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Some Hypergeometric Summation Theorems and Reduction Formulas via Laplace Transform Method

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Abstract

In this paper, we obtain analytical solutions of Laplace transform based some generalized class of the hyperbolic integrals in terms of hypergeometric functions ${}_3F_2(\pm1)$, ${}_4F_3(\pm1)$, ${}_5F_4(\pm1)$, ${}_6F_5(\pm1)$, ${}_7F_6(\pm1)$ and ${}_8F_7(\pm1)$ with suitable convergence conditions, by using some algebraic properties of Pochhammer symbols. In addition, reduction formulas for ${}_4F_3(1)$, ${}_7F_6(-1)$ and some new summation theorems (not recorded earlier in the literature of hypergeometric functions) for ${}_3F_2(-1)$, ${}_6F_5(\pm1)$, ${}_7F_6(\pm1)$ and ${}_8F_7(\pm1)$ are obtained.

Keywords: Generalized hypergeometric functions, Summation and multiplication theorems, Laplace transform, Beta and Gamma function

2010 MSC: 33C05, 33C20, 44A10, 33B15

1. Introduction and Preliminaries

For the sake of conciseness of this paper, we use the following notations

$\mathbb{N} := \{1, 2, ...\}$; $\quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $\quad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} = \{0, -1, -2, -3, ...\}$,

where the symbols $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of natural numbers and integers; as usual, the symbols $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers respectively.

Here the notation $(\lambda)_n$ $(\lambda, n \in \mathbb{C})$ denotes the Pochhammer's symbol (or the shifted factorial, since $(1)_n = n!$) is defined, in general, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & (n = 0\ ;\ \lambda \in \mathbb{C}\backslash\{0\}) \\ \lambda(\lambda+1)...(\lambda+n-1), & (n \in \mathbb{N}\ ;\ \lambda \in \mathbb{C}). \end{cases} \tag{1.1}$$

A natural generalization of Gauss hypergeometric series ${}_2F_1$ is the general hypergeometric series ${}_pF_q$ with p numerator parameters $\alpha_1, ..., \alpha_p$ and q denominator parameters $\beta_1, ..., \beta_q$. It is defined by (see, for example [10, p.42, eq.(1)]

$${}_pF_q\left(\begin{array}{l} \alpha_1, ..., \alpha_p\ ; \\ \beta_1, ..., \beta_q\ ; \end{array} z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n...(\alpha_p)_n}{(\beta_1)_n...(\beta_q)_n} \frac{z^n}{n!}\ , \tag{1.2}$$

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where $\alpha_i \in \mathbb{C}\ (i = 1, ..., p)$ and $\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-\ (j = 1, ..., q)$ $\left(\mathbb{Z}_0^- := \{0, -1, -2, ...\}\right)$ and $(\ p,\ q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, ...\})$. The ${}_pF_q(\cdot)$ series in eq.(1.2) is convergent for $|z| < \infty$ if $p \leq q$, and for $|z| < 1$ if $p = q + 1$. Furthermore, if we set

$$\omega = \left(\sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i\right), \tag{1.3}$$

it is known that the ${}_pF_q$ series, with $p = q + 1$, is
(i) absolutely convergent for $|z| = 1$ if $\Re(\omega) > 0$,
(ii) conditionally convergent for $|z| = 1, z \neq 1$, if $-1 < Re(\omega) \leq 0$.
The binomial function is given by

$$(1 - z)^{-a} = {}_1F_0\left(\begin{array}{l} a\ ; \\ -; \end{array}\ z\right) = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n, \tag{1.4}$$

where $|z| < 1,\ \ a \in \mathbb{C}$.
Next we collect some results that we will need in the sequel.

$$B_z(\alpha,\beta) = \int_0^z t^{\alpha-1}(1 - t)^{\beta-1}dt, \qquad 0 < z \leq 1, \tag{1.5}$$

$${}_2F_1\left(\begin{array}{l} a,\ \ b; \\ 1 + b, ; \end{array}\ z\right) = \frac{b}{z^b}B_z(b, 1 - a), \tag{1.6}$$

where B_z is incomplete beta function [3, p.87]
The Dixon's theorem for ${}_3F_2$ with positive unit argument [9, p.92, Theorem 33], is given by

$${}_3F_2\left(\begin{array}{l} a,\ \ b,\ \ c \qquad\qquad\quad ; \\ 1 + a - b,\ 1 + a - c; \end{array}\ 1\right) = \frac{\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 + \frac{a}{2})\Gamma(1 + \frac{a}{2} - b - c)}{\Gamma(1 + \frac{a}{2} - b)\Gamma(1 + \frac{a}{2} - c)\Gamma(1 + a)\Gamma(1 + a - b - c)}, \tag{1.7}$$

where $\Re(a - 2b - 2c) > -2;\ 1 + a - b,\ 1 + a - c \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
When $c = 1 + \frac{a}{2}$ in the eq.(1.7), we get

$${}_3F_2\left(\begin{array}{l} a,\ \ 1 + \frac{a}{2},\ \ b; \\ \frac{a}{2},\ \ 1 + a - b; \end{array}\ 1\right) = 0, \tag{1.8}$$

where $\Re(b) < 0;\ \frac{a}{2}, 1 + a - b \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
The summation theorem for ${}_3F_2(-1)$ [1, 9] is given by

$${}_3F_2\left(\begin{array}{l} a,\ \ 1 + \frac{a}{2},\ \ b; \\ \frac{a}{2},\ \ 1 + a - b; \end{array}\ -1\right) = \frac{\Gamma(1 + a - b)\Gamma(\frac{1+a}{2})}{\Gamma(\frac{1+a}{2} - b)\Gamma(1 + a)}, \tag{1.9}$$

where $\Re(b) < \frac{1}{2};\ \frac{a}{2}, 1 + a - b \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
The eq.(1.9) can be obtained by setting $2d = 1 + a$ and $c = b$ in the eq.(1.15).
One useful contiguous function relation [9, p.71, Q.N0.21, Part(13)] is given below:

$$(\beta - \gamma + 1)\ {}_2F_1\left(\begin{array}{l} \alpha,\ \beta; \\ \gamma\ \ \ \ ; \end{array}\ z\right) = \beta\ {}_2F_1\left(\begin{array}{l} \alpha,\ \beta + 1; \\ \gamma \qquad\quad ; \end{array}\ z\right) - (\gamma - 1)\ {}_2F_1\left(\begin{array}{l} \alpha,\ \beta; \\ \gamma - 1\ ; \end{array}\ z\right). \tag{1.10}$$

In the both sides of eq.(1.10) replace z by zt, multiply by $t^{h-1}(1 - t)^{g-h-1}$, integrate with respect to t over the interval $(0, 1)$ and using the definition of beta function, after simplification we get

$$(\beta - \gamma + 1)\ {}_3F_2\left(\begin{array}{l} \alpha,\ \beta,\ h; \\ \gamma,\ g\ \ \ \ ; \end{array}\ z\right) = \beta\ {}_3F_2\left(\begin{array}{l} \alpha,\ \beta + 1,\ h; \\ \gamma,\ g \qquad\quad ; \end{array}\ z\right) - (\gamma - 1)\ {}_3F_2\left(\begin{array}{l} \alpha,\ \beta,\ h; \\ \gamma - 1,\ g; \end{array}\ z\right). \tag{1.11}$$

In the eq.(1.11) put $\alpha = a,\ \beta = c,\ h = b,\ \gamma = 1 + d,\ g = 1 + c$, after simplification we get

$${}_3F_2\left(\begin{array}{l} a,\ b,\ c \qquad\ ; \\ 1 + c,\ 1 + d; \end{array}\ z\right) = \frac{c}{(c - d)}\ {}_2F_1\left(\begin{array}{l} a,\ b; \\ d + 1; \end{array}\ z\right) - \frac{d}{(c - d)}\ {}_3F_2\left(\begin{array}{l} a,\ b,\ c; \\ c + 1, d; \end{array}\ z\right). \tag{1.12}$$

Put $d = b$ in eq.(1.12), we get

$$ {}_3F_2\left(\begin{array}{c} a,\ b,\ c\ ; \\ 1+b,\ 1+c; \end{array} z\right) = \left(\frac{c}{c-b}\right){}_2F_1\left(\begin{array}{c} a,\ b; \\ 1+b; \end{array} z\right) - \left(\frac{b}{c-b}\right){}_2F_1\left(\begin{array}{c} a,\ c; \\ 1+c; \end{array} z\right), \tag{1.13} $$

where $c \neq b$ and $1+b, 1+c \in \mathbb{C}\backslash\mathbb{Z}_0^-$. When $z = 1$ in the eq.(1.13) and using Gauss classical summation theorem [9, p.49, Theorem 18], we get

$$ {}_3F_2\left(\begin{array}{c} a,\ b,\ c\ ; \\ 1+b,\ 1+c; \end{array} 1\right) = \frac{bc}{(c-b)}\Gamma(1-a)\left\{\frac{\Gamma(b)}{\Gamma(1+b-a)} - \frac{\Gamma(c)}{\Gamma(1+c-a)}\right\}, \tag{1.14} $$

where $c \neq b$; $\Re(a) < 1$; $1+b, 1+c \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

The classical summation theorems for hypergeometric series ${}_4F_3(\pm 1)$ [1, p.28, eq.(4.4.3)] are given by

$$ {}_4F_3\left(\begin{array}{c} a,\ 1+\frac{a}{2},\ c,\ d\ ; \\ \frac{a}{2},\ 1+a-c,\ 1+a-d; \end{array} -1\right) = \frac{\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma(1+a-c-d)}, \tag{1.15} $$

provided $\Re(a-2c-2d) > -2$; $\frac{a}{2}, 1+a-c, 1+a-d \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
When $e = (1+a)/2$ in the eq.(1.17), we get

$$ {}_4F_3\left(\begin{array}{c} a,\ 1+\frac{a}{2},\ c,\ d\ ; \\ \frac{a}{2},\ 1+a-c,\ 1+a-d; \end{array} 1\right) = \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(\frac{1+a}{2})\Gamma(\frac{1+a}{2}-c-d)}{\Gamma(1+a)\Gamma(\frac{1+a}{2}-d)\Gamma(\frac{1+a}{2}-c)\Gamma(1+a-c-d)}, \tag{1.16} $$

provided $\Re(2c+2d-a) < 1$; $\frac{a}{2}, 1+a-c, 1+a-d \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Another classical summation theorem for hypergeometric series ${}_5F_4(1)$ [1, p.27,eq.(4.4.1)], is given by

$$ {}_5F_4\left(\begin{array}{c} a,\ 1+\frac{a}{2},\ c,\ d,\ e\ ; \\ \frac{a}{2},\ 1+a-c,\ 1+a-d,\ 1+a-e; \end{array} 1\right) = \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-c-d)}, \tag{1.17} $$

provided $\Re(a-c-d-e) > -1$; $\frac{a}{2}, 1+a-c, 1+a-d, 1+a-e \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
Laplace transform of any constant k is given by

$$ \mathcal{L}[k;q] = \int_0^\infty e^{-qt} k\, dt = \frac{k}{q}, \tag{1.18} $$

provided

$$ \Re(q) > 0. \tag{1.19} $$

The Digamma function (or Psi function) is given by [4, pp. 902-903]; see also [5]

$$ \begin{aligned} \Psi(x) &= \frac{d}{dx}\{\ln\Gamma(x)\} = \frac{\Gamma'(x)}{\Gamma(x)}, \\ &= -\gamma + (x-1)\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+x)}, \end{aligned} \tag{1.20} $$

where $\gamma(= 0.57721566490...)$ being the Euler-Mascheroni constant and $\Psi'(x) = \frac{d}{dx}\{\Psi(x)\}$ is called a trigamma function, defined by

$$ \Psi'(x) = \sum_{k=0}^{\infty}\frac{1}{(x+k)^2} = \frac{1}{x^2}\,{}_3F_2\left(\begin{array}{c} 1,\ x, x\ ; \\ 1+x,\ 1+x; \end{array} 1\right). \tag{1.21} $$

