



Formulas and Relations of Special Numbers and Polynomials arising from Functional Equations of Generating Functions

Neslihan Kilar ^a, Yilmaz Simsek ^b

^aDepartment of Mathematics, Faculty of Science University of Akdeniz, TR-07058 Antalya-TURKEY

^bDepartment of Mathematics, Faculty of Science University of Akdeniz, TR-07058 Antalya-TURKEY

Abstract

The aim of this paper is to introduce and investigate some new identities and formulas involving many kinds of special numbers and polynomials with help of the some known results derived from blending special formulas, generating functions and their functional equations. By using functional equations of generating functions for special numbers and polynomials, we give some relations and identities including the Genocchi polynomials of negative order, the Euler numbers and polynomials of negative order, the Changhee numbers and polynomials of negative order, the Lah numbers, the Hermite polynomials, the central factorial numbers, the Bernoulli numbers of higher order, the Daehee numbers, the Bernstein basis functions, the Stirling numbers, and also the combinatorial numbers and polynomials. Moreover, we also give several combinatorial sums and identities associated with aforementioned numbers and polynomials. Finally, we derive some finite and infinite series representations that include the incomplete gamma function and aforementioned numbers. In addition, convenient links of identities, formulas, relations and results appointed in this paper with those in earlier and future studies come to attention in detail for readers.

Keywords: Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, Central factorial numbers, Hermite polynomials, Stirling numbers, Lah numbers, Combinatorial numbers, Generating functions, Incomplete gamma function

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1. Introduction

Special numbers including the Lah numbers, the Stirling numbers, and the central factorial numbers have many important applications in almost all areas of mathematics especially in theory of combinatorial analysis, numerical analysis, in approximation theory, and in the theory of analytic number theory. Recently using different techniques and methods, many interesting properties of negative order special numbers and polynomials involving the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Genocchi numbers and polynomials, the Changhee numbers and polynomials, and also the combinatorial numbers and polynomials have been studied.

The motivations of this paper are given as follows:

The first goal of this paper is to investigate the central factorial numbers and their generating functions with trigonometric functions. The second goal of this paper is to obtain new relations and identities for the Bernoulli

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Email addresses: neslihankilar@gmail.com (Neslihan Kilar ) , ysimsek@akdeniz.edu.tr (Yilmaz Simsek )

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*Corresponding Author: Neslihan Kilar



numbers of higher order and other well-known special numbers and polynomials. By using these generating functions with their functional equations, we give some novel formulas, identities, relations, combinatorial sums, and also finite and infinite series representations of the Genocchi polynomials of negative order, the Euler numbers and polynomials of the first kind of negative order, the Changhee numbers and polynomials of negative order, the Stirling numbers, the Hermite polynomials, the combinatorial numbers, the Bernstein basis functions, the Daehee numbers, and the Lah numbers.

Throughout of this paper, the following well-known notations, definitions, relations and formulas are used:

Let \mathbb{N} , \mathbb{Z} and \mathbb{C} denote the set of natural numbers, the set of integer numbers, the set of complex numbers, respectively, and also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We also assume that:

$$0^n = \begin{cases} 1, & n = 0 \\ 0 & n \in \mathbb{N}. \end{cases}$$

Furthermore, the falling factorial is defined by

$$(\alpha)_n = \alpha(\alpha - 1)(\alpha - 2) \dots (\alpha - n + 1),$$

$n \in \mathbb{N}$ and $(\alpha)_0 = 1$ such that

$$\binom{\alpha}{n} = \frac{(\alpha)_n}{n!},$$

where $n \in \mathbb{N}_0$. For $n \in \mathbb{N}_0$, we have

$$(-1)^n (-\alpha)_n = (\alpha + n - 1)_n$$

(cf. [1]-[48]; and references therein).

The upper incomplete gamma function $\Gamma(n, x)$ is defined by

$$\Gamma(n, x) = \int_x^\infty t^{n-1} e^{-t} dt, \tag{1.1}$$

where $|\arg(x)| < \pi$ (cf. [1, p. 262], [8], [46]; and references therein). This function is solutions to various problems in applied mathematics, astrophysics, heat conduction, nuclear physics, probability theory, statistics, engineering, and in the study of Fourier and Laplace transforms.

Putting $x = 0$ in (1.1), we have the gamma function:

$$\Gamma(n) = \Gamma(n, 0),$$

where $\operatorname{Re}(n) > 0$ (cf. [1], [8], [30], [46]).

The values of the function $\Gamma(-n, x)$ is given by

$$\Gamma(-n, x) = \frac{(-1)^n}{n!} \left(\Gamma(0, x) - \frac{e^{-x}}{x} \sum_{j=0}^{n-1} \frac{(-1)^j j!}{x^j} \right) \tag{1.2}$$

(cf. [1, p. 262]).

The Bernoulli polynomials $B_n^{(v)}(x)$ of order v are defined by means of the following generating function:

$$\left(\frac{t}{e^t - 1} \right)^v e^{xt} = \sum_{n=0}^\infty B_n^{(v)}(x) \frac{t^n}{n!} \tag{1.3}$$

(cf. [27], [39], [42], [46]; and references therein).

Here we note that the sign (v) which is given power of the polynomials $B_n^{(v)}(x)$ represents the order, not v th the derivative of $B_n(x)$.

Putting $x = 0$ in (1.3), we have the Bernoulli numbers of order v :

$$B_n^{(v)} = B_n^{(v)}(0)$$

(cf. [27], [39], [42], [46]).

When $\nu = 0$ in (1.3), we have

$$B_n^{(0)}(x) = x^n$$

and

$$B_n^{(0)} = B_n^{(0)}(0) = \begin{cases} 1, & n = 0 \\ 0 & n \in \mathbb{N}. \end{cases}$$

When $x = 0$ and $\nu = 1$ in (1.3), we have

$$B_n = B_n^{(1)}(0),$$

where B_n denotes the Bernoulli numbers (cf. [1]-[47]; and references therein).

