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Energy of Prime Graphs and its Bounds

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Abstract

The special properties of factorization of primes makes it vitally important to communication. The RSA encryption system uses prime numbers to encrypt data. Motivated by the work on Fibonacci graph by Adiga et al.[1], in this paper we define a prime graph and examine its eigenvalues and obtain upper and lower bounds for the energy of the prime graph.

Keywords: Prime graph, Energy of a graph, Bounds for energy of a graph, Ramanujan graph

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1. Introduction

Let G be a simple, undirected graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of G is denoted by n and the number of edges by m. For $v_i \in V(G)$ the degree of v_i, written by $d(v_i)$ or d_i is the number of edges incident with v_i. The adjacency matrix $A(G)$ of G is defined by its entries as $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of the adjacency matrix $A = A(G)$ of G. These eigenvalues are called the eigenvalues of G and they form its spectrum. The largest eigenvalue, namely λ_1 is called the spectral radius of G and is well known that

$$\sum_{i=1}^{n} \lambda_i = 0,$$

$$\sum_{i=1}^{n} \lambda_i^2 = 2m,$$

and

$$detA = \prod_{i=1}^{n} \lambda_i.$$

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A graph in which all the vertices are of same degree d is called a d-regular graph. Also, a graph G is said to be singular if at least one of its eigenvalues is equal to zero. For singular graphs, $detA = 0$. A graph is non-singular if all its eigenvalues are different from zero. Then, $|detA| > 0$.

The graph energy is a graph-spectrum based quantity which can be used in estimating π-electron energy of the given conjugated hydrocarbons, which was shown as the summation of absolute values of eigenvalues of the adjacency matrix of the graph. The energy of the graph G is defined by $\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|$, where λ_i, $i = 1, 2, 3...n$ are the eigenvalues of G. For details and an exhaustive list of references on graph energy one can refer the monograph [8].

Among the pioneering results of the theory of graph energy are the upper and lower bound for $\varepsilon(G)$, discovered by Mcclelland [9] given as,

$$\varepsilon(G) \leq \sqrt{2mn} \tag{1.1}$$

and

$$\varepsilon(G) \geq \sqrt{2m + n(n-1)|detA|^{2/n}}. \tag{1.2}$$

Recently K. Das et al.[4] and K. Das, Mojallal S.A., Gutman I.,[3] improved lower bounds for energy. They gave the following lower bound valid for non-singular graphs:

$$\varepsilon(G) \geq \frac{2m}{n} + (n-1) + ln|detA| - ln\left(\frac{2m}{n}\right). \tag{1.3}$$

They have also obtained the following lower bound for the largest eigenvalue

$$\lambda_1 \geq \frac{2(m-\delta)}{n-1}, \tag{1.4}$$

where m is the number of edges of the graph G with n vertices and the minimum degree of the graph is δ. Many research articles have been published on the bounds for the energy of graph. Recently Akbar Jahanabani [7] proved some lower bounds for the energy of graphs. One of them is

$$\varepsilon(G) \geq \frac{2(m-\delta)}{n-1} + n - 1 + ln|detA| - ln\left(\frac{2(m-\delta)}{n-1}\right). \tag{1.5}$$

In this era, where we are constantly using computer encryption it is obvious that we are regularly using prime numbers. Since all the modern security relies on the understanding of prime numbers it is essential that we study various characteristics of prime numbers. Study of prime numbers and its distribution have been the focus of mathematicians for centuries. During 18^{th} century both Gauss and Legendre made similar conjectures regarding $\pi(n)$ the number of primes less than or equal to n. In 1798 Legendre estimated that $\pi(x) \approx \dfrac{x}{logx - 1.08366}$. Gauss conjectured a similar result that $\pi(x) \approx \int_2^x \dfrac{1}{logt}dt$. In 1852, Chebychev stated that $\pi(x) \sim \dfrac{x}{logx}$, where $\pi(x)$ is the number of primes less than or equal to x, which is the prime number theorem proved later by Jacques Hadamard and Charles-Jean de la Vallée Poussin. The error bounds in prime number theorem are useful in many applications. P. Dusart [6] found new bounds for $\pi(x)$.

