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On a Common Fixed Point Theorem in Bicomplex Valued b-metric Space

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Abstract

The main purpose of this paper is to investigate a common fixed point theorem in bicomplex valued b-metric space satisfying some rational inequalities for two pairs of weakly compatible self contracting mappings and to extend the result obtained by Azam et al.[1]. Our result is the generalisation of the result of Mitra[21] and the application of Banach contraction principle. Also we use the concepts of Choi et al.[11] and Datta et al.[15]

Keywords: Common fixed point, Contractive type mapping, Bicomplex valued metric space, Bicomplex valued b-metric space

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1. Introduction, Definitions and Notations

Segre's[26] conceptualization of bicomplex numbers, tricomplex numbers initiated a new chapter in the growth of special algebras. Subsequently, this significant contribution motivated Price[22] to develop the bicomplex algebra and function theory. In this field an impressive body of work by different researchers can be noticed during the last few years. One can see some of the attempts in $cf.$ [7], [8], [12], [17], [19] & [23].

Azam et al.[1] introduced the concept of complex valued metric spaces and established some common fixed point theorems for a pair of self contracting mappings. Rouzkard & Imdad[24] generalized the result obtained by Azam et al.[1] and proved another common fixed point theorem satisfying some rational inequalities in complex valued metric space. The main tool which is used to prove the fixed point theorem is the Banach[4] contraction principle. In this connection Choudhury et al.[9]&[10] obtained some fixed point results in partially ordered complex valued metric spaces for rational type expressions. Also one can see the attempts in $cf.$ [2], [3], [5], [6], [28] & [29].

The concept of complex-valued b-metric spaces was introduced by Rao et al.[25] and they proved a common fixed point theorem in complex valued b-metric spaces. Mukheimer[20] substantiated some common fixed point theorems in complex-valued b-metric spaces.

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Dubey et al.[13] proved some common fixed point theorems for contractive mappings in complex-valued b-metric spaces. Here the contributions of Singh et al.[27] and Mitra[21] should also be mentioned.

The bicomplex valued metric was defined by Choi et al.[11]. They also proved some common fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces.

We write the set of real, complex and bicomplex numbers respectively as $\mathbb{C}_0, \mathbb{C}_1$ and $\mathbb{C}_2$.

1.1. Bicomplex Number

Segre [26] defined the bicomplex number as

$$\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2,$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$ and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. We denote $i_1 i_2 = j$, which is known as the hyperbolic unit and such that $j^2 = 1$, $i_1 j = j i_1 = -i_2$, $i_2 j = j i_2 = -i_1$. Also $\mathbb{C}_2$ is defined as

$$\mathbb{C}_2 = \{\xi : \xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in \mathbb{C}_0\}$$

i.e.,

$$\mathbb{C}_2 = \{\xi : \xi = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}_1\},$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$.

If $\xi = z_1 + i_2 z_2$ and $\eta = w_1 + i_2 w_2$ be any two bicomplex numbers then the sum is $\xi \pm \eta = (z_1 + i_2 z_2) \pm (w_1 + i_2 w_2) = (z_1 \pm w_1) + i_2 (z_2 \pm w_2)$ and the product is $\xi.\eta = (z_1 + i_2 z_2).(w_1 + i_2 w_2) = (z_1 w_1 - z_2 w_2) + i_2 (z_1 w_2 + z_2 w_1)$.

1.1.1. Norm of a bicomplex number

The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a positive real valued function and $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined by

$$\begin{aligned} \|\xi\| &= \|z_1 + i_2 z_2\| = \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} = \left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)^{\frac{1}{2}}, \end{aligned}$$

where $\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

The linear space $\mathbb{C}_2$ with respect to defined norm is a norm linear space, also $\mathbb{C}_2$ is complete; therefore $\mathbb{C}_2$ is the Banach space. If $\xi, \eta \in \mathbb{C}_2$ then $\|\xi\eta\| \leq \sqrt{2}\|\xi\|\|\eta\|$ holds instead of $\|\xi\eta\| \leq \|\xi\|\|\eta\|$. Therefore $\mathbb{C}_2$ is not the Banach algebra.

