Montes Taurus J. Pure Appl. Math. 3 (1), 1–6, 2021

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Analogues of Faulhaber's Formula for Poly-Bernoulli and Type 2 Poly-Bernoulli Polynomials

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Abstract

Faulhaber's formula expresses sums of powers of consecutive integers in terms of Bernoulli polynomials. Here we would like to find analogous ones to the Faulhaber's formula for poly-Bernoulli and type 2 poly-Bernoulli polynomials.

Keywords: Analogues of Faulhaber's formula, Poly-Bernoulli polynomials, Type 2 poly-Bernoulli polynomials

2010 MSC: 11B68, 11B75, 11B83

1. Introduction

As is well known, the Bernoulli polynomials are defined by

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad \text{(see [5, 6, 7, 8, 10])}. \tag{1.1}$$

When $x=0$, $B_n=B_n(0)$, $(n\geq 0)$, are called the Bernoulli numbers.

From (1.1), we note that

$$\begin{aligned}\sum_{k=0}^{n-1}e^{kt}&=\frac{1}{e^t-1}(e^{nt}-1)=\frac{1}{t}\frac{t}{e^t-1}(e^{nt}-1)\\&=\frac{1}{t}\sum_{m=0}^{\infty}\Big(B_m(n)-B_m\Big)\frac{t^m}{m!}=\sum_{m=0}^{\infty}\left(\frac{B_{m+1}(n)-B_{m+1}}{m+1}\right)\frac{t^m}{m!},\end{aligned} \tag{1.2}$$

where n is the positive integer.

†*Article ID:* MTJPAM-D-20-00033

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Received:9 October 2020, Accepted:28 October 2020

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On the other hand,

$$\sum_{k=0}^{n-1} e^{kt} = \sum_{m=0}^{\infty}\Big(\sum_{k=0}^{n-1} k^m\Big)\frac{t^m}{m!}, (n \in \mathbb{N}). \tag{1.3}$$

By (1.2) and (1.3), we get

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1}(B_{m+1}(n) - B_{m+1}), \quad \text{(see [1, 2, 5, 9, 11, 12])}, \tag{1.4}$$

where $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.

The equation (1.4) is equivalent to the Faulhaber's formula which is given by

$$\sum_{k=1}^{n} k^m = \frac{1}{m+1}\sum_{j=0}^{m}(-1)^j\binom{m+1}{j}B_j n^{m+1-j},$$

where $m \in \mathbb{N}$.

For $k \in \mathbb{Z}$, the polylogarithm functions are defined as

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^k}, \quad \text{(see [3])}. \tag{1.5}$$

Note that $Li_1(z) = -\log(1-z)$.

In terms of the polylogarithm functions in (1.5), the poly-Bernoulli polynomials are defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}, \quad \text{(see [3, 4, 7])}. \tag{1.6}$$

Note that $B_n^{(1)}(x) = B_n(x), (n \geq 0)$.
When $x = 0$, $B_n^{(k)} = B_n^{(k)}(0), (n \geq 0)$, are called the poly-Bernoulli numbers.

The modified Hardy's polyexponential functions are defined by

$$\mathrm{Ei}_k(z) = \sum_{n=1}^{\infty}\frac{z^n}{(n-1)!n^k}, (k \in \mathbb{Z}), \quad \text{(see [4, 6, 8])}. \tag{1.7}$$

Note that $Ei_1(z) = e^z - 1$.

Recently, the type 2 poly-Bernoulli polynomials are defined by using the modified Hardy's polyexponential functions as

$$\frac{\mathrm{Ei}_k(\log(1+t))}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty}\beta_n^{(k)}(x)\frac{t^n}{n!}, \quad \text{(see [4])}. \tag{1.8}$$

Note that $\beta_n^{(1)}(x) = B_n(x), (n \geq 0)$.
When $x = 0$, $\beta_n^{(k)} = \beta_n^{(k)}(0)$ are called the type 2 poly-Bernoulli numbers.

In light of (1.4), we may ask the following natural question.

Question 1.1. What are analogous formulas to (1.4) for the poly-Bernoulli and the type 2 poly-Bernoulli polynomials?

The aim of this paper is to answer to the above question. Indeed, we will find an analogue of the Faulhaber's formula for the poly-Bernoulli polynomials in Theorem 2 and that for the type 2 poly-Bernoulli polynomials in Theorem 4.