In this article we mainly focus on an interesting family of graphs called prime graphs. A prime graph $P_{\pi(n),2n}$ is a graph with vertex set $V = V_1 \cup V_2$, where

$$V_1 = \{v_1, v_2 \ldots v_n\},$$

$$V_2 = \{v_{n+1}, v_{n+2} \ldots v_{2n}\}$$

and $v_i v_j$ for $1 \leq i \leq n$ and $n+1 \leq j \leq 2n$ is an edge if $j - i + 1$ or $j - i + 1 - n$ is a member of the set

$$S = \{p_1, p_2 \ldots p_{\pi(n)}\}.$$

Here $\pi(n)$ is the number of primes less than or equal to n.

The below graph is a prime graph $P_{3,12}$ with 12 vertices and degree of each vertex being 3.

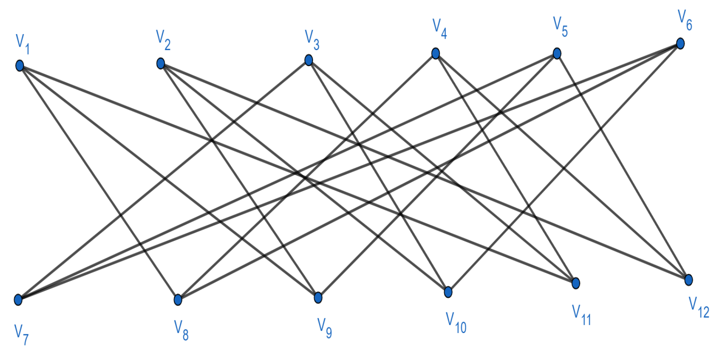

Figure 1. Prime graph $P_{3,12}$

The rest of the paper is organized as follows. In Section 2 of this paper we establish eigenvalues of the prime graph $P_{\pi(n),2n}$ and in Section 3, we compute some lower and upper bounds for the energy of the prime graph in terms of the number of vertices, determinant of the adjacency matrix and the degree of the prime graph.

2. Eigenvalues of the prime Graph

In this section we establish the eigenvalues of the prime graph. The prime graph $P_{\pi(n),2n}$ is a connected, bipartite, regular graph of degree $\pi(n)$ and in consequence, its spectra have the following distinctive features:

1. $\pi(n)$ is an eigenvalue of $P_{\pi(n),2n}$.
2. The multiplicity of $\pi(n)$ is 1.
3. For any eigenvalue λ of $P_{\pi(n),2n}$, we have $|\lambda| \leq \pi(n)$.
4. The spectrum of $P_{\pi(n),2n}$ is symmetric about 0, that is , if λ is an eigenvalue of $P_{\pi(n),2n}$, then $-\lambda$ is also an eigenvalue of $P_{\pi(n),2n}$.

Definition 2.1. A circulant matrix of order "n"is a square matrix of order "n"in which all the rows are obtainable by successive cyclic shifts of one of its rows (usually taken as the first row) and so any circulant matrix is determined by its first row.

Lemma 2.2. *Let C be a circulant matrix of order "n"with first row $(a_1, a_2 \ldots a_n)$ then, the eigenvalues of C are*

$$\lambda_r = \sum_{j=1}^{n} a_j \omega^{(j-1)r}, \quad r = 0, 1, 2 \ldots (n-1).$$

Here, $\omega = exp(2\pi i/n)$.