It can be also written by

$$ {}_3F_2\left(\begin{array}{c} 1,\ x, x\ ; \\ 1+x,\ 1+x; \end{array} 1\right) = x^2\Psi'(x), \tag{1.22} $$

and

$$ {}_3F_2\left(\begin{matrix} 1,\ a, b & ; \\ 1+a,\ 1+b; \end{matrix}\ 1\right) = \frac{ab}{(b-a)}\left[\Psi(b) - \Psi(a)\right], \quad b \neq a. \tag{1.23} $$

Properties of Digamma function [10, p.25]; see also [7]

$$ \Psi(1+x) = \Psi(x) + \frac{1}{x}, \tag{1.24} $$

$$ \Psi(1-x) = \Psi(x) + \pi\cot(\pi x), \tag{1.25} $$

where $x \neq 0, \pm 1, \pm 2, \pm 3, ...$

$$ \Psi\left(\frac{1}{2}+x\right) - \Psi\left(\frac{1}{2}-x\right) = \pi\tan(\pi x), \tag{1.26} $$

where $x \neq \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, ...$

Lower case beta function of one variable which is related with Digamma function [4, pp.906-907]; see also [5], is given by

$$ \beta(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+x)} = \frac{1}{x}\,{}_2F_1\left(\begin{matrix} 1,\ x; \\ 1+x; \end{matrix}\ -1\right), \tag{1.27} $$

$$ = \frac{1}{2}\left[\Psi\left(\frac{1+x}{2}\right) - \Psi\left(\frac{x}{2}\right)\right], \qquad x \in \mathbb{C}\backslash\mathbb{Z}_0^-, \tag{1.28} $$

and

$$ {}_3F_2\left(\begin{matrix} 1,\ a, b & ; \\ 1+a,\ 1+b; \end{matrix}\ -1\right) = \frac{ab}{(b-a)}\left[\beta(a) - \beta(b)\right], \quad b \neq a, \tag{1.29} $$

$$ {}_3F_2\left(\begin{matrix} 1,\ x, x & ; \\ 1+x,\ 1+x; \end{matrix}\ -1\right) = -x^2\frac{d}{dx}\{\beta(x)\} = -x^2\beta'(x). \tag{1.30} $$

Hypergeometric forms of some trigonometric ratios [6, pp.137-138, eq.(4.2.6); eq.(4.2.9); eq.(4.2.3)] and [8, pp.48-49, eqns(20,23,26)] , are given by

$$ \tan(z) = \frac{8z}{(\pi^2-4z^2)}\ {}_3F_2\left(\begin{matrix} 1,\ \frac{1}{2}+\frac{z}{\pi},\ \frac{1}{2}-\frac{z}{\pi}; \\ \frac{3}{2}+\frac{z}{\pi},\ \frac{3}{2}-\frac{z}{\pi}\quad ; \end{matrix}\ 1\right), \tag{1.31} $$

where $z \in \mathbb{C} \setminus \{\pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2}, ...\}$,

$$ \sec(z) = \frac{4\pi}{(\pi^2-4z^2)}\ {}_4F_3\left(\begin{matrix} 1,\ \frac{3}{2},\ \frac{1}{2}+\frac{z}{\pi},\ \frac{1}{2}-\frac{z}{\pi}; \\ \frac{1}{2},\ \frac{3}{2}+\frac{z}{\pi},\ \frac{3}{2}-\frac{z}{\pi}\quad ; \end{matrix}\ -1\right), \tag{1.32} $$

where $z \in \mathbb{C} \setminus \{\pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2}, ...\}$,

$$ \sec^2(z) = \frac{4}{(2z-\pi)^2}\ {}_3F_2\left(\begin{matrix} 1,\ \frac{1}{2}-\frac{z}{\pi},\ \frac{1}{2}-\frac{z}{\pi}; \\ \frac{3}{2}-\frac{z}{\pi},\ \frac{3}{2}-\frac{z}{\pi}\ ; \end{matrix}\ 1\right) + \frac{4}{(2z+\pi)^2}\ {}_3F_2\left(\begin{matrix} 1,\ \frac{1}{2}+\frac{z}{\pi},\ \frac{1}{2}+\frac{z}{\pi}; \\ \frac{3}{2}+\frac{z}{\pi},\ \frac{3}{2}+\frac{z}{\pi}\ ; \end{matrix}\ 1\right), \tag{1.33} $$

where $z \in \mathbb{C} \setminus \{\pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2}, ...\}$.

The plan of this paper is as follows, we find two reduction formulas and some new summation theorems given in `sections 2-3` by comparing the similar integrals. Next, we obtain generalized class and analytical solutions of some integral formulas including hyperbolic functions in terms of ${}_6F_5(\pm 1)$, ${}_7F_6(\pm 1)$, ${}_8F_7(\pm 1)$ shown in `section 4`. The special class of some integral formulas including hyperbolic functions in terms of ${}_3F_2(\pm 1), {}_4F_3(\pm 1), {}_5F_4(\pm 1)$ are given in `section 5`. We apply suitable product formulas associated with hyperbolic function in special class of hyperbolic integrals given in `section 6`.

2. Some new summation theorems

We have given in this section the following summation theorems for the hypergeometric functions ${}_3F_2(-1), {}_6F_5(\pm1), {}_7F_6(\pm1), {}_8F_7(\pm1)$. Its applications may be for calculation of more suitable results in the hypergeometric functions.

Theorem 2.1.

$$
{}_6F_5\left(\begin{array}{c} v,\ 1+\frac{v}{2},\ \frac{v}{2}-\frac{a}{2c}-\frac{b}{2c},\ \frac{v}{2}+\frac{a}{2c}+\frac{b}{2c},\ \frac{v}{2}-\frac{a}{2c}+\frac{b}{2c},\ \frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}\ ; \\ \frac{v}{2},\ 1+\frac{v}{2}-\frac{a}{2c}-\frac{b}{2c},\ 1+\frac{v}{2}+\frac{a}{2c}+\frac{b}{2c},\ 1+\frac{v}{2}-\frac{a}{2c}+\frac{b}{2c},\ 1+\frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}\ ; \end{array} -1\right)
$$

$$
= \frac{\{(vc)^2-(a+b)^2\}\{(vc)^2-(a-b)^2\}}{16vabc^2\Gamma(v)}\Big[\Gamma\left(\frac{vc+a+b}{2c}\right)\Gamma\left(\frac{vc-a-b}{2c}\right) - \Gamma\left(\frac{vc+a-b}{2c}\right)\Gamma\left(\frac{vc-a+b}{2c}\right)\Big], \quad (2.1)
$$

where $\Re(v) < 4,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2},\ 1+\frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-.$

Proof. Comparing the two equations (4.1) and (6.4), we get a summation theorem for ${}_6F_5(-1)$. □

Theorem 2.2.

$$
{}_6F_5\left(\begin{array}{c} v,\ 1+\frac{v}{2},\ \frac{v}{2}-\frac{a}{2c}-\frac{b}{2c},\ \frac{v}{2}+\frac{a}{2c}+\frac{b}{2c},\ \frac{v}{2}-\frac{a}{2c}+\frac{b}{2c},\ \frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}\ ; \\ \frac{v}{2},\ 1+\frac{v}{2}-\frac{a}{2c}-\frac{b}{2c},\ 1+\frac{v}{2}+\frac{a}{2c}+\frac{b}{2c},\ 1+\frac{v}{2}-\frac{a}{2c}+\frac{b}{2c},\ 1+\frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}\ ; \end{array} 1\right)
$$

$$
= \frac{\{(vc)^2-(a+b)^2\}\{(vc)^2-(a-b)^2\}}{16vabc^2\Gamma(v)}\Big[\Gamma\left(\frac{vc+a+b}{2c}\right)\Gamma\left(\frac{vc-a-b}{2c}\right)\frac{\cos\left(\frac{(a+b)\pi}{2c}\right)}{\cos(\frac{v\pi}{2})} - \Gamma\left(\frac{vc+a-b}{2c}\right)\Gamma\left(\frac{vc-a+b}{2c}\right)\frac{\cos\left(\frac{(a-b)\pi}{2c}\right)}{\cos(\frac{v\pi}{2})}\Big], \quad (2.2)
$$

where $\Re(v) < 3,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2},\ 1+\frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-.$

Proof. Comparing the two equations (4.2) and (6), we get a summation theorem for ${}_6F_5(1)$. □

Theorem 2.3.

$$
{}_6F_5\left(\begin{array}{c} 1, \frac{3}{2}, \frac{1}{2}-\frac{a}{2b}-\frac{c}{2b}, \frac{1}{2}-\frac{a}{2b}+\frac{c}{2b}, \frac{1}{2}+\frac{a}{2b}+\frac{c}{2b}, \frac{1}{2}+\frac{a}{2b}-\frac{c}{2b}\ ; \\ \frac{1}{2}, \frac{3}{2}-\frac{a}{2b}-\frac{c}{2b}, \frac{3}{2}-\frac{a}{2b}+\frac{c}{2b}, \frac{3}{2}+\frac{a}{2b}+\frac{c}{2b}, \frac{3}{2}+\frac{a}{2b}-\frac{c}{2b}; \end{array} -1\right)
$$

$$
= \frac{\pi(b-a-c)(b+a+c)(b-a+c)(b+a-c)}{2^2acb^2}\frac{\sin\left(\frac{a\pi}{2b}\right)\sin\left(\frac{c\pi}{2b}\right)}{\{\cos\left(\frac{c\pi}{b}\right)+\cos\left(\frac{a\pi}{b}\right)\}}, \quad (2.3)
$$

where $\Re(b) > 0,\ \Re(b \pm a \pm c) > 0,\ \frac{3}{2} \pm \frac{a}{2b} \pm \frac{c}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-, \frac{a\pm c}{b} \in \mathbb{C}\backslash\{\pm1, \pm3, \pm5, ...\}.$

Proof. Comparing the two equations (6.7) and (6.5), we get a summation theorem for ${}_6F_5(-1)$. □

Theorem 2.4.

$$
{}_7F_6\left(\begin{array}{c} v, 1+\frac{v}{2}-\frac{\sqrt{a^2-b^2}}{2c}, 1+\frac{v}{2}+\frac{\sqrt{a^2-b^2}}{2c}, \frac{v}{2}-\frac{a}{2c}-\frac{b}{2c}, \\ \frac{v}{2}-\frac{a}{2c}+\frac{b}{2c}, \frac{v}{2}+\frac{a}{2c}+\frac{b}{2c}, \frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}\,; \\ \frac{v}{2}-\frac{\sqrt{a^2-b^2}}{2c}, \frac{v}{2}+\frac{\sqrt{a^2-b^2}}{2c}, 1+\frac{v}{2}-\frac{a}{2c}-\frac{b}{2c}, \\ 1+\frac{v}{2}-\frac{a}{2c}+\frac{b}{2c}, 1+\frac{v}{2}+\frac{a}{2c}+\frac{b}{2c}, 1+\frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}; \end{array} 1\right)
$$

$$
= \frac{\{(vc)^2-(a+b)^2\}\{(vc)^2-(a-b)^2\}}{16(v^2ac^3-a^3c+ab^2c)\Gamma(v)}\Bigg[\Gamma\left(\frac{vc+a+b}{2c}\right)\Gamma\left(\frac{vc-a-b}{2c}\right)\frac{\sin\left(\frac{(a+b)\pi}{2c}\right)}{\sin(\frac{v\pi}{2})}
+\Gamma\left(\frac{vc+a-b}{2c}\right)\Gamma\left(\frac{vc-a+b}{2c}\right)\frac{\sin\left(\frac{(a-b)\pi}{2c}\right)}{\sin(\frac{v\pi}{2})}\Bigg], \quad (2.4)
$$

where $\Re(v) < 2,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2} \pm \frac{\sqrt{a^2-b^2}}{2c},\ 1+\frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Proof. Comparing the two equations (4.4) and (6.13), we get a summation theorem for ${}_7F_6(1)$. □

Theorem 2.5.

$$
{}_7F_6\left(\begin{array}{c} 1, \frac{3}{2}-\frac{\sqrt{a^2-c^2}}{2b}, \frac{3}{2}+\frac{\sqrt{a^2-c^2}}{2b}, \frac{1}{2}-\frac{a}{2b}-\frac{c}{2b}, \\ \frac{1}{2}-\frac{a}{2b}+\frac{c}{2b}, \frac{1}{2}+\frac{a}{2b}+\frac{c}{2b}, \frac{1}{2}+\frac{a}{2b}-\frac{c}{2b}\,; \\ \frac{1}{2}-\frac{\sqrt{a^2-c^2}}{2b}, \frac{1}{2}+\frac{\sqrt{a^2-c^2}}{2b}, \frac{3}{2}-\frac{a}{2b}-\frac{c}{2b}, \\ \frac{3}{2}-\frac{a}{2b}+\frac{c}{2b}, \frac{3}{2}+\frac{a}{2b}+\frac{c}{2b}, \frac{3}{2}+\frac{a}{2b}-\frac{c}{2b}; \end{array} 1\right)
= \frac{\pi(b-a-c)(b+a+c)(b-a+c)(b+a-c)}{4(ab^3-a^3b+abc^2)}\frac{\sin\left(\frac{a\pi}{b}\right)}{\{\cos\left(\frac{c\pi}{b}\right)+\cos\left(\frac{a\pi}{b}\right)\}}, \quad (2.5)
$$

where $\Re(b) > 0,\ \Re(b \pm a \pm c) > 0;\ \frac{1}{2} \pm \frac{\sqrt{a^2-c^2}}{2b},\ \frac{3}{2} \pm \frac{a}{2b} \pm \frac{c}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ \frac{a\pm c}{b} \in \mathbb{C}\backslash\{\pm1, \pm3, \pm5, ...\}$.