Now we define the partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ as follows:

Let $\mathbb{C}_2$ be the set of bicomplex numbers and $\xi = z_1 + i_2 z_2, \eta = w_1 + i_2 w_2 \in \mathbb{C}_2$ then $\xi \precsim_{i_2} \eta$ if and only if $z_1 \precsim w_1$ and $z_2 \precsim w_2$.

Thus $\xi \precsim_{i_2} \eta$ if one of the following conditions is satisfied:

$$\begin{aligned} &(1) \quad z_1 = w_1,\ z_2 = w_2, \\ &(2) \quad z_1 \prec w_1,\ z_2 = w_2, \\ &(3) \quad z_1 = w_1,\ z_2 \prec w_2 \text{ and} \\ &(4) \quad z_1 \prec w_1,\ z_2 \prec w_2. \end{aligned}$$

In particular we can write $\xi \precnsim_{i_2} \eta$ if $\xi \precsim_{i_2} \eta$ and $\xi \neq \eta$, i.e. one of (2), (3) and (4) is satisfied and we will write $\xi \prec_{i_2} \eta$ if only (4) is satisfied.

For any two bicomplex numbers $\xi, \eta \in \mathbb{C}_2$ we can verify the followings:

(i) $\xi \precsim_{i_2} \eta \Rightarrow \|\xi\| \leq \|\eta\|$,

(ii) $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$,

(iii) $\|a\xi\| = a\|\xi\|$ if a is a non negative real number,

(iv) $\|\xi\eta\| \leq \sqrt{2}\|\xi\|\|\eta\|$ and the equality holds only when at least one of ξ and η is degenerated, *1.1.2. Bicomplex valued metric space*

Choi et al.[11] defines the bicomplex valued metric space as

Definition 1.1. Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions
1. $0 \precsim_{i_2} d(x,y)$ for all $x,y \in X$,
2. $d(x,y) = 0$ if and only if $x = y$,
3. $d(x,y) = d(y,x)$ for all $x,y \in X$ and
4. $d(x,y) \precsim_{i_2} d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a bicomplex valued metric on X and (X,d) is called the bicomplex valued metric space.

Definition 1.2. [15] Let X be a nonempty set and let $s \geq 1$. Suppose the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions
1. $0 \precsim_{i_2} d(x,y)$ for all $x,y \in X$,
2. $d(x,y) = 0$ if and only if $x = y$,
3. $d(x,y) = d(y,x)$ for all $x,y \in X$ and
4. $d(x,y) \precsim_{i_2} s\left[d(x,z) + d(z,y)\right]$ for all $x,y,z \in X$.

Then d is called a bicomplex valued b-metric on X and (X,d) is called the bicomplex valued b-metric space.

Definition 1.3. For a bicomplex valued metric space (X,d)

(i). A sequence $\{x_n\}$ in X is said to be a convergent sequence and converges to a point x if for any $0 \prec_{i_2} r \in \mathbb{C}_2$ there exists a natural number $n_0 \in \mathbb{N}$ such that $d(x_n,x) \prec_{i_2} r$, for all $n > n_0$. And we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii). A sequence $\{x_n\}$ in X is said to be a Cauchy sequence in (X,d) if for any $0 \prec_{i_2} r \in \mathbb{C}_2$ there exists a natural number $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec_{i_2} r$, for all $m,n \in \mathbb{N}$ and $n > n_0$.

(iii). If every Cauchy sequence in X is convergent in X then (X,d) is said to be a complete bicomplex valued metric space.

