new results. These new representations, combined with other rules involving fractional derivatives, will be studied in the future.

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