Proof. Comparing the two equations (6.16) and (6.14), we get a summation theorem for ${}_7F_6(1)$. □

Theorem 2.6.

$$
{}_7F_6\left(\begin{array}{c} 1, \frac{3}{2}-\frac{\sqrt{a^2-c^2}}{2b}, \frac{3}{2}+\frac{\sqrt{a^2-c^2}}{2b}, \frac{1}{2}-\frac{a}{2b}-\frac{c}{2b}, \\ \frac{1}{2}-\frac{a}{2b}+\frac{c}{2b}, \frac{1}{2}+\frac{a}{2b}+\frac{c}{2b}, \frac{1}{2}+\frac{a}{2b}-\frac{c}{2b}\,; \\ \frac{1}{2}-\frac{\sqrt{a^2-c^2}}{2b}, \frac{1}{2}+\frac{\sqrt{a^2-c^2}}{2b}, \frac{3}{2}-\frac{a}{2b}-\frac{c}{2b}, \\ \frac{3}{2}-\frac{a}{2b}+\frac{c}{2b}, \frac{3}{2}+\frac{a}{2b}+\frac{c}{2b}, \frac{3}{2}+\frac{a}{2b}-\frac{c}{2b}; \end{array} -1\right)
$$

$$
= \frac{\pi(b-a-c)(b+a+c)(b-a+c)(b+a-c)}{2(ab^3-a^3b+abc^2)}\frac{\cos\left(\frac{a\pi}{2b}\right)\cos\left(\frac{c\pi}{2b}\right)}{\{\cos\left(\frac{c\pi}{b}\right)+\cos\left(\frac{a\pi}{b}\right)\}}
- \frac{(b-a-c)(b+a+c)(b-a+c)(b+a-c)}{4(ab^3-a^3b+abc^2)}\left[\beta\left(\frac{(a+b+c)}{2b}\right)+\beta\left(\frac{(a+b-c)}{2b}\right)\right], \quad (2.6)
$$

where $\Re(b) > 0,\ \Re(b \pm a \pm c) > 0;\ \frac{1}{2} \pm \frac{\sqrt{a^2-c^2}}{2b},\ \frac{3}{2} \pm \frac{a}{2b} \pm \frac{c}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-, \frac{a\pm c}{b} \in \mathbb{C}\backslash\{\pm1, \pm3, \pm5, ...\}$.

Proof. Comparing the two equations (6.12) and (6.10), we get a summation theorem for ${}_7F_6(-1)$. □

Theorem 2.7.

$$
{}_8F_7\left(\begin{array}{l} v, 1+\frac{v}{2}, 1+\frac{v}{2}-\frac{\sqrt{a^2+b^2}}{2c}, 1+\frac{v}{2}+\frac{\sqrt{a^2+b^2}}{2c}, \\ \qquad \frac{v}{2}-\frac{a}{2c}-\frac{b}{2c}, \frac{v}{2}-\frac{a}{2c}+\frac{b}{2c}, \frac{v}{2}+\frac{a}{2c}+\frac{b}{2c}, \frac{v}{2}+\frac{a}{2c}-\frac{b}{2c} ; \\ \frac{v}{2}, \frac{v}{2}-\frac{\sqrt{a^2+b^2}}{2c}, \frac{v}{2}+\frac{\sqrt{a^2+b^2}}{2c}, 1+\frac{v}{2}-\frac{a}{2c}-\frac{b}{2c}, \\ \qquad 1+\frac{v}{2}-\frac{a}{2c}+\frac{b}{2c}, 1+\frac{v}{2}+\frac{a}{2c}+\frac{b}{2c}, 1+\frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}; \end{array} -1\right)
$$

$$
= \frac{\{(vc)^2-(a+b)^2\}\{(vc)^2-(a-b)^2\}}{8(v^3c^4-a^2vc^2-b^2vc^2)\Gamma(v)}\left[\Gamma\left(\frac{vc+a+b}{2c}\right)\Gamma\left(\frac{vc-a-b}{2c}\right) + \Gamma\left(\frac{vc+a-b}{2c}\right)\Gamma\left(\frac{vc-a+b}{2c}\right)\right], \quad (2.7)
$$

where $\Re(v) < 2,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2},\ \frac{v}{2} \pm \frac{\sqrt{a^2+b^2}}{2c},\ 1+\frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Proof. Comparing the two equations (4.5) and (6.17), we get a summation theorem for ${}_8F_7(-1)$. □

Theorem 2.8.

$$
{}_8F_7\left(\begin{array}{l} v, 1+\frac{v}{2}, 1+\frac{v}{2}-\frac{\sqrt{a^2+b^2}}{2c}, 1+\frac{v}{2}+\frac{\sqrt{a^2+b^2}}{2c}, \\ \qquad \frac{v}{2}-\frac{a}{2c}-\frac{b}{2c}, \frac{v}{2}-\frac{a}{2c}+\frac{b}{2c}, \frac{v}{2}+\frac{a}{2c}+\frac{b}{2c}, \frac{v}{2}+\frac{a}{2c}-\frac{b}{2c} ; \\ \frac{v}{2}, \frac{v}{2}-\frac{\sqrt{a^2+b^2}}{2c}, \frac{v}{2}+\frac{\sqrt{a^2+b^2}}{2c}, 1+\frac{v}{2}-\frac{a}{2c}-\frac{b}{2c}, \\ \qquad 1+\frac{v}{2}-\frac{a}{2c}+\frac{b}{2c}, 1+\frac{v}{2}+\frac{a}{2c}+\frac{b}{2c}, 1+\frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}; \end{array} 1\right)
$$

$$
= \frac{\{(vc)^2-(a+b)^2\}\{(vc)^2-(a-b)^2\}}{8(v^3c^4-a^2vc^2-b^2vc^2)\Gamma(v)}\left[\Gamma\left(\frac{vc+a+b}{2c}\right)\Gamma\left(\frac{vc-a-b}{2c}\right)\frac{\cos\left(\frac{(a+b)\pi}{2c}\right)}{\cos(\frac{v\pi}{2})} + \Gamma\left(\frac{vc+a-b}{2c}\right)\Gamma\left(\frac{vc-a+b}{2c}\right)\frac{\cos\left(\frac{(a-b)\pi}{2c}\right)}{\cos(\frac{v\pi}{2})}\right], \quad (2.8)
$$

where $Re(v) < 1,\ Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2},\ \frac{v}{2} \pm \frac{\sqrt{a^2+b^2}}{2c},\ 1+\frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Proof. Comparing the two equations (4.6) and (6), we get a summation theorem for ${}_8F_7(1)$. □

Theorem 2.9.

$$
{}_8F_7\left(\begin{array}{l} 1, \frac{3}{2}, \frac{3}{2}-\frac{\sqrt{a^2+c^2}}{2b}, \frac{3}{2}+\frac{\sqrt{a^2+c^2}}{2b}, \\ \qquad \frac{1}{2}-\frac{a}{2b}-\frac{c}{2b}, \frac{1}{2}-\frac{a}{2b}+\frac{c}{2b}, \frac{1}{2}+\frac{a}{2b}+\frac{c}{2b}, \frac{1}{2}+\frac{a}{2b}-\frac{c}{2b} ; \\ \frac{1}{2}, \frac{1}{2}-\frac{\sqrt{a^2+c^2}}{2b}, \frac{1}{2}+\frac{\sqrt{a^2+c^2}}{2b}, \frac{3}{2}-\frac{a}{2b}-\frac{c}{2b}, \\ \qquad \frac{3}{2}-\frac{a}{2b}+\frac{c}{2b}, \frac{3}{2}+\frac{a}{2b}+\frac{c}{2b}, \frac{3}{2}+\frac{a}{2b}-\frac{c}{2b}; \end{array} -1\right)
$$

$$
= \frac{\pi(b-a-c)(b+a+c)(b-a+c)(b+a-c)}{2(b^4-a^2b^2-c^2b^2)}\frac{\cos\left(\frac{a\pi}{2b}\right)\cos\left(\frac{c\pi}{2b}\right)}{\{\cos\left(\frac{c\pi}{b}\right)+\cos\left(\frac{a\pi}{b}\right)\}}, \quad (2.9)
$$

where $\Re(b) > 0,\ \Re(b \pm a \pm c) > 0;\ \frac{1}{2} \pm \frac{\sqrt{a^2+c^2}}{2b},\ \frac{3}{2} \pm \frac{a}{2b} \pm \frac{c}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-, \frac{a\pm c}{b} \in \mathbb{C}\backslash\{\pm1, \pm3, \pm5, ...\}$.

Proof. Comparing the two equations (6.20) and (6.18), we get a summation theorem for ${}_8F_7(-1)$. □

Theorem 2.10.

$$
{}_3F_2\left(\begin{array}{l} 1,\ \frac{1}{2}-\frac{a}{2b},\ \frac{1}{2}+\frac{a}{2b}; \\ \frac{3}{2}-\frac{a}{2b},\ \frac{3}{2}+\frac{a}{2b}; \end{array} -1\right) = \frac{(b^2-a^2)}{2ab}\left[\frac{\pi}{2}\sec\left(\frac{\pi a}{2b}\right)-\beta\left(\frac{a+b}{2b}\right)\right] \quad (2.10)
$$

$$
= \frac{(b^2-a^2)}{8ab}\left[\Psi\left(\frac{3b-a}{4b}\right)-\Psi\left(\frac{b-a}{4b}\right)-\Psi\left(\frac{3b+a}{4b}\right)+\Psi\left(\frac{b+a}{4b}\right)\right], \quad (2.11)
$$

where $\Re(b) > 0,\ \Re(b \pm a) > 0,\ \frac{3}{2} \pm \frac{a}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ \frac{a}{b} \in \mathbb{C}\backslash\{\pm1, \pm3, \pm5, ...\}$.

Proof. The summation formula with unit negative argument (2.10) is obtained by comparing the two solutions of the integral $\int_0^\infty \frac{\sinh(ax)}{\cosh(bx)}dx$, given in (5.24), (5.22) and is not available in the literature of the hypergeometric summation theorem. The above eq.(2.11) is obtained by using properties of beta function of one variable (1.28) and (1.29). □

3. Some reduction formulas

Here, we have given two reduction formulae in terms of hypergeometric functions ${}_4F_3(1)$ and ${}_7F_6(-1)$. It can be applicable for calculation of more suitable results.