Definition 1.4. Let (X,d) be a bicomplex valued metric space and $S,T : X \to X$ be two self-mappings then S and T are said to be compatible if $\lim_{n\to\infty} d(STx, TSx_n) = 0,$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u$ for some $u \in X$.

Definition 1.5. Let $S,T : X \to X$ be two self-mappings then S and T are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$ for all $x \in X$.

Definition 1.6. Let $S,T : X \to X$ be two self-mappings then S and T are said to be commuting if $TSx = STx$ for all $x \in X$.

Definition 1.7. Let (X,d) be a bicomplex valued metric space and $S,T : X \to X$ be two self-mappings then S and T are said to be weakly commuting if $d(STx, TSx) \precsim_{i_2} d(Sx, Tx)$ for all $x \in X$.

2. Lemmas.

In this section we present two lemmas which will be needed in the sequel.

Lemma 2.1. *[11] Let (X,d) be a bicomplex valued metric space and a sequence $\{x_n\}$ in X is said to be convergent to a point x if and only if* $\lim_{n\to\infty} \|d(x_n,x)\| = 0$.

Lemma 2.2. *[11] Let (X,d) be a bicomplex valued metric space and a sequence $\{x_n\}$ in X is said to be a Cauchy sequence in X if and only if* $\lim_{n\to\infty} \|d(x_n, x_{n+m})\| = 0$.

3. Main Theorem.

In this section we prove a fixed point theorem on bicomplex valued b-metric space for four self contracting mappings.

Theorem 3.1. *Let (X, d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$. Let $S, T, f, g : X \to X$ be mappings such that*

$$S(X) \subseteq g(X) \text{ and } T(X) \subseteq f(X) \tag{3.1}$$

also satisfying the condition

$$d(Sx, Ty) \precsim_{i_2} Ad(fx, gy) + Bd(fx, Sx) + Cd(gy, Ty) + D[d(gy, Sx) + d(fx, Ty)] \tag{3.2}$$

for all $x, y \in X$, where A, B, C and D are non negative real numbers such that $A + sB + C + 2sD < 1$. Suppose the pairs $\{f, S\}$ and $\{g, T\}$ be weakly compatible and $f(X)$ or $g(X)$ be a complete subspace of X. Then S, T, f and g have a unique common fixed point in X.

Proof. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$$y_{2k} = Sx_{2k} = gx_{2k+1}, \quad y_{2k+1} = Tx_{2k+1} = fx_{2k+2}, \quad k = 0, 1, 2, ...$$

where x_0 is an arbitrary fixed point in X

Therefore by using (3.2) we obtain that

$$\begin{aligned} d(y_{2k}, y_{2k+1}) &= d(Sx_{2k}, Tx_{2k+1}) \\ &\precsim_{i_2} Ad(fx_{2k}, gx_{2k+1}) + Bd(fx_{2k}, Sx_{2k}) + Cd(gx_{2k+1}, Tx_{2k+1}) \\ &\quad + D[d(gx_{2k+1}, Sx_{2k}) + d(fx_{2k}, Tx_{2k+1})] \\ &\precsim_{i_2} Ad(y_{2k-1}, y_{2k}) + Bd(y_{2k-1}, y_{2k}) + Cd(y_{2k}, y_{2k+1}) \\ &\quad + D[d(y_{2k}, y_{2k}) + d(y_{2k-1}, y_{2k+1})] \\ &\precsim_{i_2} Ad(y_{2k-1}, y_{2k}) + Bd(y_{2k-1}, y_{2k}) + Cd(y_{2k}, y_{2k+1}) \\ &\quad + Dd(y_{2k-1}, y_{2k+1}) \\ &\precsim_{i_2} Ad(y_{2k-1}, y_{2k}) + Bd(y_{2k-1}, y_{2k}) + Cd(y_{2k}, y_{2k+1}) \\ &\quad + Ds[d(y_{2k-1}, y_{2k}) + d(y_{2k}, y_{2k+1})] \end{aligned}$$