Using the above result we now compute the eigenvalues of $P_{\pi(n),2n}$. The adjacency matrix A of $P_{\pi(n),2n}$ is,

$$\text{A}=\begin{pmatrix} 0 & C \\ C^T & 0 \end{pmatrix}_{2n\times 2n}.$$

Here, $C = C_{n\times n}$ is a circulant matrix whose first row is $\{a_1, a_2 \ldots a_n\}$, where

$$a_i = \begin{cases} 1 & \text{if} \quad i = p_1, p_2 \ldots p_{\pi(n)}, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $P_{\pi(n),2n}$ is

$$|\lambda I_{2n} - A| = |\lambda^2 I_n - CC^T| = |\lambda^2 I_n - D|,$$

where $D = CC^T$ and I_n is the identity matrix of order n. We have,

$$\text{D}=\begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \ldots & a_{n-2} & a_{n-1} \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ a_2 & a_3 & a_4 & \ldots & a_n & a_1 \end{pmatrix}\begin{pmatrix} a_1 & a_n & a_{n-1} & \ldots & a_2 \\ a_2 & a_1 & a_n & \ldots & a_3 \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_1 \end{pmatrix}.$$

Observe that C^T is a circulant matrix. Hence $D = CC^T$ is also circulant.

Let $(d_1, d_2, d_3 \ldots d_n)$ be the first row of the circulant matrix D. Since,

$${a_i}^2 = \begin{cases} 1 & \text{if} \quad i = p_1, p_2 \ldots p_{\pi(n)}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $d_1 = {a_1}^2 + {a_2}^2 + \ldots + {a_n}^2 = \pi(n)$.
For $j \geq 2$,

$$d_j = a_1 a_{n-(j-2)} + a_2 a_{n-(j-3)} + \ldots + a_j a_1 + a_{j+1} a_2 + \ldots + a_n a_{n-(j-1)}. \tag{2.1}$$

Each term in the right side of the above equation is either one or zero. It is one, precisely when both suffices of a term are prime numbers. Thus, d_j is equal to the number of prime pairs in the set

$$S_1 = \{(n-(j-2), 1), (n-(j-3), 2) \ldots (n, j-1), (1, j) \ldots (n-(j-1), n)\}.$$

Similarly d_{n-j+2} is the number of prime pairs in the set

$$S_2 = \{(j, 1), (j+1, 2) \ldots (n, n-j+1), (1, n-j+2) \ldots (n, j-1)\}.$$

As $S_1 = S_2$, it follows that

$$d_j = d_{n-j+2}, \quad for \qquad j \geq 2. \tag{2.2}$$

Therefore by Lemma 2.3, the eigenvalues of $P_{\pi(n),2n}$ are

$$\pm \sqrt{\pi(n) + d_2\omega^r + d_3\omega^{2r} + \ldots + d_n\omega^{r(n-1)}}, r = 0, 1, 2 \ldots n-1, \tag{2.3}$$

where

$$\omega = e^{2\pi i/n} = cos(2\pi/n) + isin(2\pi/n).$$

From (2.2), we have

$$d_2 = d_n = 1, \quad d_3 = d_{n-1} \ldots d_{(n+1)/2} = d_{(n+3)/2}, \text{ for odd n,} \tag{2.4}$$

and

$$d_2 = d_n = 1, d_3 = d_{n-1} \ldots d_{n/2} = d_{(n+4)/2}, \text{ for even n.} \tag{2.5}$$

Suppose n is odd. Using (2.4) in (2.3) and the fact that $\omega^n = 1$, we see that the eigenvalues of $P_{\pi(n),2n}$ are

$$\pm \sqrt{\pi(n) + d_2(\omega^r + \omega^{-r}) + d_3(\omega^{2r} + \omega^{-2r}) + \ldots + d_{(n+1)/2}(\omega^{(n-1)r/2} + \omega^{-(n-1)r/2})}$$

$$= \pm \sqrt{\pi(n) + 2\sum_{j=1}^{(n-1)/2} d_{j+1}cos(2\pi jr/n)}, \quad r = 0, 1, 2 \ldots (n-1).$$

Similarly for even n one can show that the eigenvalues of $P_{\pi(n),2n}$ are

$$\pm\sqrt{\pi(n)+2\sum_{j=1}^{(n-2)/2} d_{j+1}cos(2\pi jr/n)+d_{(n+2)/2}(-1)^r},\quad r=0,1,2\ldots(n-1).$$

Thus we have proved the following theorem.