Theorem 3.1.

$${}_4F_3\left(\begin{matrix} 2,2,\ \frac{1}{2}+\frac{a}{2b},\ \frac{1}{2}-\frac{a}{2b}; \\ 1,\ \frac{5}{2}+\frac{a}{2b},\ \frac{5}{2}-\frac{a}{2b}\ \ ; \end{matrix}\ 1\right) = \frac{(9b^2-a^2)}{8b^2}\ {}_3F_2\left(\begin{matrix} 1,\ \frac{1}{2}-\frac{a}{2b},\ \frac{1}{2}+\frac{a}{2b}; \\ \frac{3}{2}-\frac{a}{2b},\ \frac{3}{2}+\frac{a}{2b}\ \ ; \end{matrix}\ 1\right), \tag{3.1}$$

where $\Re(b) > 0,\ \Re(b \pm a) > 0,\ \frac{3}{2} \pm \frac{a}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \frac{a}{b} \in \mathbb{C}\backslash\{\pm1, \pm3, \pm5, ...\}$.

Proof. The reduction formula (3.1) is obtained by comparing the two integrals (5.13) and (5.3). □

Theorem 3.2.

$${}_7F_6\left(\begin{matrix} v, 1+\frac{v}{2}-\frac{\sqrt{a^2-b^2}}{2c}, 1+\frac{v}{2}+\frac{\sqrt{a^2-b^2}}{2c}, \frac{v}{2}-\frac{a}{2c}-\frac{b}{2c}, \\ \frac{v}{2}+\frac{a}{2c}+\frac{b}{2c}, \frac{v}{2}-\frac{a}{2c}+\frac{b}{2c}, \frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}; \\ \frac{v}{2}-\frac{\sqrt{a^2-b^2}}{2c}, \frac{v}{2}+\frac{\sqrt{a^2-b^2}}{2c}, 1+\frac{v}{2}-\frac{a}{2c}-\frac{b}{2c}, \\ 1+\frac{v}{2}+\frac{a}{2c}+\frac{b}{2c}, 1+\frac{v}{2}-\frac{a}{2c}+\frac{b}{2c}, 1+\frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}; \end{matrix}\ -1\right)$$

$$= \frac{\{(vc)^2-(a+b)^2\}\{(vc)^2-(a-b)^2\}}{4(v^2ac^2-a^3+ab^2)} \times \left[\frac{1}{(vc-a-b)}\ {}_2F_1\left(\begin{matrix} v,\ \frac{vc-a-b}{2c}\ ; \\ 1+\frac{vc-a-b}{2c}; \end{matrix}\ -1\right) - \frac{1}{(vc+a+b)}\ {}_2F_1\left(\begin{matrix} v,\ \frac{vc+a+b}{2c}\ ; \\ 1+\frac{vc+a+b}{2c}; \end{matrix}\ -1\right)\right.$$
$$\left. + \frac{1}{(vc-a+b)}\ {}_2F_1\left(\begin{matrix} v,\ \frac{vc-a+b}{2c}\ ; \\ 1+\frac{vc-a+b}{2c}; \end{matrix}\ -1\right) - \frac{1}{(vc+a-b)}\ {}_2F_1\left(\begin{matrix} v,\ \frac{vc+a-b}{2c}\ ; \\ 1+\frac{vc+a-b}{2c}; \end{matrix}\ -1\right)\right], \tag{3.2}$$

where $\Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2} \pm \frac{\sqrt{a^2-b^2}}{2c},\ 1+\frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Proof. Comparing the integrals (4.3) and (6.9), we get the reduction formula (3.2). The integral representation of ${}_2F_1(-1)$ type hypergeometric functions involved in reduction formula (3.2) is given below

$${}_2F_1\left(\begin{matrix} a,\ b,; \\ 1+b; \end{matrix}\ -1\right) = b\int_0^1 \frac{t^{b-1}}{(1+t)^a}dt, \qquad \Re(b) > 0. \tag{3.3}$$

The definite integral (3.3) can be solved by using suitable numerical methods (for example composite trapezoidal rule, composite Simpson's 1/3 rule, composite Simpson's 3/8 rule, composite Boole rule, composite Midris two rules, composite Weddle rule, composite Sadiq rule, Gauss-Legendre three points formula, Gauss-Chebyshev three points formula, Radau three points formula, Lobatto three points formula etc). □

4. Generalized class of some integral formulas including hyperbolic functions in terms of ${}_6F_5(\pm1)$, ${}_7F_6(\pm1)$ and ${}_8F_7(\pm1)$

Many authors have studied some definite integrals containing the integrands as a quotient of hyperbolic functions. Mainly, V. H. Moll et.al evaluated some definite integrals given in the table of Gradshteyn and Ryzhik [4, 5], by using the change of independent variables. We have given generalizations and analytical solutions of some integral formulas including hyperbolic functions, using hypergeometric approach and Laplace transform method.

Theorem 4.1.

$$\int_0^\infty \frac{\sinh(ax)\sinh(bx)}{\cosh^v(cx)}dx = \frac{2^{v+1}vabc}{(vc-a-b)(vc+a+b)(vc-a+b)(vc+a-b)}$$
$$\times\ {}_6F_5\left(\begin{matrix} v,\ 1+\frac{v}{2},\ \frac{v}{2}-\frac{a}{2c}-\frac{b}{2c},\ \frac{v}{2}+\frac{a}{2c}+\frac{b}{2c},\ \frac{v}{2}-\frac{a}{2c}+\frac{b}{2c},\ \frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}\ ; \\ \frac{v}{2},\ 1+\frac{v}{2}-\frac{a}{2c}-\frac{b}{2c},\ 1+\frac{v}{2}+\frac{a}{2c}+\frac{b}{2c},\ 1+\frac{v}{2}-\frac{a}{2c}+\frac{b}{2c},\ 1+\frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}\ ; \end{matrix}\ -1\right), \tag{4.1}$$

where $\Re(v) < 4,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2},\ 1+\frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Theorem 4.2.

$$\int_0^\infty \frac{\sinh(ax)\sinh(bx)}{\sinh^v(cx)}dx = \frac{2^{v+1}vabc}{(vc-a-b)(vc+a+b)(vc-a+b)(vc+a-b)}$$
$$\times\ {}_6F_5\left(\begin{array}{c} v,\ 1+\frac{v}{2},\ \frac{v}{2}-\frac{a}{2c}-\frac{b}{2c},\ \frac{v}{2}+\frac{a}{2c}+\frac{b}{2c},\ \frac{v}{2}-\frac{a}{2c}+\frac{b}{2c},\ \frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}\ ; \\ \frac{v}{2},\ 1+\frac{v}{2}-\frac{a}{2c}-\frac{b}{2c},\ 1+\frac{v}{2}+\frac{a}{2c}+\frac{b}{2c},\ 1+\frac{v}{2}-\frac{a}{2c}+\frac{b}{2c},\ 1+\frac{v}{2}+\frac{a}{2c}-\frac{b}{2c}\ ; \end{array} 1\right), \quad (4.2)$$

where $\Re(v) < 3,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2},\ 1+\frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Theorem 4.3.

$$\int_0^\infty \frac{\sinh(ax)\cosh(bx)}{\cosh^v(cx)}dx = \frac{2^v(v^2ac^2-a^3+ab^2)}{(vc-a-b)(vc+a+b)(vc-a+b)(vc+a-b)}$$
$$\times\ {}_7F_6\left(\begin{array}{c} v, 1+\sigma_1,\ 1+\sigma_2,\ \sigma_3,\ \sigma_4, \sigma_5,\ \sigma_6 \qquad ; \\ \sigma_1, \sigma_2, 1+\sigma_3,\ 1+\sigma_4,\ 1+\sigma_5,\ 1+\sigma_6; \end{array} -1\right), \quad (4.3)$$

where

$$\sigma_1 = \frac{v}{2} - \frac{\sqrt{a^2-b^2}}{2c},\ \sigma_2 = \frac{v}{2} + \frac{\sqrt{a^2-b^2}}{2c},\sigma_3 = \frac{v}{2} - \frac{a}{2c} - \frac{b}{2c},$$
$$\sigma_4 = \frac{v}{2} - \frac{a}{2c} + \frac{b}{2c},\ \sigma_5 = \frac{v}{2} + \frac{a}{2c} + \frac{b}{2c},\ \sigma_6 = \frac{v}{2} + \frac{a}{2c} - \frac{b}{2c},$$

and $\Re(v) < 3,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2} \pm \frac{\sqrt{a^2-b^2}}{2c},\ 1+\frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Theorem 4.4.

$$\int_0^\infty \frac{\sinh(ax)\cosh(bx)}{\sinh^v(cx)}dx = \frac{2^v(v^2ac^2-a^3+ab^2)}{(vc-a-b)(vc+a+b)(vc-a+b)(vc+a-b)}$$
$$\times\ {}_7F_6\left(\begin{array}{c} v, 1+\sigma_1,\ 1+\sigma_2,\ \sigma_3,\ \sigma_4, \sigma_5,\ \sigma_6 \qquad ; \\ \sigma_1, \sigma_2, 1+\sigma_3,\ 1+\sigma_4,\ 1+\sigma_5,\ 1+\sigma_6; \end{array} 1\right), \quad (4.4)$$

where

$$\sigma_1 = \frac{v}{2} - \frac{\sqrt{a^2-b^2}}{2c},\ \sigma_2 = \frac{v}{2} + \frac{\sqrt{a^2-b^2}}{2c},\sigma_3 = \frac{v}{2} - \frac{a}{2c} - \frac{b}{2c},$$
$$\sigma_4 = \frac{v}{2} - \frac{a}{2c} + \frac{b}{2c},\ \sigma_5 = \frac{v}{2} + \frac{a}{2c} + \frac{b}{2c},\ \sigma_6 = \frac{v}{2} + \frac{a}{2c} - \frac{b}{2c},$$

and $\Re(v) < 2,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2} \pm \frac{\sqrt{a^2-b^2}}{2c},\ 1+\frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Theorem 4.5.

$$\int_0^\infty \frac{\cosh(ax)\cosh(bx)}{\cosh^v(cx)}dx = \frac{2^v(v^3c^3-a^2vc-b^2vc)}{(vc-a-b)(vc+a+b)(vc-a+b)(vc+a-b)}$$
$$\times\ {}_8F_7\left(\begin{array}{c} v,\ 1+\frac{v}{2},\ 1+\lambda_1,\ 1+\lambda_2,\ \sigma_3,\ \sigma_4, \sigma_5,\ \sigma_6\ ; \\ \frac{v}{2},\ \lambda_1,\ \lambda_2, 1+\sigma_3,\ 1+\sigma_4,\ 1+\sigma_5,\ 1+\sigma_6\ ; \end{array} -1\right), \quad (4.5)$$

where

$$\lambda_1 = \frac{v}{2} - \frac{\sqrt{a^2+b^2}}{2c},\ \lambda_2 = \frac{v}{2} + \frac{\sqrt{a^2+b^2}}{2c},\sigma_3 = \frac{v}{2} - \frac{a}{2c} - \frac{b}{2c},$$
$$\sigma_4 = \frac{v}{2} - \frac{a}{2c} + \frac{b}{2c},\ \sigma_5 = \frac{v}{2} + \frac{a}{2c} + \frac{b}{2c},\ \sigma_6 = \frac{v}{2} + \frac{a}{2c} - \frac{b}{2c},$$

and $\Re(v) < 2,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0; \frac{v}{2},\ \frac{v}{2} \pm \frac{\sqrt{a^2+b^2}}{2c},\ 1+\frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Theorem 4.6.