With the help of (1.3), we get

$$B_n^{(n+1)} = (-1)^n n! \tag{1.4}$$

(cf. [5, Eq. (2.14)]).

The Bernoulli polynomials $B_n^{(-k)}(x)$ of order $-k$ are defined by means of the following generating function:

$$H_{NB}(t, x, k) = \left(\frac{e^t - 1}{t}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(-k)}(x) \frac{t^n}{n!} \tag{1.5}$$

(cf. [7], [27], [39], [42], [46]; and references therein).

Putting $x = 0$ in (1.5), we have the Bernoulli numbers of order $-k$:

$$B_n^{(-k)} = B_n^{(-k)}(0)$$

(cf. [7], [27], [39], [42], [46]).

With the help of (1.5), an explicit formula for the Bernoulli polynomials of negative order is given as follows:

$$B_n^{(-k)}(x) = \frac{1}{\binom{n+k}{k} k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (x+j)^{n+k}$$

(cf. [7, Eq. (3.20)]). Putting $x = 0$ in the above equation, we have the following well-known identity [45, Eq. (7.17)]:

$$B_n^{(-k)} = \frac{1}{\binom{n+k}{k}} S_2(n+k, k), \tag{1.6}$$

where $S_2(n, k)$ denotes the Stirling numbers of the second kind which are defined by means of the following generating function:

$$H_S(t, k) = \frac{(e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n, k) \frac{t^n}{n!} \tag{1.7}$$

(cf. [1]-[48]; and references therein). By using (1.7), an explicit formula for the numbers $S_2(n, k)$ is given as follows:

$$S_2(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n \tag{1.8}$$

and for $k > n$,

$$S_2(n, k) = 0$$

(cf. [1]-[48]).

The Euler polynomials $E_n^{(\nu)}(x)$ of the first kind of order ν are defined by means of the following generating function:

$$\left(\frac{2}{e^t + 1}\right)^\nu e^{xt} = \sum_{n=0}^{\infty} E_n^{(\nu)}(x) \frac{t^n}{n!} \tag{1.9}$$

(cf. [27], [38], [39], [46]; and references therein).

Putting $x = 0$ in (1.9), we have the Euler numbers of the first kind of order v :

$$E_n^{(v)} = E_n^{(v)}(0)$$

(cf. [27], [38], [39], [46]).

When $v = 0$ in (1.9), we have

$$E_n^{(0)}(x) = x^n$$

and

$$E_n^{(0)} = E_n^{(0)}(0) = \begin{cases} 1, & n = 0 \\ 0 & n \in \mathbb{N}. \end{cases}$$

When $x = 0$ and $v = 1$ in (1.9), we have

$$E_n = E_n^{(1)}(0),$$

where E_n denotes the Euler numbers of the first kind (cf. [1]-[47]; and references therein).

The Euler polynomials $E_n^{(-k)}(x)$ of the first kind of order $-k$ are defined by means of the following generating function:

$$H_{NE}(t, x, k) = \left(\frac{e^t + 1}{2}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_n^{(-k)}(x) \frac{t^n}{n!} \tag{1.10}$$

(cf. [27], [38], [39], [46]; and references therein).

Putting $x = 0$ in (1.10), we have the Euler numbers of the first kind of order $-k$:

$$E_n^{(-k)} = E_n^{(-k)}(0)$$

(cf. [27], [38], [39], [46]).

By using (1.10), a computation formula for the Euler polynomials of the first kind of negative order is given as follows:

$$E_n^{(-k)}(x) = \sum_{j=0}^n \binom{n}{j} x^{n-j} \sum_{d=0}^j \binom{d-k-1}{d} \frac{d!(-1)^d}{2^d} S_2(j, d)$$

(cf. [27], [38], [39], [46]).

The Euler numbers $E_n^{*(-k)}$ of the second kind of order $-k$ are defined by means of the following generating function:

$$\left(\frac{e^t + e^{-t}}{2}\right)^k = \sum_{n=0}^{\infty} E_n^{*(-k)} \frac{t^n}{n!} \tag{1.11}$$

(cf. [38], [42]; and references therein).

By using (1.11), an explicit formula for the Euler polynomials of the second kind of negative order is given as follows:

$$E_n^{*(-k)} = 2^{n-k} \sum_{j=0}^k \binom{k}{j} \left(j - \frac{k}{2}\right)^n$$

(cf. [38], [42]).

The Genocchi polynomials $G_n^{(-k)}(x)$ of order $-k$ are defined by means of the following generating function:

$$H_{NG}(t, x, k) = \left(\frac{e^t + 1}{2t}\right)^k e^{xt} = \sum_{n=0}^{\infty} G_n^{(-k)}(x) \frac{t^n}{n!} \tag{1.12}$$

(cf. [28], [46]; and references therein).

Putting $x = 0$ in (1.12), we have the Genocchi numbers of order $-k$:

$$G_n^{(-k)} = G_n^{(-k)}(0)$$

(cf. [28], [46]).

By using (1.12), a computation formula for the Genocchi polynomials of negative order is given as follows:

$$G_n^{(-k)}(x) = \binom{n+k}{k}^{-1} \sum_{j=0}^{n+k} \binom{n+k}{j} x^{n+k-j} \sum_{d=0}^k \frac{S_2(j, d)}{2^d (k-d)!} \quad (1.13)$$

(cf. [28], [46]).

The numbers $B(n, k)$ are defined by the following combinatorial sum:

$$B(n, k) = \sum_{j=0}^k \binom{k}{j} j^n \quad (1.14)$$

which satisfies the following differential equation:

$$B(n, k) = \frac{d^n}{dt^n} (e^t + 1)^k \Big|_{t=0}$$

(cf. [14]).