Theorem 2.3. *If "n"is odd, then the eigenvalues of $P_{\pi(n),2n}$ are*

$$\pm\sqrt{\pi(n)+2\sum_{j=1}^{(n-1)/2} d_{j+1}cos(2\pi jr/n)},\qquad r=0,1,2\ldots(n-1).$$

and, if "n"is even then the eigenvalues of $P_{\pi(n),2n}$ are

$$\pm\sqrt{\pi(n)+2\sum_{j=1}^{(n-2)/2} d_{j+1}cos(2\pi jr/n)+d_{(n+2)/2}(-1)^r},\qquad r=0,1,2\ldots(n-1).$$

For example, the eigenvalues of $P_{3,10}$ are ± 3, $\pm\frac{\sqrt{5}+1}{2}$, $\pm\frac{\sqrt{5}+1}{2}$, $\pm\frac{\sqrt{5}-1}{2}$, $\pm\frac{\sqrt{5}-1}{2}$ and its energy is $\varepsilon(P_{3,10})=6+4\sqrt{5}$.
Eigenvalues of $P_{5,24}$ are ±5, ±3, $\pm\sqrt{5}$, $\pm\sqrt{5}$, ±2, ±2, $\pm\sqrt{2}$, $\pm\sqrt{2}$, $\pm\sqrt{2}$, $\pm\sqrt{2}$, 0, 0, 0, 0, and its energy is $\varepsilon(P_{5,24})=24+4\sqrt{5}+8\sqrt{2}$.

3. Lower and Upper Bounds for the Energy of prime Graph

In this section, we give lower and upper bounds for the energy $\varepsilon(P_{\pi(n),2n})$ in terms of n, $\pi(n)$ and the determinant of the adjacency matrix of $P_{\pi(n),2n}$.

Theorem 3.1. *Let $P_{\pi(n),2n}$ be the prime graph with $2n$ vertices and m edges. Let $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_{2n-1}\geq\lambda_{2n}$ be a non-increasing arrangement of eigenvalues of $P_{\pi(n),2n}$. Then the following inequality is valid.*

$$\varepsilon(P_{\pi(n),2n})\leq\sqrt{n^2\pi(n)(\pi(n)+4)}.$$

Proof. Let $a_1,a_2...a_{2n}$ and $b_1,b_2...b_{2n}$ be real numbers for which there exist real constants a, b, A and B, such that for each $i=1,2...2n$, $a\leq a_i\leq A$ and $b\leq b_i\leq B$. Then the following inequality [2] is valid.

$$\left|2n\sum_{i=1}^{2n}a_ib_i-\sum_{i=1}^{2n}a_i\sum_{i=1}^{2n}b_i\right|\leq\alpha(2n)(A-a)(B-b),$$

where $\alpha(2n)=2n^2\left(1-\frac{1}{2n}\left[\frac{2n}{2}\right]\right)=2n^2\left(1-\frac{1}{2}\right)=n^2$. Equality holds if and only if $a_1=a_2=...=a_{2n}$ and $b_1=b_2=...=b_{2n}$.

Let $\lambda_1\geq\lambda_2\geq...\geq\lambda_n\geq\lambda_{n+1}\geq...\geq\lambda_{2n-1}\geq\lambda_{2n}$ be the eigenvalues of $P_{\pi(n),2n}$. Since $P_{\pi(n),2n}$ is a bipartite, regular graph, the spectrum of $P_{\pi(n),2n}$ is symmetric with respect to zero, that is, if λ is an eigenvalue of $P_{\pi(n),2n}$, then $-\lambda$ is also an eigenvalue of $P_{\pi(n),2n}$.
Thus, $|\lambda_1|=|\lambda_{2n}|,|\lambda_2|=|\lambda_{2n-1}|...|\lambda_n|=|\lambda_{n+1}|$.
Define $\mu_1=|\lambda_1|$, $\mu_2=|\lambda_{2n}|$, $\mu_3=|\lambda_2|$, $\mu_4=|\lambda_{2n-1}|...\mu_{2n-1}=|\lambda_n|$, $\mu_{2n}=|\lambda_{n+1}|$.
Note that $0\leq\lambda_n\leq\mu_i\leq\lambda_1$ for $i=1,2...2n$. Setting $a_i=|\mu_i|$, $b_i=|\mu_i|$, $a=b=\lambda_n$ and $A=B=\lambda_1$ in the above inequality, we obtain