2. Analogues of Faulhaber's formula

From (1.6), we note that

$$\begin{aligned}\mathrm{Li}_k(1-e^{-t}) &= \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^t - \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}\\ &= \sum_{n=0}^{\infty}(B_n^{(k)}(1)-B_n^{(k)})\frac{t^n}{n!}.\end{aligned} \tag{2.1}$$

By (2.1),we get

$$\sum_{m=1}^{n}\frac{(-1)^{n-m}m!}{m^k}S_2(n,m) = B_n^{(k)}(1)-B_n^{(k)}, \quad (n\in\mathbb{N}), \tag{2.2}$$

where $S_2(n,m)$ are the Stirling numbers of the second kind.

When $k=1$, we note

$$\sum_{m=1}^{n}(-1)^{n-m}(m-1)!S_2(n,m) = \delta_{n,1}, \quad (n\in\mathbb{N}), \tag{2.3}$$

where $\delta_{n,1}$ is the Kronecker's symbol.

For $x\in\mathbb{N}$, we observe that

$$\begin{aligned}\sum_{i=0}^{x-1}e^{it}\mathrm{Li}_k(1-e^{-t}) &= \sum_{i=0}^{x-1}e^{it}\sum_{j=1}^{\infty}\frac{(1-e^{-t})^j}{j^k}\\ &= \sum_{l=0}^{\infty}\sum_{i=0}^{x-1}i^l\frac{t^l}{l!}\sum_{j=1}^{\infty}\frac{j!(-1)^j}{j^k}\sum_{m=j}^{\infty}S_2(m,j)\frac{(-1)^m t^m}{m!}\\ &= \sum_{l=0}^{\infty}\sum_{i=0}^{x-1}i^l\frac{t^l}{l!}\sum_{m=1}^{\infty}\sum_{j=1}^{m}\frac{(j-1)!}{j^{k-1}}(-1)^{m-j}S_2(m,j)\frac{t^m}{m!}\\ &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\sum_{j=1}^{m}\sum_{i=0}^{x-1}\binom{n}{m}\frac{(j-1)!}{j^{k-1}}(-1)^{m-j}S_2(m,j)i^{n-m}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

On the other hand,

$$\begin{aligned}\sum_{i=0}^{x-1}e^{it}\mathrm{Li}_k(1-e^{-t}) &= \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}(e^{xt}-1)\\ &= \sum_{n=0}^{\infty}(B_n^{(k)}(x)-B_n^{(k)})\frac{t^n}{n!}\\ &= \sum_{n=1}^{\infty}(B_n^{(k)}(x)-B_n^{(k)})\frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

Therefore, by (2.4) and (2.5), we obtain the following analogue of the Faulhaber's formula for the poly-Bernoulli polynomials.

Theorem 2.1. *For $n,x\in\mathbb{N}$, we have*

$$\sum_{i=0}^{x-1}\sum_{m=1}^{n}\sum_{j=1}^{m}\binom{n}{m}\frac{(j-1)!}{j^{k-1}}(-1)^{m-j}S_2(m,j)i^{n-m} = B_n^{(k)}(x)-B_n^{(k)}, \tag{2.6}$$

where $S_2(m,j)$ are the Stirling numbers of the second kind.

By taking $k = 1$ in Theorem 2 and making use of (2.3), we obtain the following corollary.

Corollary 2.2. *For $n, x \in \mathbb{N}$, we have*

$$\sum_{i=0}^{x-1} i^{n-1} = \frac{1}{n}(B_n(x) - B_n). \tag{2.7}$$

From (1.6), we note that

$$\begin{aligned}\frac{e^t \mathrm{Li}_k(1-e^{-t})}{e^t-1} &= \sum_{m=1}^{\infty} \frac{1}{m^k}(1-e^{-t})^{m-1} \\ &= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k}(1-e^{-t})^m \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m m!}{(m+1)^k} \sum_{n=m}^{\infty} S_2(n,m)\frac{(-1)^n}{n!}t^n \\ &= \sum_{n=0}^{\infty} \Big(\sum_{m=0}^{n} \frac{(-1)^{n-m} m!}{(m+1)^k} S_2(n,m) \Big) \frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

Thus, by (1.6) and (2.8), we have

$$B_n^{(k)}(1) = \sum_{m=0}^{n} \frac{(-1)^{n-m} m!}{(m+1)^k} S_2(n,m),$$

where n is a nonnegative integer.