$$\int_0^\infty \frac{\cosh(ax)\cosh(bx)}{\sinh^v(cx)}dx = \frac{2^v(v^3c^3 - a^2vc - b^2vc)}{(vc-a-b)(vc+a+b)(vc-a+b)(vc+a-b)} \times {}_8F_7\left(\begin{matrix} v,\ 1+\frac{v}{2},\ 1+\lambda_1,\ 1+\lambda_2,\ \sigma_3,\ \sigma_4, \sigma_5,\ \sigma_6\ ; \\ \frac{v}{2},\ \lambda_1,\ \lambda_2, 1+\sigma_3,\ 1+\sigma_4,\ 1+\sigma_5,\ 1+\sigma_6\ ; \end{matrix}\ 1\right), \quad (4.6)$$

where

$$\lambda_1 = \frac{v}{2} - \frac{\sqrt{a^2+b^2}}{2c},\ \lambda_2 = \frac{v}{2} + \frac{\sqrt{a^2+b^2}}{2c}, \sigma_3 = \frac{v}{2} - \frac{a}{2c} - \frac{b}{2c},$$

$$\sigma_4 = \frac{v}{2} - \frac{a}{2c} + \frac{b}{2c},\ \sigma_5 = \frac{v}{2} + \frac{a}{2c} + \frac{b}{2c},\ \sigma_6 = \frac{v}{2} + \frac{a}{2c} - \frac{b}{2c},$$

and $\Re(v) < 1,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2},\ \frac{v}{2} \pm \frac{\sqrt{a^2+b^2}}{2c},\ 1 + \frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Proof of (4.5). Suppose left hand side of eq.(4.5) is denoted by $\Upsilon(a, b, c, v)$ and using the product formula of hyperbolic function in the left hand side of eq.(4.5), we get

$$\Upsilon(a,b,c,v) = \frac{1}{2}\int_0^\infty \frac{\cosh\{(a+b)x\}}{\cosh^v(cx)}dx + \frac{1}{2}\int_0^\infty \frac{\cosh\{(a-b)x\}}{\cosh^v(cx)}dx = \mathbf{L}_1 + \mathbf{L}_2, \quad (4.7)$$

where $\mathbf{L}_1$ and $\mathbf{L}_2$ are given by

$$\mathbf{L}_1 = \frac{1}{2}\int_0^\infty \frac{\cosh\{(a+b)x\}}{\cosh^v(cx)}dx \ \textit{ and } \ \mathbf{L}_2 = \frac{1}{2}\int_0^\infty \frac{\cosh\{(a-b)x\}}{\cosh^v(cx)}dx. \quad (4.8)$$

Using exponential definition of hyperbolic functions in the integral $\mathbf{L}_1$, which yields

$$\begin{aligned} \mathbf{L}_1 &= 2^{v-2}\int_0^\infty e^{-vcx}\Big[e^{(a+b)x} + e^{-(a+b)x}\Big]\Big(1+e^{-2cx}\Big)^{-v}dx. \quad (4.9) \\ &= 2^{v-2}\int_0^\infty e^{-vcx}\Big[e^{(a+b)x} + e^{-(a+b)x}\Big]{}_1F_0\left(\begin{matrix} v\ ; \\ -; \end{matrix}\ -e^{-2cx}\right)dx, \end{aligned}$$

when $\Re(c) > 0$, then $|-e^{-2cx}| < 1$ for all real $x > 0$. It is the convergence condition of above binomial function ${}_1F_0(\cdot)$ in eq.(4.10), then it yields

$$\mathbf{L}_1 = 2^{v-2}\sum_{r=0}^{\infty}\frac{(v)_r}{r!}(-1)^r\Big[\int_0^\infty e^{-\{(vc-a-b)+2cr\}x}dx + \int_0^\infty e^{-\{(vc+a+b)+2cr\}x}dx\Big], \quad (4.10)$$

where $\Re(vc-a-b) > 0,\ \Re(vc+a+b) > 0,\ \Re(c) > 0$, it is the convergence conditions of Laplace transform of unity in the integral (4.10). Then applying Laplace transformation formula (1.18) in the eq.(4.10), we obtain

$$\mathbf{L}_1 = 2^{v-2}\sum_{r=0}^{\infty}\frac{(v)_r}{r!}(-1)^r\Big[\frac{1}{\{(vc-a-b)+2cr\}} + \frac{1}{\{(vc+a+b)+2cr\}}\Big], \quad (4.11)$$

where $\Re(vc-a-b) > 0,\ \Re(vc+a+b) > 0,\ \Re(c) > 0$.
Similarly, proof of $\mathbf{L}_2$ is given by

$$\mathbf{L}_2 = 2^{v-2}\sum_{r=0}^{\infty}\frac{(v)_r}{r!}(-1)^r\Big[\frac{1}{\{(vc-a+b)+2cr\}} + \frac{1}{\{(vc+a-b)+2cr\}}\Big], \quad (4.12)$$

where $\Re(vc-a+b) > 0,\ \Re(vc+a-b) > 0,\ \Re(c) > 0$.
Making use of the eqns (4.11) and (4.12) in the above eq. (4.7), we obtain

$$\Upsilon(a,b,c,v) = 2^{v-2}\sum_{r=0}^{\infty}\frac{(v)_r}{r!}(-1)^r\Big[\frac{1}{\{(vc-a-b)+2cr\}} + \frac{1}{\{(vc+a+b)+2cr\}} + \frac{1}{\{(vc-a+b)+2cr\}} + \frac{1}{\{(vc+a-b)+2cr\}}\Big], \quad (4.13)$$

$$= 2^{v-1}\sum_{r=0}^{\infty}\frac{(v)_r}{r!}(-1)^r\Big[\frac{(vc-a+2cr)}{\{(vc-a-b)+2cr\}\{(vc-a+b)+2cr\}}+\frac{(vc+a+2cr)}{\{(vc+a+b)+2cr\}\{(vc+a-b)+2cr\}}\Big]$$

where $\Re(vc \pm a \pm b) > 0,\ \Re(c) > 0$. After simplifications we obtain

$$\Upsilon(a,b,c,v) = 2^{v-1}\sum_{r=0}^{\infty}\frac{(v)_r}{r!}(-1)^r\Big[\frac{16c^3r^3+24vc^3r^2+(12v^2c^3-4a^2c-4b^2c)r+(2v^3c^3-2va^2c-2b^2vc)}{\{(vc-a-b)+2cr\}\{(vc+a+b)+2cr\}\{(vc-a+b)+2cr\}\{(vc+a-b)+2cr\}}\Big], \tag{4.14}$$

$$= 2^{v-1}\sum_{r=0}^{\infty}\frac{(v)_r}{r!}(-1)^r\Big[\frac{16c^3\Big\{r-\Big(\frac{-vc+\sqrt{a^2+b^2}}{2c}\Big)\Big\}\Big\{r-\Big(\frac{-vc-\sqrt{a^2+b^2}}{2c}\Big)\Big\}\Big\{r+\frac{v}{2}\Big\}}{\{(vc-a-b)+2cr\}\{(vc+a+b)+2cr\}\{(vc-a+b)+2cr\}\{(vc+a-b)+2cr\}}\Big],$$

where $\Re(vc \pm a \pm b) > 0,\ \Re(c) > 0$.
Employ algebraic properties of Pochhammer symbol in the eq.(4.14), after simplifications, we obtain

$$\Upsilon(a,b,c,v) = \frac{2^v(v^3c^3-a^2vc-b^2vc)}{(vc-a-b)(vc+a+b)(vc-a+b)(vc+a-b)}\sum_{r=0}^{\infty}\Big[\frac{(v)_r\left(1+\frac{v}{2}\right)_r}{\left(\frac{v}{2}\right)_r r!}$$
$$\times\frac{\left(1+\frac{v}{2}-\frac{\sqrt{a^2+b^2}}{2c}\right)_r\left(1+\frac{v}{2}+\frac{\sqrt{a^2+b^2}}{2c}\right)_r\left(\frac{vc-a-b}{2c}\right)_r\left(\frac{vc+a+b}{2c}\right)_r\left(\frac{vc-a+b}{2c}\right)_r\left(\frac{vc+a-b}{2c}\right)_r(-1)^r}{\left(\frac{v}{2}-\frac{\sqrt{a^2+b^2}}{2c}\right)_r\left(\frac{v}{2}+\frac{\sqrt{a^2+b^2}}{2c}\right)_r\left(\frac{vc-a-b+2c}{2c}\right)_r\left(\frac{vc+a+b+2c}{2c}\right)_r\left(\frac{vc-a+b+2c}{2c}\right)_r\left(\frac{vc+a-b+2c}{2c}\right)_r}\Big], \tag{4.15}$$

$$= \frac{2^v(v^3c^3-a^2vc-b^2vc)}{(vc-a-b)(vc+a+b)(vc-a+b)(vc+a-b)}\times$$
$$\times {}_8F_7\left(\begin{matrix} v,\ 1+\frac{v}{2},\ 1+\lambda_1,\ 1+\lambda_2,\ \sigma_3,\ \sigma_4,\sigma_5,\ \sigma_6\ ; \\ \frac{v}{2},\ \lambda_1,\ \lambda_2, 1+\sigma_3,\ 1+\sigma_4,\ 1+\sigma_5,\ 1+\sigma_6\ ; \end{matrix}\ -1\right),$$

where

$$\lambda_1 = \frac{v}{2}-\frac{\sqrt{a^2+b^2}}{2c},\ \lambda_2 = \frac{v}{2}+\frac{\sqrt{a^2+b^2}}{2c},\sigma_3 = \frac{v}{2}-\frac{a}{2c}-\frac{b}{2c},$$
$$\sigma_4 = \frac{v}{2}-\frac{a}{2c}+\frac{b}{2c},\ \sigma_5 = \frac{v}{2}+\frac{a}{2c}+\frac{b}{2c},\ \sigma_6 = \frac{v}{2}+\frac{a}{2c}-\frac{b}{2c},$$

and $\Re(c) > 0,\ \Re(vc \pm a \pm b) > 0; \frac{v}{2},\ \frac{v}{2} \pm \frac{\sqrt{a^2+b^2}}{2c},\ 1+\frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$. Similarly, proofs of the integrals (4.1),(4.2),(4.3),(4.4) and (4.6) are much akin to that of the integral (4.5), which we have already discussed in a detailed manner. □

5. Special class of some hyperbolic integrals in terms of ${}_3F_2(\pm1), {}_4F_3(\pm1)$ and ${}_5F_4(\pm1)$

We have given special class of some integral formulas including hyperbolic functions in terms of hypergeometric functions ${}_3F_2(\pm1), {}_4F_3(\pm1), {}_5F_4(\pm1)$, and each of the following hyperbolic definite integrals holds true
• When $v = 2$ and $c = b$ in the eq.(4.1), we get

$$\int_0^{\infty}\frac{\sinh(ax)\sinh(bx)}{\cosh^2(bx)}dx = \frac{16ab^2}{(b^2-a^2)(9b^2-a^2)}\ {}_4F_3\left(\begin{matrix} 2,2,\ \frac{1}{2}+\frac{a}{2b},\ \frac{1}{2}-\frac{a}{2b}; \\ 1,\ \frac{5}{2}+\frac{a}{2b},\ \frac{5}{2}-\frac{a}{2b}\ \ ; \end{matrix}\ -1\right), \tag{5.1}$$
$$= \frac{a\pi}{2b^2}\sec\left(\frac{\pi a}{2b}\right), \tag{5.2}$$

where $\Re(b) > 0,\ \Re(b \pm a) > 0,\ \Re(3b \pm a) > 0,\ \frac{5}{2} \pm \frac{a}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \frac{a}{b} \in \mathbb{C}\backslash\{\pm1, \pm3, \pm5, ...\}$. Using hypergeometric form of $\sec(z)$ function (1.32) [when $z = \frac{\pi a}{2b}$] in the eq. (5.1), we obtain right hand side of (5.2). Also, right hand side of eq.(5.2) can be obtained by using summation theorem (1.15) and recurrence relation for gamma function in the hypergeometric series (5.1).

• When $v = 2$ and $c = b$ in the eq.(4.2), we get

$$\int_0^\infty \frac{\sinh(ax)}{\sinh(bx)}dx = \frac{16ab^2}{(b^2-a^2)(9b^2-a^2)}\ {}_4F_3\left(\begin{array}{c} 2,2,\ \frac{1}{2}+\frac{a}{2b},\ \frac{1}{2}-\frac{a}{2b};\\ 1,\ \frac{5}{2}+\frac{a}{2b},\ \frac{5}{2}-\frac{a}{2b}\ ; \end{array} 1\right), \tag{5.3}$$

$$= \frac{\pi}{2b}\tan\left(\frac{\pi a}{2b}\right), \tag{5.4}$$

where $\Re(b) > 0,\ \Re(b\pm a) > 0,\ \Re(3b\pm a) > 0,\ \frac{5}{2}\pm\frac{a}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \frac{a}{b} \in \mathbb{C}\backslash\{\pm1,\pm3,\pm5,...\}$. The right hand side of eq.(5.4) can be obtained by using summation theorem (1.16), recurrence relations for gamma function in the hypergeometric series (5.3).