In the work of Spivey [48, Identity 12.], one has the following relation:

$$B(n, k) = \sum_{j=0}^n \binom{k}{j} j! 2^{k-j} S_2(n, j).$$

Recently, Simsek [39, Eq. (29)] gave a relation between the numbers $E_n^{(-k)}$ and the numbers $B(n, k)$ as follows:

$$E_n^{(-k)} = 2^{-k} B(n, k). \quad (1.15)$$

The central factorial numbers of the second kind $T(n, k)$ are defined by means of the following generating function:

$$H_T(t, k) = \frac{(e^t + e^{-t} - 2)^k}{(2k)!} = \sum_{n=0}^{\infty} T(n, k) \frac{t^{2n}}{(2n)!} \quad (1.16)$$

(cf. [4], [9], [37], [39]; and references therein).

In [9], Cigler gave an explicit formula for the numbers $T(n, k)$ as follows:

$$T(n, k) = \frac{1}{(2k)!} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (j-k)^{2n}.$$

Moreover, the central factorial numbers are also associated with hyperbolic functions and this relationship is given as follows:

$$H_T(t, k) = \frac{2^k}{(2k)!} (\cosh(t) - 1)^k \quad (1.17)$$

(cf. [37]).

The Hermite polynomials $H_n(x)$ are defined by means of the following generating function:

$$G_H(t, x) = e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \quad (1.18)$$

(cf. [30, p. 187], [33]; and references therein).

By using (1.18), an explicit formula for the polynomials $H_n(x)$ is given as follows:

$$H_n(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j n! (2x)^{n-2j} \frac{1}{j!(n-2j)!}$$

(cf. [30], [33]).

The Lah numbers $L(n, k)$ are defined by means of the following generating function:

$$\left(\frac{t}{1-t}\right)^k = k! \sum_{n=0}^{\infty} L(n, k) \frac{t^n}{n!} \tag{1.19}$$

(cf. [10, p. 156], [18], [32]; and references therein).

By using (1.19), an explicit formula for the numbers $L(n, k)$ is given as follows:

$$L(n, k) = (-1)^n \frac{n!}{k!} \binom{n-1}{k-1}, \tag{1.20}$$

where $n, k \in \mathbb{N}$ with $1 \leq k \leq n$ (cf. [10], [32]).

The Lah numbers are also related to the falling factorial, and this relation is given as follows:

$$(-x)_n = \sum_{k=0}^n L(n, k) (x)_k \tag{1.21}$$

so that

$$(x)_n = \sum_{k=0}^n L(n, k) (-x)_k$$

(cf. [10, p. 156], [32]).

The Catalan numbers C_n are defined by means of the following generating function:

$$\frac{1 - \sqrt{1-4t}}{2t} = \sum_{n=0}^{\infty} C_n t^n,$$

where $0 < |t| \leq \frac{1}{4}$ and $C_0 = 1$ (cf. [24]). The explicit formula for the Catalan numbers is given as follows:

$$C_n = \frac{1}{n+1} \binom{2n}{n} \tag{1.22}$$

(cf. [24]).

The Daehee numbers $D_n^{(\nu)}$ of order ν are defined by means of the following generating function:

$$\left(\frac{\log(1+t)}{t}\right)^\nu = \sum_{n=0}^{\infty} D_n^{(\nu)} \frac{t^n}{n!} \tag{1.23}$$

(cf. [22]).

When $\nu = 1$ in (1.23), we have

$$D_n = D_n^{(1)},$$

where D_n denotes the Daehee numbers (cf. [32, p. 45], [11], [19]). By using (1.23), we get

$$D_n = (-1)^n \frac{n!}{n+1} \tag{1.24}$$

(cf. [19], [31]).

The Changhee numbers $Ch_n^{(\nu)}$ of order ν are defined by means of the following generating function:

$$\left(\frac{2}{2+t}\right)^\nu = \sum_{n=0}^{\infty} Ch_n^{(\nu)} \frac{t^n}{n!} \tag{1.25}$$

(cf. [20], [23]).

When $\nu = 1$ in (1.25), we have

$$Ch_n = Ch_n^{(1)},$$

where Ch_n denotes the Changhee numbers (cf. [20], [21], [23]). By using (1.25), we get

$$Ch_n = (-1)^n \frac{n!}{2^n} \tag{1.26}$$

(cf. [21]).

The Changhee polynomials $Ch_n^{(-k)}(x)$ of order $-k$ are defined by means of the following generating function:

$$H_{NCh}(t, x, k) = \left(\frac{2+t}{2}\right)^k (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(-k)}(x) \frac{t^n}{n!} \tag{1.27}$$

(cf. [23]).

Putting $x = 0$ in (1.27), we have the Changhee numbers of order $-k$:

$$Ch_n^{(-k)} = Ch_n^{(-k)}(0)$$

(cf. [23]).

By using (1.27), a computation formula for the polynomials $Ch_n^{(-k)}(x)$ is given as follows:

$$Ch_n^{(-k)}(x) = \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} (x+j)_n$$

(cf. [23, Eq. (30)]). By aid of (1.7), (1.10) and (1.27), we get

$$E_n^{(-k)}(x) = \sum_{j=0}^n Ch_j^{(-k)}(x) S_2(n, j) \tag{1.28}$$

(cf. [23, Eq. (22)]).

The numbers $Y_n^{(\nu)}(\lambda)$ of order ν are defined by means of the following generating function:

$$\left(\frac{2}{\lambda(1+\lambda t) - 1}\right)^\nu = \sum_{n=0}^{\infty} Y_n^{(\nu)}(\lambda) \frac{t^n}{n!} \tag{1.29}$$

(cf. [25]).

Putting $\nu = 1$ in (1.29), we have the numbers $Y_n(\lambda)$:

$$Y_n(\lambda) = Y_n^{(1)}(\lambda)$$

(cf. [41]).

By using (1.29), an explicit formula for the numbers $Y_n(\lambda)$ is given as follows:

$$Y_n(\lambda) = \frac{2(-1)^n n!}{\lambda - 1} \left(\frac{\lambda^2}{\lambda - 1}\right)^n \tag{1.30}$$

(cf. [41]).