For $x \in \mathbb{N}$, we note that

$$\begin{aligned}&\sum_{i=0}^{x-1} e^{it}\mathrm{Ei}_k(\log(1+t)) = \frac{\mathrm{Ei}_k(\log(1+t)}{e^t-1}(e^{xt}-1) \\ &= \sum_{n=0}^{\infty} (\beta_n^{(k)}(x) - \beta_n^{(k)})\frac{t^n}{n!} = \sum_{n=1}^{\infty} (\beta_n^{(k)}(x) - \beta_n^{(k)})\frac{t^n}{n!}.\end{aligned} \tag{2.9}$$

On the other hand,

$$\begin{aligned}\sum_{i=0}^{x-1} e^{it}\mathrm{Ei}_k(\log(1+t)) &= \sum_{j=0}^{\infty}\sum_{i=0}^{x-1} i^j \frac{t^j}{j!} \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{m^k (m-1)!} \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{x-1} i^j \frac{t^j}{j!} \sum_{m=1}^{\infty} \frac{1}{m^{k-1}} \sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!} \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{x-1} i^j \frac{t^j}{j!} \sum_{l=1}^{\infty}\sum_{m=1}^{l} \frac{S_1(l,m)}{m^{k-1}}\frac{t^l}{l!} \\ &= \sum_{n=1}^{\infty} \Big(\sum_{l=1}^{n}\sum_{m=1}^{l}\sum_{i=0}^{x-1} \binom{n}{l} \frac{S_1(l,m)}{m^{k-1}} i^{n-l} \Big)\frac{t^n}{n!},\end{aligned} \tag{2.10}$$

where $S_1(n,l)$ are the Stirling numbers of the first kind.

Therefore, by (2.9) and (2.10), we obtain the following analogue of the Faulhaber's formula for the type 2 poly-Bernoulli polynomials.

Theorem 2.3. *For $n, x \in \mathbb{N}$, we have*

$$\sum_{i=0}^{x-1}\sum_{l=1}^{n}\sum_{m=1}^{l}\binom{n}{l}\frac{S_1(l,m)}{m^{k-1}}i^{n-l} = \beta_n^{(k)}(x) - \beta_n^{(k)},$$

where $S_1(l,m)$ are the Stirling numbers of the first kind.

Now, we observe that

$$\begin{aligned}\mathrm{Ei}_k(\log(1+t)) &= \frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1}e^t - \frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1} \\ &= \sum_{n=0}^{\infty}(\beta_n^{(k)}(1) - \beta_n^{(k)})\frac{t^n}{n!}.\end{aligned} \tag{2.11}$$

On the other hand,

$$\begin{aligned}\mathrm{Ei}_k(\log(1+t)) &= \sum_{m=1}^{\infty}\frac{1}{m^k(m-1)!}(\log(1+t))^m \\ &= \sum_{m=1}^{\infty}\frac{1}{m^{k-1}}\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty}\Big(\sum_{m=1}^{n}\frac{1}{m^{k-1}}S_1(n,m)\Big)\frac{t^n}{n!}.\end{aligned} \tag{2.12}$$

Thus, by (2.11) and (2.12), we get

$$\beta_n^{(k)}(1) - \beta_n^{(k)} = \sum_{m=1}^{n}\frac{1}{m^{k-1}}S_1(n,m).$$

When $k = 1$, we have

$$\begin{aligned}\sum_{m=1}^{n}S_1(n,m) &= \beta_n^{(1)}(1) - \beta_n^{(1)} \\ &= B_n(1) - B_n \\ &= \delta_{n,1}, \quad (n \in \mathbb{N}).\end{aligned} \tag{2.13}$$

From Theorem 4 and (2.13), we note that

$$\beta_n^{(1)}(x) - \beta_n^{(1)} = n\sum_{i=0}^{x-1}i^{n-1}, \quad (n, x \in \mathbb{N}).$$

Acknowledgments

The authors would like to thank Jangjeon Research Institute for Mathematical Sciences for the support of this research.

Competing interests

The authors declare no conflict of interest.

Consent for publication

All authors agreed to publish this paper in this journal.

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