• When $v = 2,\ b = a,\ c = 1$ in the eq.(4.2), we get

$$\frac{1}{2}\int_{-\infty}^\infty \frac{\sinh^2(ax)}{\sinh^2(x)}dx = \int_0^\infty \frac{\sinh^2(ax)}{\sinh^2(x)}dx = \frac{a^2}{(1-a^2)}\ {}_3F_2\left(\begin{array}{c} 1,\ 1-a,\ 1+a;\\ 2-a,\ 2+a\ \ ; \end{array} 1\right), \tag{5.5}$$

$$= \frac{a}{2}[\Psi(1+a)-\Psi(1-a)], \tag{5.6}$$

$$= \frac{1}{2}[1-a\pi\cot(a\pi)], \tag{5.7}$$

where $a \neq 0,\pm1,\pm2,\pm3,...$

The right hand side of eq.(5.7) can be obtained by using the properties of Digamma function (1.23)-(1.25) in the hypergeometric series (5.5).

• In the eq.(4.3) we interchange a and b; $v = 2$, then put $c = b$, we get

$$\int_0^\infty \frac{\cosh(ax)\sinh(bx)}{\cosh^2(bx)}dx = \frac{(12b^3+4a^2b)}{(b^2-a^2)(9b^2-a^2)}\ {}_5F_4\left(\begin{array}{c} 2,\ 2-\frac{\sqrt{b^2-a^2}}{2b},\ 2+\frac{\sqrt{b^2-a^2}}{2b},\ \frac{1}{2}-\frac{a}{2b},\ \frac{1}{2}+\frac{a}{2b};\\ 1-\frac{\sqrt{b^2-a^2}}{2b},\ 1+\frac{\sqrt{b^2-a^2}}{2b},\ \frac{5}{2}+\frac{a}{2b},\ \frac{5}{2}-\frac{a}{2b}\ \ ; \end{array} -1\right), \tag{5.8}$$

where $\Re(b) > 0,\ \Re(b\pm a) > 0,\ \Re(3b\pm a) > 0;\ 1\pm\frac{\sqrt{b^2-a^2}}{2b},\ \frac{5}{2}\pm\frac{a}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

• When $b = 0$ in the eq.(4.3), we get

$$\int_0^\infty \frac{\sinh(ax)}{\cosh^v(cx)}dx = \frac{2^v a}{[(vc)^2-a^2]}\ {}_3F_2\left(\begin{array}{c} v,\ \frac{v}{2}-\frac{a}{2c},\ \frac{v}{2}+\frac{a}{2c};\\ 1+\frac{v}{2}-\frac{a}{2c},\ 1+\frac{v}{2}+\frac{a}{2c}; \end{array} -1\right), \tag{5.9}$$

$$= \frac{2^{v-1}}{(vc-a)}\ {}_2F_1\left(\begin{array}{c} v,\ \frac{v}{2}-\frac{a}{2c}\ ;\\ 1+\frac{v}{2}-\frac{a}{2c}; \end{array} -1\right) - \frac{2^{v-1}}{(vc+a)}\ {}_2F_1\left(\begin{array}{c} v,\ \frac{v}{2}+\frac{a}{2c}\ ;\\ 1+\frac{v}{2}+\frac{a}{2c}; \end{array} -1\right), \tag{5.10}$$

where $\Re(v) < 3,\ \Re(c) > 0,\ \Re(vc\pm a) > 0,\ \ 1+\frac{v}{2}\pm\frac{a}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

• When $b = 0$ in the eq.(4.4), we get

$$\int_0^\infty \frac{\sinh(ax)}{\sinh^v(cx)}dx = \frac{2^v a}{[(vc)^2-a^2]}\ {}_3F_2\left(\begin{array}{c} v,\ \frac{v}{2}-\frac{a}{2c},\ \frac{v}{2}+\frac{a}{2c};\\ 1+\frac{v}{2}-\frac{a}{2c},\ 1+\frac{v}{2}+\frac{a}{2c}; \end{array} 1\right), \tag{5.11}$$

$$= \frac{2^{v-2}}{(c)\Gamma(v)}\Gamma\left(\frac{v}{2}+\frac{a}{2c}\right)\Gamma\left(\frac{v}{2}-\frac{a}{2c}\right)\frac{\sin\left(\frac{a\pi}{2c}\right)}{\sin\left(\frac{v\pi}{2}\right)}, \tag{5.12}$$

where $\Re(v) < 2,\ \Re(c) > 0,\ \Re(vc\pm a) > 0,\ \ 1+\frac{v}{2}\pm\frac{a}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $v \neq 0,\pm2,\pm4,\pm6,...$

• When $v = 1$ and $b = 0$, then $c = b$ in the eq.(4.4), we get

$$\int_0^\infty \frac{\sinh(ax)}{\sinh(bx)}dx = \frac{2a}{(b^2-a^2)}\ {}_3F_2\left(\begin{array}{c} 1,\ \frac{1}{2}-\frac{a}{2b},\ \frac{1}{2}+\frac{a}{2b};\\ \frac{3}{2}-\frac{a}{2b},\ \frac{3}{2}+\frac{a}{2b}\ \ ; \end{array} 1\right), \tag{5.13}$$

$$= \frac{\pi}{2b}\tan\left(\frac{\pi a}{2b}\right), \tag{5.14}$$

where $\Re(b) > 0,\ \Re(b \pm a) > 0,\ \frac{3}{2} \pm \frac{a}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \frac{a}{b} \in \mathbb{C}\backslash\{\pm1, \pm3, \pm5, ...\}$.
Using hypergeometric form of $\tan(z)$ function (1.31) [when $z = \frac{\pi a}{2b}$] in the eq. (5.13), we obtain right hand side of eq.(5.14). Also, right hand side of eq.(5.14) can be obtained by using the Dixon's theorem ${}_3F_2(1)$ (1.7) in the eq. (5.13).
• When $b = 0$ in the eq.(4.5), we get

$$\int_0^\infty \frac{\cosh(ax)}{\cosh^v(cx)}dx = \frac{2^v(vc)}{[(vc)^2 - a^2]}\ {}_4F_3\left(\begin{array}{l} v,\ 1+\frac{v}{2}, \frac{v}{2}-\frac{a}{2c},\ \frac{v}{2}+\frac{a}{2c}; \\ \frac{v}{2}, 1+\frac{v}{2}-\frac{a}{2c},\ 1+\frac{v}{2}+\frac{a}{2c}\quad ; \end{array} -1\right), \tag{5.15}$$

$$= \frac{2^{v-2}}{(c)\Gamma(v)}\Gamma\left(\frac{v}{2}+\frac{a}{2c}\right)\Gamma\left(\frac{v}{2}-\frac{a}{2c}\right), \tag{5.16}$$

where $\Re(v) < 2,\ \Re(c) > 0,\ \Re(vc \pm a) > 0;\ \frac{v}{2}, 1+\frac{v}{2} \pm \frac{a}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
• When $v = 1$ and $b = 0$, then $c = b$ in the eq.(4.5), we get

$$\int_0^\infty \frac{\cosh(ax)}{\cosh(bx)}dx = \frac{2b}{(b^2 - a^2)}\ {}_4F_3\left(\begin{array}{l} 1,\ \frac{3}{2},\ \frac{1}{2}-\frac{a}{2b},\ \frac{1}{2}+\frac{a}{2b}; \\ \frac{1}{2},\ \frac{3}{2}-\frac{a}{2b},\ \frac{3}{2}+\frac{a}{2b}\quad ; \end{array} -1\right), \tag{5.17}$$

$$= \frac{\pi}{2b}\sec\left(\frac{\pi a}{2b}\right), \tag{5.18}$$

where $\Re(b) > 0,\ \Re(b \pm a) > 0,\ \frac{3}{2} \pm \frac{a}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \frac{a}{b} \in \mathbb{C}\backslash\{\pm1, \pm3, \pm5, ...\}$.
Using hypergeometric form of $\sec(z)$ function (1.32) [when $z = \frac{\pi a}{2b}$] in the eq. (5.17), we obtain right hand side of eq. (5.18). Also, right hand side of eq.(5.18) can be obtained by using the classical summation theorem ${}_4F_3(-1)$ (1.15) in the eq. (5.17).
• In the eq.(5.18) replacing $a \to 2ia$ and $b = \pi$, we get a known result of Ramanujan [3, p.11,eq.(1.5.1(27))]

$$\int_0^\infty \frac{\cos(2ax)}{\cosh(\pi x)}dx = \frac{1}{2}sech\,(a), \qquad |\Im(a)| < \frac{\pi}{2}. \tag{5.19}$$

• When $b = 0$ in the eq.(4.6), we get

$$\int_0^\infty \frac{\cosh(ax)}{\sinh^v(cx)}dx = \frac{2^v(vc)}{[(vc)^2 - a^2]}\ {}_4F_3\left(\begin{array}{l} v,\ 1+\frac{v}{2}, \frac{v}{2}-\frac{a}{2c},\ \frac{v}{2}+\frac{a}{2c}; \\ \frac{v}{2}, 1+\frac{v}{2}-\frac{a}{2c},\ 1+\frac{v}{2}+\frac{a}{2c}\quad ; \end{array} 1\right), \tag{5.20}$$

$$= \frac{2^{v-2}}{(c)\Gamma(v)}\Gamma\left(\frac{v}{2}+\frac{a}{2c}\right)\Gamma\left(\frac{v}{2}-\frac{a}{2c}\right)\frac{\cos\left(\frac{a\pi}{2c}\right)}{\cos\left(\frac{v\pi}{2}\right)}, \tag{5.21}$$

where $\Re(v) < 1,\ \Re(c) > 0,\ \Re(vc \pm a) > 0;\ \frac{v}{2}, 1+\frac{v}{2} \pm \frac{a}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $v \neq \pm1, \pm3, \pm5, ...$
• When $v = 1$ and $b = 0$, then $c = b$ in the eq.(4.3), we get

$$\int_0^\infty \frac{\sinh(ax)}{\cosh(bx)}dx = \frac{2a}{(b^2 - a^2)}{}_3F_2\left(\begin{array}{l} 1,\ \frac{1}{2}-\frac{a}{2b},\ \frac{1}{2}+\frac{a}{2b}; \\ \frac{3}{2}-\frac{a}{2b},\ \frac{3}{2}+\frac{a}{2b}\quad ; \end{array} -1\right), \tag{5.22}$$

$$= \frac{\pi}{2b}\sec\left(\frac{\pi a}{2b}\right) - \frac{1}{b}\beta\left(\frac{a+b}{2b}\right), \tag{5.23}$$

$$= \frac{\pi}{2b}\sec\left(\frac{\pi a}{2b}\right) - \frac{1}{2b}\left[\Psi\left(\frac{a+3b}{4b}\right) - \Psi\left(\frac{a+b}{4b}\right)\right], \tag{5.24}$$

$$= \frac{(b^2 - a^2)}{8ab}\left[\Psi\left(\frac{3b-a}{4b}\right) - \Psi\left(\frac{b-a}{4b}\right) - \Psi\left(\frac{3b+a}{4b}\right) + \Psi\left(\frac{b+a}{4b}\right)\right], \tag{5.25}$$

where $\Re(b) > 0,\ \Re(b \pm a) > 0,\ \frac{3}{2} \pm \frac{a}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \frac{a}{b} \in \mathbb{C}\backslash\{\pm1, \pm3, \pm5, ...\}$.