In the special case of (1.30) when $\lambda = -1$, we get

$$Y_n(-1) = (-1)^{n+1} Ch_n \tag{1.31}$$

(cf. [47]).

The numbers $Y_n^{(-k)}(\lambda)$ of order $-k$ are defined by means of the following generating function:

$$\left(\frac{\lambda(1+\lambda t)-1}{2}\right)^k = \sum_{n=0}^{\infty} Y_n^{(-k)}(\lambda) \frac{t^n}{n!}, \tag{1.32}$$

(cf. [26]).

By using (1.32), we get

$$Y_n^{(-k)}(\lambda) = n! (-1)^{k-n} 2^{-k} \lambda^n B_{n,k}(\lambda), \tag{1.33}$$

where $\lambda \in [0, 1]$ and $B_{v,n}(x)$ denotes the Bernstein basis functions which is defined by means of the following generating function:

$$\frac{(xt)^v e^{(1-x)t}}{v!} = \sum_{n=0}^{\infty} B_{v,n}(x) \frac{t^n}{n!}$$

(cf. [2], [36], [44]). For $x \in [0, 1]$, using the above generating function, one has

$$B_{v,n}(x) = \binom{n}{v} x^v (1-x)^{n-v},$$

where $v = 0, 1, 2, \dots, n$; $n \in \mathbb{N}_0$ and also $B_{v,n}(x) = 0$ if $v > n$ (cf. [2], [3], [13], [12], [36], [44]).

In the special case of (1.33) when $k = n$, we obtain

$$Y_n^{(-n)}(\lambda) = 2^{-n} n! \lambda^{2n} \tag{1.34}$$

(cf. [26]).

The numbers $y_6(m, n; \lambda, p)$ are defined by the following generating function:

$$\frac{1}{n!} \sum_{k=0}^n \binom{n}{k}^p \lambda^k e^{tk} = \sum_{m=0}^{\infty} y_6(m, n; \lambda, p) \frac{t^m}{m!} \tag{1.35}$$

(cf. [43]).

By using (1.35), an explicit formula for the numbers $y_6(m, n; \lambda, p)$ is given as follows:

$$y_6(m, n; \lambda, p) = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k}^p k^m \lambda^k \tag{1.36}$$

(cf. [43]).

In the special case of (1.36) when $m = 0$, $\lambda = 1$ and $p = 2$, we get a relation among the numbers $y_6(m, n; \lambda, p)$, the Daehee numbers and the Catalan numbers as follows:

$$y_6(0, n; 1, 2) = (-1)^n \frac{C_n}{D_n} \tag{1.37}$$

(cf. [43]).

This paper is organized as follows:

In Section 2, by using generating functions and functional equations techniques, we derive some new identities and combinatorial sums related to the central factorial numbers, the Stirling numbers, the Genocchi polynomials of negative order, the Euler numbers and polynomials of negative order, the Changhee numbers and polynomials of negative order, the Lah numbers, the numbers $B(n, k)$, and the Hermite polynomials.

In Section 3, we give many identities and formulas including the Bernoulli numbers of higher order, the Euler numbers of the first kind of negative order, the Catalan numbers, the Lah numbers, the Changhee numbers, the Daehee numbers, the Bernstein basis functions, and other combinatorial numbers and polynomials. Moreover, we derive some finite and infinite series representations related to the incomplete gamma function and aforementioned numbers.

Finally, in Section 4, we give conclusion and observations on our results.

2. Combinatorial sums and formulas for central factorial numbers, special numbers and polynomials

In this section, using generating functions and functional equations techniques, some of which are related to hyperbolic functions, we give some interesting identities and combinatorial sums related to the central factorial numbers, the Stirling numbers of the second kind, the Genocchi polynomials of negative order, the Euler numbers and polynomials of the first kind of negative order, the Changhee numbers and polynomials of negative order, the Lah numbers, the numbers $B(n, k)$, the numbers $y_6(m, n; \lambda, p)$, and the Hermite polynomials.

Theorem 2.1. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$T(n, k) = \frac{2^{2n+k}}{(2k)!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} E_{2n}^{(-j)} \left(\frac{-j}{2} \right) \tag{2.1}$$

and

$$\sum_{j=0}^k (-1)^{-j} \binom{k}{j} E_{2n+1}^{(-j)} \left(\frac{-j}{2} \right) = 0. \tag{2.2}$$

Proof. Using binomial theorem in (1.17), we have

$$\sum_{n=0}^{\infty} T(n, k) \frac{t^{2n}}{(2n)!} = \frac{2^k}{(2k)!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (\cosh(t))^j. \tag{2.3}$$

Since

$$\cosh(t) = \frac{e^t + e^{-t}}{2}$$

equation (2.3) reduces to the following functional equation:

$$\sum_{n=0}^{\infty} T(n, k) \frac{t^{2n}}{(2n)!} = \frac{2^k}{(2k)!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} H_{AE} \left(2t, \frac{-j}{2}, j \right).$$

Combining the above equation with (1.10), we get

$$\sum_{n=0}^{\infty} T(n, k) \frac{t^{2n}}{(2n)!} = \frac{2^k}{(2k)!} \sum_{n=0}^{\infty} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} 2^n E_n^{(-j)} \left(\frac{-j}{2} \right) \frac{t^n}{n!}.$$

Comparing the coefficients on both sides of the above equation, we arrive at the desired result. □

Remark 2.2. Note that Simsek [40] recently gave a relation between the numbers $E_n^{*(-k)}$ and the numbers $T(n, k)$ by the following result:

$$T(n, k) = \frac{2^k}{(2k)!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} E_{2n}^{*(-j)}$$

(cf. [40, Eq. (13)]).