$$\left|2n\sum_{i=1}^{2n}\mu_i^2-\sum_{i=1}^{2n}\mu_i\sum_{i=1}^{2n}\mu_i\right|\leq n^2(\lambda_1-\lambda_n)^2.$$

Since, $\sum_{i=1}^{2n} \mu_i^2 = 2m = 2\pi(n)n$, we have,

$$\left|4n^2\pi(n) - \varepsilon^2(P_{\pi(n),2n})\right| \leq n^2\pi^2(n).$$

Or equivalently,

$$|\varepsilon^2(P_{\pi(n),2n})| - |4n^2\pi(n)| \leq n^2\pi^2(n),$$

and hence,

$$\varepsilon(P_{\pi(n),2n}) \leq \sqrt{n^2\pi(n)(\pi(n)+4)}.$$

□

Theorem 3.2. *For the prime graph $P_{\pi(n),2n}$, the following inequality holds good:*

$$\varepsilon(P_{\pi(n),2n}) \geq n(1+\lambda_n).$$

Proof. Let $a_1, a_2...a_{2n}$ and $b_1, b_2...b_{2n}$ be real numbers for which there exist real constants r and R such that for each i, $i = 1, 2...2n$, $ra_i \leq b_i \leq Ra_i$. Then the following inequality [5] is valid.

$$\sum_{i=1}^{2n} b_i^2 + rR\sum_{i=1}^{2n} a_i^2 \leq (r+R)\sum_{i=1}^{2n} a_i b_i.$$

Equality holds if and only if, for at least one i, $1 \leq i \leq 2n$, $ra_i = b_i = Ra_i$ holds. Using the above inequality, the definition of μ_i from the previous theorem and on substituting $b_i = |\mu_i|, a_i = 1, r = |\lambda_n| = \lambda_n, R = |\mu_1|$ for $i = 1, 2...2n$, we get,

$$\sum_{i=1}^{2n} |\mu_i|^2 + |\lambda_1||\lambda_n|\sum_{i=1}^{2n} 1 \leq (|\lambda_n| + |\lambda_1|)\sum_{i=1}^{2n} |\mu_i|.$$

Since,

$$\sum_{i=1}^{2n} |\mu_i|^2 = \sum_{i=1}^{2n} |\lambda_i|^2 = 2n\pi(n),$$

$$\sum_{i=1}^{2n} 1 = 2n,$$

and

$$\sum_{i=1}^{2n} |\mu_i| = \sum_{i=1}^{2n} |\lambda_i| = \varepsilon(P_{\pi(n),2n}),$$

we deduce that,

$$2n\pi(n) + \pi(n)\lambda_n 2n \leq 2\pi(n)\varepsilon(P_{\pi(n),2n}).$$

This completes the proof. □

Theorem 3.3. *Let $P_{\pi(n),2n}$ be a non-empty and non-singular graph. Then for* n > 60184 *we have,*

$$\varepsilon(P_{\pi(n),2n}) \geq \frac{n}{ln\,n} + (2n-1) + ln|detA| - ln\left(\frac{n}{ln\,n - 1.1}\right).$$

Proof. Consider the function $f(x) = x - 1 - lnx$ for $x > 0$. It is elementary to prove that $f(x)$ is increasing for $x \geq 1$ and decreasing for $0 < x \leq 1$.