Independent proofs of (5.22)-(5.25). Taking left hand side of eq.(5.22) and suppose it is denoted by $\Phi(a, b)$ upon using the well known result of hyperbolic function , we get

$$\Phi(a, b) = \int_0^\infty \left(\frac{e^{ax}}{e^{bx} + e^{-bx}}\right)dx - \int_0^\infty \left(\frac{e^{-ax}}{e^{bx} + e^{-bx}}\right)dx = \mathbf{Y}_1 - \mathbf{Y}_2, \tag{5.26}$$

where $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are given by

$$\mathbf{Y}_1 = \int_0^\infty \left(\frac{e^{ax}}{e^{bx}+e^{-bx}}\right)dx, \ \ and \ \ \mathbf{Y}_2 = \int_0^\infty \left(\frac{e^{-ax}}{e^{bx}+e^{-bx}}\right)dx. \tag{5.27}$$

From the above integral $\mathbf{Y}_1$ can also written by

$$\begin{aligned} \mathbf{Y}_1 &= \int_0^\infty e^{-(b-a)x}\Big(1+e^{-2bx}\Big)^{-1}dx, \\ &= \int_0^\infty \Big[e^{-(b-a)x}\, {}_1F_0\left(\begin{matrix} 1\,; \\ -; \end{matrix} \ -e^{-2bx}\right)\Big]dx, \end{aligned} \tag{5.28}$$

when $\Re(b) > 0$, then $|-e^{-2bx}| < 1$ for all $x > 0$. It is the convergence conditions of above binomial function ${}_1F_0(\cdot)$ in eq.(5.28), then it yields

$$\begin{aligned} \mathbf{Y}_1 &= \int_0^\infty \Big[e^{-(b-a)x}\sum_{r=0}^\infty (-1)^r\, e^{-2brx}\Big]dx, \\ &= \sum_{r=0}^\infty (-1)^r\Big[\int_0^\infty e^{-\{(b-a)+2br\}x}dx\Big], \end{aligned} \tag{5.29}$$

where $\Re(b-a) > 0$, $\Re(b) > 0$, it is the convergence condition of Laplace transform of unity in the integral (5.29). Then applying formula (1.18) in the eq.(5.29), we obtain

$$\mathbf{Y}_1 = \sum_{r=0}^\infty (-1)^r\Big[\frac{1}{(b-a)+2br}\Big], \qquad \Re(b-a) > 0, \ \Re(b) > 0. \tag{5.30}$$

Similarly, proof of $\mathbf{Y}_2$ is given by

$$\mathbf{Y}_2 = \sum_{r=0}^\infty (-1)^r\Big[\frac{1}{(b+a)+2br}\Big], \qquad \Re(b+a) > 0, \ \Re(b) > 0. \tag{5.31}$$

Making use of the eqns (5.30) and (5.31) in the above eq. (5.26), we obtain

$$\Phi(a,b) = \sum_{r=0}^\infty \Big[\frac{(-1)^r}{(b-a)+2br} - \frac{(-1)^r}{(b+a)+2br}\Big], \tag{5.32}$$

where $\Re(b \pm a) > 0$, $\Re(b) > 0$.
Employ algebraic properties of Pochhammer symbol in the eq.(5.32), after simplifications, we obtain

$$\begin{aligned} \Phi(a,b) &= \frac{2a}{b^2-a^2}\sum_{r=0}^\infty \Big[\frac{\left(\frac{b-a}{2b}\right)_r\left(\frac{b+a}{2b}\right)_r(-1)^r}{\left(\frac{3b-a}{2b}\right)_r\left(\frac{3b+a}{2b}\right)_r}\Big], \\ &= \frac{2a}{(b^2-a^2)}\ {}_3F_2\left(\begin{matrix} 1, \ \frac{1}{2}-\frac{a}{2b}, \ \frac{1}{2}+\frac{a}{2b}; \\ \frac{3}{2}-\frac{a}{2b}, \ \frac{3}{2}+\frac{a}{2b}; \end{matrix} \ -1\right), \end{aligned} \tag{5.33}$$

where $\Re(b \pm a) > 0$, $\Re(b) > 0$, $\frac{a}{b} \in \mathbb{C}\backslash\{\pm1, \pm3, \pm5, ...\}$, $\frac{3}{2} \pm \frac{a}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-$. Now, proof of the eq.(5.23) is obtained by using the eq. (5.32) with addition and substraction of its second term, is given below

$$\Phi(a,b) = \sum_{r=0}^\infty \Big[\frac{(-1)^r}{(b-a)+2br} + \frac{(-1)^r}{(b+a)+2br} - \frac{2(-1)^r}{(b+a)+2br}\Big], \tag{5.34}$$

$$= \sum_{r=0}^\infty (-1)^r\Big[\frac{(1+2r)}{\{(b-a)+2br\}\{(b+a)+2br\}}\Big] - \frac{1}{b}\sum_{r=0}^\infty \frac{(-1)^r}{\{\left(\frac{b+a}{2b}\right)+r\}}. \tag{5.35}$$

Employ algebraic properties of Pochhammer symbol and Lower case beta function (1.27) in the eq.(5.35), after simplifications, we obtain

$$\begin{aligned}\Phi(a,b) &= \frac{2b}{b^2-a^2}\sum_{r=0}^{\infty}\Big[\frac{\left(\frac{3}{2}\right)_r\left(\frac{b-a}{2b}\right)_r\left(\frac{b+a}{2b}\right)_r(-1)^r}{\left(\frac{1}{2}\right)_r\left(\frac{3b-a}{2b}\right)_r\left(\frac{3b+a}{2b}\right)_r}\Big]-\frac{1}{b}\beta\left(\frac{a+b}{2b}\right),\\ &= \frac{2b}{(b^2-a^2)}\;{}_4F_3\left(\begin{array}{l}1,\ \frac{3}{2},\ \frac{1}{2}-\frac{a}{2b},\ \frac{1}{2}+\frac{a}{2b};\\ \frac{1}{2},\ \frac{3}{2}-\frac{a}{2b},\ \frac{3}{2}+\frac{a}{2b};\end{array}\ -1\right)-\frac{1}{b}\beta\left(\frac{a+b}{2b}\right),\end{aligned} \tag{5.36}$$

where $\Re(b\pm a)>0,\ \Re(b)>0,\ \frac{a}{b}\in\mathbb{C}\backslash\{\pm1,\pm3,\pm5,...\},\ \frac{3}{2}\pm\frac{a}{2b}\in\mathbb{C}\backslash\mathbb{Z}_0^-$. We obtain, upon using hypergeometric form of sec(z) function (1.32) [when $z=\frac{\pi a}{2b}$] in the eq. (5.36).

$$\Phi(a,b)=\frac{\pi}{2b}\sec\left(\frac{\pi a}{2b}\right)-\frac{1}{b}\beta\left(\frac{a+b}{2b}\right). \tag{5.37}$$

Using the classical summation theorem ${}_4F_3(-1)$ (1.15) in the right hand side of eq.(5.36), after simplification we get eq.(5.37) or (5.23). In the eq.(5.22) apply the properties of Digamma functions (1.28) and (1.29), we get the result (5.25). □

6. Applications of product formulas in special class of hyperbolic integrals

We get some applications by applying suitable product formulas associated with hyperbolic function in the special class of hyperbolic integrals. The well-known product formulas of hyperbolic functions are given as follows

$$\sinh(A)\cosh(B)=\frac{1}{2}[\sinh(A+B)+\sinh(A-B)], \tag{6.1}$$

$$\sinh(A)\sinh(B)=\frac{1}{2}[\cosh(A+B)-\cosh(A-B)], \tag{6.2}$$

$$\cosh(A)\cosh(B)=\frac{1}{2}[\cosh(A+B)+\cosh(A-B)]. \tag{6.3}$$

Each of the following hyperbolic definite integrals holds true:

• In the eq.(4.1) using product formula and applying the result (5.16), we get

$$\int_0^{\infty}\frac{\sinh(ax)\sinh(bx)}{\cosh^{v}(cx)}dx=\frac{2^{v-3}}{(c)\Gamma(v)}\Big[\Gamma\left(\frac{vc+a+b}{2c}\right)\Gamma\left(\frac{vc-a-b}{2c}\right)-\Gamma\left(\frac{vc+a-b}{2c}\right)\Gamma\left(\frac{vc-a+b}{2c}\right)\Big], \tag{6.4}$$

where $\Re(v)<4,\ \Re(c)>0,\ \Re(vc\pm a\pm b)>0;\ \frac{v}{2},\ 1+\frac{v}{2}\pm\frac{a}{2c}\pm\frac{b}{2c}\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

• In the eq.(4.1) put $v=1$ and interchange b and c, we get

$$\begin{aligned}\int_0^{\infty}\frac{\sinh(ax)\sinh(cx)}{\cosh(bx)}dx=&\frac{2^2acb}{(b-a-c)(b+a+c)(b-a+c)(b+a-c)}\\ &\times{}_6F_5\left(\begin{array}{l}1,\frac{3}{2},\frac{1}{2}-\frac{a}{2b}-\frac{c}{2b},\frac{1}{2}-\frac{a}{2b}+\frac{c}{2b},\frac{1}{2}+\frac{a}{2b}+\frac{c}{2b},\frac{1}{2}+\frac{a}{2b}-\frac{c}{2b};\\ \frac{1}{2},\frac{3}{2}-\frac{a}{2b}-\frac{c}{2b},\frac{3}{2}-\frac{a}{2b}+\frac{c}{2b},\frac{3}{2}+\frac{a}{2b}+\frac{c}{2b},\frac{3}{2}+\frac{a}{2b}-\frac{c}{2b};\end{array}\ -1\right),\end{aligned} \tag{6.5}$$

where $\Re(b)>0,\ \Re(b\pm a\pm c)>0,\ \frac{3}{2}\pm\frac{a}{2b}\pm\frac{c}{2b}\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

• Using the product formula of hyperbolic function, then applying eq.(5.18), we get

$$\int_0^{\infty}\frac{\sinh(ax)\sinh(cx)}{\cosh(bx)}dx = \frac{\pi}{4b}\left[\sec\left(\frac{\pi(a+c)}{2b}\right)-\sec\left(\frac{\pi(a-c)}{2b}\right)\right], \tag{6.6}$$

$$= \left(\frac{\pi}{b}\right)\frac{\sin\left(\frac{a\pi}{2b}\right)\sin\left(\frac{c\pi}{2b}\right)}{\left\{\cos\left(\frac{c\pi}{b}\right)+\cos\left(\frac{a\pi}{b}\right)\right\}}, \tag{6.7}$$

where $\Re(b) > 0,\ \Re(b \pm a \pm c) > 0,\ \frac{3}{2} \pm \frac{a}{2b} \pm \frac{c}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

• In the eq.(4.2) using product formula and applying the result (5.21), we get

$$\int_0^\infty \frac{\sinh(ax)\sinh(bx)}{\sinh^v(cx)}dx = \frac{2^{v-3}}{(c)\Gamma(v)}\Bigg[\Gamma\left(\frac{vc+a+b}{2c}\right)\Gamma\left(\frac{vc-a-b}{2c}\right)\frac{\cos\left(\frac{(a+b)\pi}{2c}\right)}{\cos(\frac{v\pi}{2})}$$

$$-\Gamma\left(\frac{vc+a-b}{2c}\right)\Gamma\left(\frac{vc-a+b}{2c}\right)\frac{\cos\left(\frac{(a-b)\pi}{2c}\right)}{\cos(\frac{v\pi}{2})}\Bigg], \tag{6.8}$$

where $\Re(v) < 3,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2},\ 1 + \frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

• In the eq.(4.3) using product formula and then applying the result (5.10), we get

$$\int_0^\infty \frac{\sinh(ax)\cosh(bx)}{\cosh^v(cx)}dx = 2^{v-2}\Bigg[\frac{1}{(vc-a-b)}\,{}_2F_1\left(\begin{matrix} v, \frac{vc-a-b}{2c} & ; \\ 1+\frac{vc-a-b}{2c} & ; \end{matrix}\ -1\right) - \frac{1}{(vc+a+b)}\,{}_2F_1\left(\begin{matrix} v, \frac{vc+a+b}{2c} & ; \\ 1+\frac{vc+a+b}{2c} & ; \end{matrix}\ -1\right)$$

$$+\frac{1}{(vc-a+b)}\,{}_2F_1\left(\begin{matrix} v, \frac{vc-a+b}{2c} & ; \\ 1+\frac{vc-a+b}{2c} & ; \end{matrix}\ -1\right) - \frac{1}{(vc+a-b)}\,{}_2F_1\left(\begin{matrix} v, \frac{vc+a-b}{2c} & ; \\ 1+\frac{vc+a-b}{2c} & ; \end{matrix}\ -1\right)\Bigg], \tag{6.9}$$

where $\Re(v) < 3,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2} \pm \frac{\sqrt{a^2-b^2}}{2c},\ 1 + \frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