Combining (2.1) with (1.15), we arrive at the following theorem:

Theorem 2.3. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$T(n, k) = \frac{2^k}{(2k)!} \sum_{j=0}^k \sum_{v=0}^{2n} (-1)^{k-j} \binom{k}{j} \binom{2n}{v} 2^{v-j} (-j)^{2n-v} B(v, j).$$

With the help of (2.2), (1.15) and (1.14), we also obtain the following combinatorial sum:

Corollary 2.4. Let $n, k \in \mathbb{N}_0$. Then we have

$$\sum_{v=0}^{2n+1} \sum_{j=0}^k \sum_{m=0}^j (-1)^{j+v} \binom{k}{j} \binom{j}{m} \binom{2n+1}{v} 2^{v-j} j^{2n+1-v} m^v = 0. \tag{2.4}$$

By combining (2.4) with (1.36), we arrive at the following corollary:

Corollary 2.5. Let $n, k \in \mathbb{N}_0$. Then we have

$$\sum_{v=0}^{2n+1} \sum_{j=0}^k (-1)^{j+v} j! \binom{2n+1}{v} \binom{k}{j} 2^{v-j} j^{2n+1-v} y_6(v, j; 1, 1) = 0.$$

Combining Theorem 2.1 with (1.28), we arrive at the following result:

Corollary 2.6. Let $n, k \in \mathbb{N}_0$. Then we have

$$T(n, k) = \frac{2^{2n+k}}{(2k)!} \sum_{j=0}^k \sum_{v=0}^{2n} (-1)^{k-j} \binom{k}{j} Ch_v^{(-j)} \left(\frac{-j}{2} \right) S_2(2n, v)$$

and

$$\sum_{j=0}^k \sum_{v=0}^{2n+1} (-1)^{-j} \binom{k}{j} Ch_v^{(-j)} \left(\frac{-j}{2} \right) S_2(2n+1, v) = 0.$$

Theorem 2.7. Let $n, k \in \mathbb{N}_0$. Then we have

$$T(n, k) = \frac{2^{2n+k}}{(2k)!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (2n)_j G_{2n-j}^{(-j)} \left(\frac{-j}{2} \right) \tag{2.5}$$

and

$$\sum_{j=0}^k (-1)^{-j} \binom{k}{j} (2n+1)_j G_{2n+1-j}^{(-j)} \left(\frac{-j}{2} \right) = 0. \tag{2.6}$$

Proof. By using (2.3), we have the following functional equation:

$$\sum_{n=0}^{\infty} T(n, k) \frac{t^{2n}}{(2n)!} = \frac{2^k}{(2k)!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (2t)^j H_{AG} \left(2t, \frac{-j}{2}, j \right).$$

Combining the above equation with (1.12), we have

$$\sum_{n=0}^{\infty} T(n, k) \frac{t^{2n}}{(2n)!} = \frac{2^k}{(2k)!} \sum_{n=0}^{\infty} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (n)_j 2^n G_{n-j}^{(-j)} \left(\frac{-j}{2} \right) \frac{t^n}{n!}.$$

Comparing the coefficients on both sides of the above equation, we arrive at the desired result. □

By combining (2.5) with (1.13), we arrive at the following result:

Corollary 2.8. Let $n, k \in \mathbb{N}_0$. Then we have

$$T(n, k) = \sum_{m=0}^{2n} \sum_{j=0}^k \sum_{v=0}^j (-1)^{k-j-m} \binom{k}{j} \binom{j}{v} \binom{2n}{m} \frac{j^{2n-m} 2^{m+k-v} v!}{(2k)!} S_2(m, v).$$

With the help of (2.6), (1.13) and (1.8), we also obtain the following combinatorial sum:

Corollary 2.9. Let $n, k \in \mathbb{N}_0$. Then we have

$$\sum_{m=0}^{2n+1} \sum_{j=0}^k \sum_{v=0}^j \sum_{d=0}^v (-1)^{j+d+1-m} \binom{k}{j} \binom{v}{d} \binom{2n+1}{m} j^{2n+1-m} \frac{(2v-2d)^m}{2^v} = 0.$$

Replacing x by $-x$ into (1.21), then combining the final equation with Theorem 2.7, we arrive at the following result:

Corollary 2.10. Let $n, k \in \mathbb{N}_0$. Then we have

$$T(n, k) = \frac{2^{2n+k}}{(2k)!} \sum_{j=0}^k \sum_{v=0}^j (-1)^{k-j} \binom{k}{j} L(j, v) (-2n)_v G_{2n-j}^{(-j)} \left(\frac{-j}{2}\right)$$

and

$$\sum_{j=0}^k \sum_{v=0}^j (-1)^j \binom{k}{j} L(j, v) (-2n-1)_v G_{2n+1-j}^{(-j)} \left(\frac{-j}{2}\right) = 0.$$

Theorem 2.11. Let $n \in \mathbb{N}_0$. Then we have

$$\sum_{j=0}^k \sum_{v=0}^{\lfloor \frac{n}{2} \rfloor} \binom{k}{j} \binom{n}{2v} 2^{-j} (2j)! j^{n-2v} T(v, j) = \sum_{m=0}^n \binom{2m}{m} m! Ch_m^{(-k)} S_2(n, 2m).$$

Proof. Putting $x = 0$ and using binomial theorem in (1.27), we have

$$\sum_{j=0}^k \binom{k}{j} 2^{-j} t^j = \sum_{m=0}^{\infty} Ch_m^{(-k)} \frac{t^m}{m!}. \tag{2.7}$$

Substituting $t = (e^u - 1)^2$ into the above equation, we get the following functional equation:

$$\sum_{j=0}^k \binom{k}{j} 2^{-j} (2j)! e^{uj} H_T(u, j) = \sum_{m=0}^{\infty} \frac{(2m)!}{m!} Ch_m^{(-k)} H_S(u, 2m).$$