Consequently, $f(x) \geq f(1) = 0$ implying that $x \geq 1 + lnx$ for $x > 0$, with equality holds if and only if $x = 1$.
Using the above result we get

$$\begin{aligned}
\varepsilon(P_{\pi(n),2n}) &= \lambda_1 + \sum_{i=2}^{2n} |\lambda_i| \\
&\geq \lambda_1 + \sum_{i=2}^{2n} 1 + \sum_{i=2}^{2n} ln|\lambda_i| \\
&= \lambda_1 + (2n-1) + ln\prod_{i=1}^{2n} |\lambda_i| - ln\lambda_1 \\
&= \lambda_1 + (2n-1) + ln|detA| - ln\lambda_1.
\end{aligned}$$

P. Dusart [6], proved in 2010 that

$$\frac{x}{lnx - 1} < \pi(x) < \frac{x}{lnx - 1.1} \text{ for } x > 60184.$$

Using this result, we get the required lower bound. □

Lemma 3.4. *[7] If G is a graph with n-vertices, m edges and minimum degree δ, then*

$$\lambda_1(G) \geq \frac{2(m-\delta)}{n-1}.$$

A family of d-regular graphs G with $d \geq 3$ is called an expander graph if all nontrivial eigenvalues of the adjacency matrices are uniformly bounded away from d. A connected d-regular graph is a Ramanujan graph if $|\lambda_2| \leq 2\sqrt{d-1}$.

Ramanujan graphs have been the focus of study in theoretical Computer Science and Mathematics as they are optimal expanders. They also have wide range of applications in coding theory, Neural connections and many other types of networks. Expander graphs were first defined by Bassalygo and Pinsker, and their existence first proved by Pinsker.

Theorem 3.5. *Let $P_{\pi(n),2n}$ be a non-singular graph, which is not a Ramanujan graph and if the second highest eigenvalue is ≥ 1, then using Lemma 3.4 we have,*

$$\varepsilon(P_{\pi(n),2n}) \geq \frac{2(n-1)\pi(n)}{2n-1} + 2\sqrt{\pi(n)-1} + 2(n-1) + ln|detA| - 2ln\left(\frac{n}{ln\,n - 1.1}\right).$$

Proof. Since G is non-singular, $|\lambda_i| > 0$ for $i = 1, 2...2n$. Using the function defined in the proof of previous theorem, we have,

$$\begin{aligned}
\varepsilon(P_{\pi(n),2n}) &= \sum_{i=1}^{2n} |\lambda_i| \\
&= \lambda_1 + \lambda_2 + \sum_{i=3}^{2n} |\lambda_i| \\
&\geq \lambda_1 + \lambda_2 + \sum_{i=3}^{2n} 1 + \sum_{i=3}^{2n} ln|\lambda_i| \\
&\geq \frac{2(n\pi(n) - \pi(n))}{2n-1} + 2\sqrt{\pi(n)-1} + (2n-2) + ln|detA| - ln(\lambda_1) - ln(\lambda_2) \\
&= \frac{2(n-1)\pi(n)}{2n-1} + 2\sqrt{\pi(n)-1} + (2n-2) + ln|detA| - 2ln\left(\frac{n}{ln\,n - 1.1}\right).
\end{aligned}$$

□

Theorem 3.6. *If $P_{\pi(n),2n}$ is a non-singular graph, then*

$$\varepsilon(P_{\pi(n),2n}) \geq 2\sqrt{\pi(n)} + (2n-2)\left(\frac{|detA|}{\pi(n)}\right)^{\frac{1}{2(n-1)}}.$$

Proof. We know that

$$\begin{aligned}\varepsilon(P_{\pi(n),2n}) &= |\lambda_1| + |\lambda_{2n}| + \sum_{i=1, i\neq 1,2n}^{2n} |\lambda_i| \\ &\geq 2\lambda_1 + 2(n-1)\left(\frac{\prod_{i=1}^{2n}|\lambda_i|}{\lambda_1^2}\right)^{\frac{1}{2(n-1)}}.\end{aligned}$$