• In the eq.(4.3) put $v = 1$ and interchange b and c, we get

$$\int_0^\infty \frac{\sinh(ax)\cosh(cx)}{\cosh(bx)}dx = \frac{2(ab^2-a^3+ac^2)}{(b-a-c)(b+a+c)(b-a+c)(b+a-c)}$$

$$\times\,{}_7F_6\left(\begin{matrix} 1, \frac{3}{2}-\frac{\sqrt{a^2-c^2}}{2b}, \frac{3}{2}+\frac{\sqrt{a^2-c^2}}{2b}, \frac{1}{2}-\frac{a}{2b}-\frac{c}{2b}, \\ \frac{1}{2}-\frac{a}{2b}+\frac{c}{2b}, \frac{1}{2}+\frac{a}{2b}+\frac{c}{2b}, \frac{1}{2}+\frac{a}{2b}-\frac{c}{2b}\ ; \\ \frac{1}{2}-\frac{\sqrt{a^2-c^2}}{2b}, \frac{1}{2}+\frac{\sqrt{a^2-c^2}}{2b}, \frac{3}{2}-\frac{a}{2b}-\frac{c}{2b}, \\ \frac{3}{2}-\frac{a}{2b}+\frac{c}{2b}, \frac{3}{2}+\frac{a}{2b}+\frac{c}{2b}, \frac{3}{2}+\frac{a}{2b}-\frac{c}{2b}; \end{matrix}\ -1\right), \tag{6.10}$$

where $\Re(b) > 0,\ \Re(b \pm a \pm c) > 0;\ \frac{1}{2} \pm \frac{\sqrt{a^2-c^2}}{2b},\ \frac{3}{2} \pm \frac{a}{2b} \pm \frac{c}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

• Using the product formula of hyperbolic function , then applying eq.(5.23), we get

$$\int_0^\infty \frac{\sinh(ax)\cosh(cx)}{\cosh(bx)}dx = \frac{\pi}{4b}\left[\sec\left(\frac{\pi(a+c)}{2b}\right)+\sec\left(\frac{\pi(a-c)}{2b}\right)\right] - \frac{1}{2b}\left[\beta\left(\frac{(a+b+c)}{2b}\right)+\beta\left(\frac{(a+b-c)}{2b}\right)\right], \tag{6.11}$$

$$= \left(\frac{\pi}{b}\right)\frac{\cos\left(\frac{a\pi}{2b}\right)\cos\left(\frac{c\pi}{2b}\right)}{\left\{\cos\left(\frac{c\pi}{b}\right)+\cos\left(\frac{a\pi}{b}\right)\right\}} - \frac{1}{2b}\left[\beta\left(\frac{(a+b+c)}{2b}\right)+\beta\left(\frac{(a+b-c)}{2b}\right)\right], \tag{6.12}$$

where $\Re(b) > 0,\ \Re(b \pm a \pm c) > 0;\ \frac{1}{2} \pm \frac{\sqrt{a^2-c^2}}{2b},\ \frac{3}{2} \pm \frac{a}{2b} \pm \frac{c}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

• In the eq.(4.4) using product formula and applying the result (5.12), we get

$$\int_0^\infty \frac{\sinh(ax)\cosh(bx)}{\sinh^v(cx)}dx = \frac{2^{v-3}}{(c)\Gamma(v)}\Bigg[\Gamma\left(\frac{vc+a+b}{2c}\right)\Gamma\left(\frac{vc-a-b}{2c}\right)\frac{\sin\left(\frac{(a+b)\pi}{2c}\right)}{\sin(\frac{v\pi}{2})} \tag{6.13}$$

$$+\Gamma\left(\frac{vc+a-b}{2c}\right)\Gamma\left(\frac{vc-a+b}{2c}\right)\frac{\sin\left(\frac{(a-b)\pi}{2c}\right)}{\sin(\frac{v\pi}{2})}\Bigg],$$

where $\Re(v) < 2,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2} \pm \frac{\sqrt{a^2-b^2}}{2c},\ 1 + \frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

• In the eq.(4.4) put $v = 1$ and interchange b and c, we get

$$\int_0^\infty \frac{\sinh(ax)\cosh(cx)}{\sinh(bx)}dx = \frac{2(ab^2 - a^3 + ac^2)}{(b-a-c)(b+a+c)(b-a+c)(b+a-c)} \times {}_7F_6\left(\begin{array}{c} 1, \frac{3}{2} - \frac{\sqrt{a^2-c^2}}{2b}, \frac{3}{2} + \frac{\sqrt{a^2-c^2}}{2b}, \frac{1}{2} - \frac{a}{2b} - \frac{c}{2b}, \\ \frac{1}{2} - \frac{a}{2b} + \frac{c}{2b}, \frac{1}{2} + \frac{a}{2b} + \frac{c}{2b}, \frac{1}{2} + \frac{a}{2b} - \frac{c}{2b}\ ; \\ \frac{1}{2} - \frac{\sqrt{a^2-c^2}}{2b}, \frac{1}{2} + \frac{\sqrt{a^2-c^2}}{2b}, \frac{3}{2} - \frac{a}{2b} - \frac{c}{2b}, \\ \frac{3}{2} - \frac{a}{2b} + \frac{c}{2b}, \frac{3}{2} + \frac{a}{2b} + \frac{c}{2b}, \frac{3}{2} + \frac{a}{2b} - \frac{c}{2b}; \end{array} 1\right), \tag{6.14}$$

where $\Re(b) > 0,\ \Re(b \pm a \pm c) > 0;\ \frac{1}{2} \pm \frac{\sqrt{a^2-c^2}}{2b},\ \frac{3}{2} \pm \frac{a}{2b} \pm \frac{c}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

• Using the product formula of hyperbolic function , then applying eq.(5.14), we get

$$\int_0^\infty \frac{\sinh(ax)\cosh(cx)}{\sinh(bx)}dx = \frac{\pi}{4b}\left[\tan\left(\frac{\pi(a+c)}{2b}\right) + \tan\left(\frac{\pi(a-c)}{2b}\right)\right], \tag{6.15}$$

$$= \left(\frac{\pi}{2b}\right)\frac{\sin\left(\frac{a\pi}{b}\right)}{\left\{\cos\left(\frac{c\pi}{b}\right) + \cos\left(\frac{a\pi}{b}\right)\right\}}, \tag{6.16}$$

where $\Re(b) > 0,\ \Re(b \pm a \pm c) > 0;\ \frac{1}{2} \pm \frac{\sqrt{a^2-c^2}}{2b},\ \frac{3}{2} \pm \frac{a}{2b} \pm \frac{c}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

• In the eq.(4.5) using product formula and applying the result (5.16), we get

$$\int_0^\infty \frac{\cosh(ax)\cosh(bx)}{\cosh^v(cx)}dx = \frac{2^{v-3}}{(c)\Gamma(v)}\left[\Gamma\left(\frac{vc+a+b}{2c}\right)\Gamma\left(\frac{vc-a-b}{2c}\right) + \Gamma\left(\frac{vc+a-b}{2c}\right)\Gamma\left(\frac{vc-a+b}{2c}\right)\right], \tag{6.17}$$

where $\Re(v) < 2,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0; \frac{v}{2},\ \frac{v}{2} \pm \frac{\sqrt{a^2+b^2}}{2c},\ 1 + \frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

• In the eq.(4.5) put $v = 1$ and interchange b and c, we get

$$\int_0^\infty \frac{\cosh(ax)\cosh(cx)}{\cosh(bx)}dx = \frac{2(b^3 - a^2b - c^2b)}{(b-a-c)(b+a+c)(b-a+c)(b+a-c)} \times {}_8F_7\left(\begin{array}{c} 1, \frac{3}{2}, \frac{3}{2} - \frac{\sqrt{a^2+c^2}}{2b}, \frac{3}{2} + \frac{\sqrt{a^2+c^2}}{2b}, \\ \frac{1}{2} - \frac{a}{2b} - \frac{c}{2b}, \frac{1}{2} - \frac{a}{2b} + \frac{c}{2b}, \frac{1}{2} + \frac{a}{2b} + \frac{c}{2b}, \frac{1}{2} + \frac{a}{2b} - \frac{c}{2b}\ ; \\ \frac{1}{2}, \frac{1}{2} - \frac{\sqrt{a^2+c^2}}{2b}, \frac{1}{2} + \frac{\sqrt{a^2+c^2}}{2b}, \frac{3}{2} - \frac{a}{2b} - \frac{c}{2b}, \\ \frac{3}{2} - \frac{a}{2b} + \frac{c}{2b}, \frac{3}{2} + \frac{a}{2b} + \frac{c}{2b}, \frac{3}{2} + \frac{a}{2b} - \frac{c}{2b}; \end{array} -1\right), \tag{6.18}$$

where $\Re(b) > 0,\ \Re(b \pm a \pm c) > 0;\ \frac{1}{2} \pm \frac{\sqrt{a^2+c^2}}{2b},\ \frac{3}{2} \pm \frac{a}{2b} \pm \frac{c}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

• Using the product formula of hyperbolic function, then applying eq.(5.18), we get

$$\int_0^\infty \frac{\cosh(ax)\cosh(cx)}{\cosh(bx)}dx = \frac{\pi}{4b}\left[\sec\left(\frac{\pi(a+c)}{2b}\right) + \sec\left(\frac{\pi(a-c)}{2b}\right)\right], \tag{6.19}$$

$$= \left(\frac{\pi}{b}\right)\frac{\cos\left(\frac{a\pi}{2b}\right)\cos\left(\frac{c\pi}{2b}\right)}{\left\{\cos\left(\frac{c\pi}{b}\right) + \cos\left(\frac{a\pi}{b}\right)\right\}}, \tag{6.20}$$

where $\Re(b) > 0,\ \Re(b \pm a \pm c) > 0;\ \frac{1}{2} \pm \frac{\sqrt{a^2+c^2}}{2b},\ \frac{3}{2} \pm \frac{a}{2b} \pm \frac{c}{2b} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

• In the eq.(4.6) using product formula and applying the result (5.21), we get

$$\int_0^\infty \frac{\cosh(ax)\cosh(bx)}{\sinh^v(cx)}dx = \frac{2^{v-3}}{(c)\Gamma(v)}\left[\Gamma\left(\frac{vc+a+b}{2c}\right)\Gamma\left(\frac{vc-a-b}{2c}\right)\frac{\cos\left(\frac{(a+b)\pi}{2c}\right)}{\cos(\frac{v\pi}{2})}\right.$$
$$\left. + \Gamma\left(\frac{vc+a-b}{2c}\right)\Gamma\left(\frac{vc-a+b}{2c}\right)\frac{\cos\left(\frac{(a-b)\pi}{2c}\right)}{\cos(\frac{v\pi}{2})}\right], \tag{6.21}$$

where $\Re(v) < 1,\ \Re(c) > 0,\ \Re(vc \pm a \pm b) > 0;\ \frac{v}{2},\ \frac{v}{2} \pm \frac{\sqrt{a^2+b^2}}{2c},\ 1 + \frac{v}{2} \pm \frac{a}{2c} \pm \frac{b}{2c} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

7. Conclusion

Here, we have described some definite integrals containing the quotients of hyperbolic functions. Thus certain integrals of hyperbolic functions, which may be different from those of presented here, can also be evaluated in a similar way. Therefore, the results presented in this paper can be expressed in terms of hypergeometric functions, trigonometric and hyperbolic functions, Digamma functions, Beta function of one variable, and Gamma function.

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