Combining the above equation with (1.16) and (1.7), we have

$$\sum_{j=0}^k \binom{k}{j} 2^{-j} (2j)! \sum_{n=0}^{\infty} j^n \frac{u^n}{n!} \sum_{n=0}^{\infty} T(n, j) \frac{u^{2n}}{(2n)!} = \sum_{m=0}^{\infty} Ch_m^{(-k)} \frac{(2m)!}{m!} \sum_{n=0}^{\infty} S_2(n, 2m) \frac{u^n}{n!}.$$

Thus,

$$\sum_{n=0}^{\infty} \sum_{j=0}^k \sum_{v=0}^{\lfloor \frac{n}{2} \rfloor} \binom{k}{j} \binom{n}{2v} 2^{-j} (2j)! j^{n-2v} T(v, j) \frac{u^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{(2m)!}{m!} Ch_m^{(-k)} S_2(n, 2m) \frac{u^n}{n!}.$$

Comparing the coefficients of $\frac{u^n}{n!}$ on the both sides of the above equation, we arrive at the desired result. □

Theorem 2.12. Let $n \in \mathbb{N}_0$. Then we have

$$\sum_{j=0}^k \binom{k}{j} 2^{-j} j! S_2(n, j) = \sum_{m=0}^n Ch_m^{(-k)} S_2(n, m).$$

Proof. Substituting $t = (e^u - 1)$ into (2.7), we have the following functional equation:

$$\sum_{j=0}^k \binom{k}{j} 2^{-j} j! H_S(u, j) = \sum_{m=0}^{\infty} Ch_m^{(-k)} H_S(u, m).$$

Combining the above equation with (1.7), we have

$$\sum_{n=0}^{\infty} \sum_{j=0}^k \binom{k}{j} 2^{-j} j! S_2(n, j) \frac{u^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^n Ch_m^{(-k)} S_2(n, m) \frac{u^n}{n!}.$$

Comparing the coefficients of $\frac{u^n}{n!}$ on the both sides of the above equation, we arrive at the desired result. □

Theorem 2.13. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^n \binom{n}{j} 2^j E_j^{(-k)} H_{n-j}(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{n-2j} (-1)^j \binom{n}{2j} \frac{2^{n-2j} (2j)!}{j!} Ch_m^{(-k)}(x) S_2(n-2j, m).$$

Proof. Substituting $t = e^{2u} - 1$ into (1.27), we get the following functional equation:

$$G_H(u, x) H_{AE}(2u, x, k) = e^{-u^2} H_{Ch}(e^{2u} - 1, x, k).$$

Combining the above equation with (1.7), (1.10), (1.18) and (1.27), we have

$$\sum_{n=0}^{\infty} H_n(x) \frac{u^n}{n!} \sum_{n=0}^{\infty} 2^n E_n^{(-k)} \frac{u^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{n!} \sum_{m=0}^{\infty} Ch_m^{(-k)}(x) \sum_{n=0}^{\infty} 2^n S_2(n, m) \frac{u^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} 2^j E_j^{(-k)} H_{n-j}(x) \frac{u^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^{n-2j} \frac{(-1)^j 2^{n-2j}}{j! (n-2j)!} Ch_m^{(-k)}(x) S_2(n-2j, m) u^n.$$

Comparing the coefficients of u^n on the both sides of the above equation, after some elementary calculations, we arrive at the desired result. □

Theorem 2.14. *Let $n \in \mathbb{N}_0$. Then we have*

$$S_2(n, k) = \frac{2^{k-n}}{k!} \sum_{m=0}^n \sum_{j=0}^k (-1)^{k-j} \binom{n}{m} \binom{k}{j} j^m E_{n-m}^{(-k)}.$$

Proof. By using (1.7) and (1.10), we have the following functional equation:

$$2^k (e^t - 1)^k H_{NE}(t, 0, k) = k! H_S(2t, k).$$

From the above equation, we get

$$2^k \sum_{n=0}^{\infty} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n \frac{t^n}{n!} \sum_{n=0}^{\infty} E_n^{(-k)} \frac{t^n}{n!} = k! \sum_{n=0}^{\infty} 2^n S_2(n, k) \frac{t^n}{n!}.$$

Thus,

$$2^k \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^m E_{n-m}^{(-k)} \frac{t^n}{n!} = k! \sum_{n=0}^{\infty} 2^n S_2(n, k) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the desired result. □

3. Identities for Bernoulli numbers of higher order and other special numbers

In this section, by using some known formulas, we give many novel identities and relations related to the Bernoulli numbers of higher order, the Bernoulli and Euler numbers of negative order, the Stirling numbers, the Catalan numbers, the Lah numbers, the numbers $B(n, k)$, the Bernstein basis functions, the Changhee numbers, the Daehee numbers, and the combinatorial numbers. Furthermore, we give some finite and infinite series representations including aforementioned numbers and incomplete gamma function.

Theorem 3.1. *Let $n \in \mathbb{N}_0$. Then we have*

$$B_n^{(-k)} = 2^{-n} \sum_{j=0}^n \binom{n}{j} B_j^{(-k)} E_{n-j}^{(-k)}. \tag{3.1}$$

Proof. By using (1.5) and (1.10), we have the following functional equation:

$$H_{NB}(t, 0, k) H_{NE}(t, 0, k) = H_{NB}(2t, 0, k).$$

From the above equation, we get

$$\sum_{n=0}^{\infty} B_n^{(-k)} \frac{t^n}{n!} \sum_{n=0}^{\infty} E_n^{(-k)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n^{(-k)} \frac{(2t)^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} B_j^{(-k)} E_{n-j}^{(-k)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} 2^n B_n^{(-k)} \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the desired result. □

Combining (3.1) with (1.15), we arrive at the following corollary:

Corollary 3.2. *Let $n \in \mathbb{N}_0$. Then we have*

$$B_n^{(-k)} = \sum_{j=0}^n \binom{n}{j} \frac{B_j^{(-k)} B(n-j, k)}{2^{k+n}}. \tag{3.2}$$