Now consider the function

$$f(x) = 2x + 2(n-1)\left(\frac{|detA|}{x^2}\right)^{\frac{1}{2(n-1)}}.$$

It is easy to check that the function $f(x)$ is an increasing function for

$$x \geq |detA|^{\frac{1}{2n}}.$$

Moreover we have $\varepsilon(G) \leq \sqrt{2mn}$, which implies

$$\frac{\varepsilon(P_{\pi(n),2n})}{2n} \leq \sqrt{\pi(n)}.$$

We know that,

$$\lambda_1 = \pi(n) \geq \sqrt{\pi(n)} \geq \frac{\varepsilon(P_{\pi(n),2n})}{2n} \geq |detA|^{\frac{1}{2n}}.$$

Thus,

$$f(\pi(n)) \geq f(\sqrt{\pi(n)}).$$

Equivalently,

$$2\pi(n) + (2n-2)\left(\frac{|detA|}{\pi^2(n)}\right)^{\frac{1}{2n-2}} \geq 2\sqrt{\pi(n)} + (2n-2)\left(\frac{|detA|}{\pi(n)}\right)^{\frac{1}{2n-2}}.$$

Hence,

$$\varepsilon(P_{\pi(n),2n}) \geq 2\sqrt{\pi(n)} + 2(n-1)\left(\frac{|detA|}{\pi(n)}\right)^{\frac{1}{2(n-1)}}.$$

□

Theorem 3.7. *If $P_{\pi(n),2n}$ is not a Ramanujan graph and the second largest eigenvalue of $P_{\pi(n),2n}$ is non- negative, then we have*

$$\varepsilon(P_{\pi(n),2n}) \leq 2\sqrt{\pi(n)-1} + \sqrt{2(2n-1)(\pi(n)(n-2)+2)}.$$

Proof. Let $(a_1, a_2...a_n)$ and $(b_1, b_2...b_n)$ be two sequences of real numbers. Then

$$\sum_{i=1}^{n} {a_i}^2 \sum_{i=1}^{n} {b_i}^2 \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

with equality holds if and only if the sequence $(a_1, a_2...a_n)$ and $(b_1, b_2...b_n)$ are proportional, i.e., there is a constant λ such that $a_k = \lambda b_k$ for each $k \in \{1, 2...n\}$. In the above inequality if we choose $a_i = |\lambda_i|$ and $b_i = 1$, and using Lemma 3.4 we get,

$$\left(\sum_{i=1,i\neq 2}^{2n} |\lambda_i|.1\right)^2 \leq \sum_{i=1,i\neq 2}^{2n} |\lambda_i|^2 \sum_{i=1,i\neq 2}^{2n} 1.$$

Taking the square root, we obtain

$$\sum_{i=1,i\neq 2}^{2n} |\lambda_i| \leq \sqrt{\sum_{i=1,i\neq 2}^{2n} |\lambda_i|^2 (2n-1)}.$$

Hence,

$$\varepsilon(P_{\pi(n),2n}) \leq \lambda_2 + \sqrt{(2m - \lambda_2^2)(2n-1)}.$$

Note that the function

$$F(x) = x + \sqrt{(2m - x^2)(2n-1)},$$

decreases for $\sqrt{\pi(n)} \leq x \leq \sqrt{2m}$.
Since $P_{\pi(n),2n}$ is not Ramanujan, we have,

$$\lambda_2 \geq 2\sqrt{\pi(n) - 1} \geq \sqrt{\pi(n)}.$$

Hence,

$$F(\lambda_2) \leq F(2\sqrt{\pi(n) - 1}).$$

This implies

$$\lambda_2 + \sqrt{(2n\pi(n) - \lambda_2^2)(2n-1)} \leq 2\sqrt{\pi(n) - 1} + \sqrt{(2n\pi(n) - 4(\pi(n) - 1))(2n-1)}.$$

Hence,

$$\varepsilon(P_{\pi(n),2n}) \leq 2\sqrt{\pi(n) - 1} + \sqrt{(2n\pi(n) - 4(\pi(n) - 1))(2n-1)}.$$

□

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