With the aid of (3.2) and (1.4), we have the following corollary:

Corollary 3.3. *Let $n \in \mathbb{N}_0$. Then we have*

$$\frac{B_n^{(-k)}}{B_n^{(n+1)}} = \sum_{j=0}^n \frac{B_j^{(-k)} B(n-j, k)}{2^{k+n} B_j^{(j+1)} B_{n-j}^{(n-j+1)}}.$$

With the aid of (3.2) and (1.6), we also have the following corollaries:

Corollary 3.4. *Let $n \in \mathbb{N}_0$. Then we have*

$$B_n^{(-k)} = \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{j+k}{k} 2^{k+n}} S_2(j+k, k) B(n-j, k).$$

Corollary 3.5. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$S_2(n+k, k) = \sum_{j=0}^n \frac{\binom{n}{j} \binom{n+k}{k}}{2^{k+n} \binom{j+k}{k}} S_2(j+k, k) B(n-j, k).$$

Combining (1.30) with (1.4), we have the following corollary:

Corollary 3.6. Let $n \in \mathbb{N}_0$. Then we have

$$B_n^{(n+1)} = \frac{(\lambda - 1)^{n+1}}{2\lambda^{2n}} Y_n(\lambda). \tag{3.3}$$

In the special case of (3.3) when $\lambda = -1$, and combining the final equation with (1.31), we also arrive at the following corollary:

Corollary 3.7. Let $n \in \mathbb{N}_0$. Then we have

$$B_n^{(n+1)} = 2^n Ch_n. \tag{3.4}$$

By combining (1.24) with (1.4), we arrive at the following corollary:

Corollary 3.8. Let $n \in \mathbb{N}_0$. Then we have

$$B_n^{(n+1)} = (n + 1) D_n. \tag{3.5}$$

By aid of (3.5) and (1.37), we also arrive at the following corollary:

Corollary 3.9. Let $n \in \mathbb{N}_0$. Then we have

$$B_n^{(n+1)} = \frac{(-1)^n (n + 1) C_n}{y_6(0, n; 1, 2)}.$$

Combining (1.34) with (1.4), we arrive at the following corollary:

Corollary 3.10. Let $n \in \mathbb{N}_0$. Then we have

$$B_n^{(n+1)} = \frac{(-2)^n}{\lambda^{2n}} Y_n^{(-n)}(\lambda). \tag{3.6}$$

By aid of (3.5) and (3.6), we also arrive at the following corollary:

Corollary 3.11. Let $n \in \mathbb{N}_0$. Then we have

$$Y_n^{(-n)}(\lambda) = (-1)^n 2^{-n} \lambda^{2n} (n + 1) D_n. \tag{3.7}$$

Combining (1.33) with (1.4) and (1.26), we arrive at the following corollary:

Corollary 3.12. Let $n, k \in \mathbb{N}_0$. Then we have

$$Y_n^{(-k)}(\lambda) = \frac{\lambda^n}{k!} Ch_k B_n^{(n+1)} B_{n,k}(\lambda).$$

With the help of the above equation and (3.5), we arrive at the following corollary:

Corollary 3.13. Let $n, k \in \mathbb{N}_0$. Then we have

$$Y_n^{(-k)}(\lambda) = \frac{\lambda^n}{k!} (n + 1) D_n Ch_k B_{n,k}(\lambda). \tag{3.8}$$

By aid of (3.8) and (1.37), we also arrive at the following corollary:

Corollary 3.14. Let $n, k \in \mathbb{N}_0$. Then we have

$$Y_n^{(-k)}(\lambda) = \frac{(-\lambda)^n (n + 1) Ch_k C_n B_{n,k}(\lambda)}{k! y_6(0, n; 1, 2)}.$$

Theorem 3.15. Let $n \in \mathbb{N}_0$. Then we have

$$L(2n + 1, n + 1) = -\frac{(2n + 1)!}{n!} C_n. \tag{3.9}$$

Proof. By using (1.20), we have

$$L(2n + 1, n + 1) = -\frac{(2n + 1)!}{(n + 1)!} \binom{2n}{n}.$$

Combining the above equation with (1.22), we arrive at the desired result. □

With the help of (3.9) and (1.37), we also arrive at the following corollary:

Corollary 3.16. *Let $n \in \mathbb{N}_0$. Then we have*

$$L(2n + 1, n + 1) = \frac{(-1)^{n+1} (2n + 1)!}{n!} y_6(0, n; 1, 2) D_n.$$

By combining (3.9) with (1.4), we have the following corollary:

Corollary 3.17. *Let $n \in \mathbb{N}_0$. Then we have*

$$L(2n + 1, n + 1) = \frac{(-1)^n B_{2n+1}^{(2n+2)} C_n}{B_n^{(n+1)}}.$$

By combining (3.9) with (1.24), we also have the following corollary:

Corollary 3.18. *Let $n \in \mathbb{N}_0$. Then we have*

$$L(2n + 1, n + 1) = (-1)^n \frac{2D_{2n+1} C_n}{D_n}.$$

3.1. Finite and infinite series representations involving special numbers and polynomials

In this subsection, we give some finite and infinite series representations which are related to the incomplete gamma function and special numbers such as the numbers $Y_n^{(-k)}(\lambda)$, the Changhee numbers, the Daehee numbers, and the Bernstein basis functions.

Theorem 3.19. *The following sum holds true:*

$$\sum_{n=0}^{\infty} (-1)^n \frac{D_n - Ch_n}{D_n Ch_n} = e(e - 2).$$

Proof. By using (1.26), we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{Ch_n} = \sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

Hence

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{Ch_n} = e^2. \tag{3.10}$$

And using (1.24), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{D_n} &= \sum_{n=0}^{\infty} \frac{n + 1}{n!} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n - 1)!} + \sum_{n=0}^{\infty} \frac{1}{n!} \\ &= 2e. \end{aligned} \tag{3.11}$$

After removing Equation (3.10) from Equation (3.11) and after some elementary calculations, the desired result is easily found. □

Theorem 3.20. We assume that

$$|\lambda^2| < 2.$$

Then we have

$$\sum_{n=0}^{\infty} \frac{Y_n^{(-n)}(\lambda)}{D_n} = \frac{4}{(2 + \lambda^2)^2}.$$

Proof. By using (3.7), we have

$$\sum_{n=0}^{\infty} \frac{Y_n^{(-n)}(\lambda)}{D_n} = \sum_{n=0}^{\infty} (-1)^n (n + 1) \left(\frac{\lambda^2}{2}\right)^n.$$

Now, assuming that

$$\left|\frac{\lambda^2}{2}\right| < 1,$$

we obtain

$$\sum_{n=0}^{\infty} \frac{Y_n^{(-n)}(\lambda)}{D_n} = \frac{-2\lambda^2}{(2 + \lambda^2)^2} + \frac{2}{2 + \lambda^2}.$$

After some elementary calculations in the above equation, proof of theorem is completed. □

Theorem 3.21. Let $\lambda \neq 0$. Then the following identity holds true:

$$\sum_{n=0}^k \frac{Y_n^{(-k)}(\lambda)}{B_{n,k}(\lambda)} = \frac{(-1)^k e^{\frac{1}{\lambda}}}{2^k \lambda} \left((-1)^k (k + 1)! \Gamma\left(-k - 1, \frac{1}{\lambda}\right) + \Gamma\left(0, \frac{1}{\lambda}\right) \right). \tag{3.12}$$

Proof. By using (1.33), we have

$$\sum_{n=0}^k \frac{Y_n^{(-k)}(\lambda)}{B_{n,k}(\lambda)} = \frac{(-1)^k}{2^k} \sum_{n=0}^k (-1)^n \lambda^n n!. \tag{3.13}$$

Combining the above equation with (1.2), and after some elementary calculations, we obtain

$$\sum_{n=0}^k \frac{Y_n^{(-k)}(\lambda)}{B_{n,k}(\lambda)} = \frac{(-1)^k e^{\frac{1}{\lambda}}}{2^k \lambda} \left((-1)^k (k + 1)! \Gamma\left(-k - 1, \frac{1}{\lambda}\right) + \Gamma\left(0, \frac{1}{\lambda}\right) \right),$$

where $\lambda \neq 0$. Thus, proof of theorem is completed. □

Observe that using the well-known formula: $\Gamma(k + 1) = k!$, ($k \in \mathbb{N}_0$), Equation (3.12) reduces to the following relation:

$$\sum_{n=0}^k \frac{Y_n^{(-k)}(\lambda)}{B_{n,k}(\lambda)} = \frac{e^{\frac{1}{\lambda}}}{2^k \lambda} \left(\Gamma(k + 2) \Gamma\left(-k - 1, \frac{1}{\lambda}\right) + (-1)^k \Gamma\left(0, \frac{1}{\lambda}\right) \right).$$

Combining (3.12) with (1.4), we arrive at the following corollary:

Corollary 3.22. Let $\lambda \neq 0$. Then the following identity holds true:

$$\sum_{n=0}^k \frac{Y_n^{(-k)}(\lambda)}{B_{n,k}(\lambda)} = \frac{(-1)^k e^{\frac{1}{\lambda}}}{2^k \lambda} \left(\Gamma\left(0, \frac{1}{\lambda}\right) - B_{k+1}^{(k+2)} \Gamma\left(-k - 1, \frac{1}{\lambda}\right) \right). \tag{3.14}$$

Combining (3.14) with (1.26), we arrive at the following corollary:

Corollary 3.23. Let $\lambda \neq 0$. Then the following identity holds true:

$$\sum_{n=0}^k \frac{Y_n^{(-k)}(\lambda)}{B_{n,k}(\lambda)} = \frac{e^{\frac{1}{\lambda}} Ch_k}{\lambda \Gamma(k + 1)} \left(\Gamma\left(0, \frac{1}{\lambda}\right) - B_{k+1}^{(k+2)} \Gamma\left(-k - 1, \frac{1}{\lambda}\right) \right).$$

By using (1.4), (1.24), (1.26) and (3.13), we get the following corollary:

Corollary 3.24. *The following identities holds true:*

$$\sum_{n=0}^k \frac{Y_n^{(-k)}(\lambda)}{B_{n,k}(\lambda)} = \frac{Ch_k}{k!} \sum_{n=0}^k (n+1) \lambda^n D_n,$$

$$\sum_{n=0}^k \frac{Y_n^{(-k)}(\lambda)}{B_{n,k}(\lambda)} = \frac{Ch_k}{k!} \sum_{n=0}^k (2\lambda)^n Ch_n,$$

and

$$\sum_{n=0}^k \frac{Y_n^{(-k)}(\lambda)}{B_{n,k}(\lambda)} = \frac{Ch_k}{k!} \sum_{n=0}^k \lambda^n B_n^{(n+1)}.$$

4. Conclusion

Generating functions and their functional equations, trigonometric functions and their applications are used in many different areas. In this paper, we used them to investigate many families of special numbers and polynomials. Using both the generating functions and their functional equations techniques and some known results, we obtained many new identities, formulas, and combinatorial sums including the Genocchi polynomials of negative order, the Euler numbers and polynomials of negative order, the Changhee numbers and polynomials of negative order, the Bernoulli numbers of higher order, the Hermite polynomials, the Lah numbers, the central factorial numbers, the Daehee numbers, the Bernstein basis functions, the Stirling numbers, and also the combinatorial numbers and polynomials. Furthermore, we derived some finite and infinite series representations that include the gamma function, the incomplete gamma function and aforementioned numbers. Consequently, the results of this paper have the potential to be used and applied in numerous areas such as theory of combinatorial analysis, approximation theory, analytic number theory, probability theory, physics, engineering, and other related areas.

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