i.e.,

$$\begin{aligned} \|d(y_{2k}, y_{2k+1})\| &\leq A\|d(y_{2k-1}, y_{2k})\| + B\|d(y_{2k-1}, y_{2k})\| + C\|d(y_{2k}, y_{2k+1})\| \\ &\quad + Ds[\|d(y_{2k-1}, y_{2k})\| + \|d(y_{2k}, y_{2k+1})\|] \end{aligned}$$

$$\begin{aligned} \text{i.e., } (1 - C - sD)\|d(y_{2k}, y_{2k+1})\| &\leq (A + B + sD)\|d(y_{2k-1}, y_{2k})\| \\ \text{i.e., } \|d(y_{2k}, y_{2k+1})\| &\leq \frac{A + B + sD}{1 - C - sD}\|d(y_{2k-1}, y_{2k})\| \\ \text{i.e., } \|d(y_{2k}, y_{2k+1})\| &\leq \delta_1 \|d(y_{2k-1}, y_{2k})\|, \end{aligned}$$

where $0 < \delta_1 = \frac{A+B+sD}{1-C-sD} < 1$, as $A + sB + C + 2sD < 1$ and $s \geq 1$.

Similarly,

$$\begin{aligned}
d(y_{2k+1}, y_{2k+2}) &= d(Tx_{2k+1}, Sx_{2k+2}) = d(Sx_{2k+2}, Tx_{2k+1}) \\
&\precsim_{i_2} Ad(fx_{2k+2}, gx_{2k+1}) + Bd(fx_{2k+2}, Sx_{2k+2}) + Cd(gx_{2k+1}, Tx_{2k+1}) \\
&\quad + D[d(gx_{2k+1}, Sx_{2k+2}) + d(fx_{2k+2}, Tx_{2k+1})] \\
&\precsim_{i_2} Ad(y_{2k+1}, y_{2k}) + Bd(y_{2k+1}, y_{2k+2}) + Cd(y_{2k}, y_{2k+1}) \\
&\quad + D[d(y_{2k}, y_{2k+2}) + d(y_{2k+1}, y_{2k+1})] \\
&\precsim_{i_2} Ad(y_{2k+1}, y_{2k}) + Bd(y_{2k+1}, y_{2k+2}) \\
&\quad + Cd(y_{2k}, y_{2k+1}) + Dd(y_{2k}, y_{2k+2}) \\
&\precsim_{i_2} Ad(y_{2k+1}, y_{2k}) + Bd(y_{2k+1}, y_{2k+2}) + Cd(y_{2k}, y_{2k+1}) \\
&\quad + Ds[d(y_{2k}, y_{2k+1}) + d(y_{2k+1}, y_{2k+2})]
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|d(y_{2k+1}, y_{2k+2})\| &\le A\|d(y_{2k}, y_{2k+1})\| + B\|d(y_{2k+1}, y_{2k+2})\| + C\|d(y_{2k}, y_{2k+1})\| \\
&\quad + Ds[\|d(y_{2k}, y_{2k+1})\| + \|d(y_{2k+1}, y_{2k+2})\|]
\end{aligned}$$

$$\begin{aligned}
i.e.,\ (1 - B - sD)\|d(y_{2k+1}, y_{2k+2})\| &\le (A + C + sD)\|d(y_{2k}, y_{2k+1})\| \\
i.e.,\ \|d(y_{2k+1}, y_{2k+2})\| &\le \frac{A + C + sD}{1 - B - sD}\|d(y_{2k}, y_{2k+1})\| \\
i.e.,\ \|d(y_{2k+1}, y_{2k+2})\| &\le \delta_2\|d(y_{2k}, y_{2k+1})\|,
\end{aligned}$$

where $0 \le \delta_2 = \frac{A+C+sD}{1-B-sD} < 1$, as $A + sB + C + 2sD < 1$ and $s \ge 1$.

Let $\delta = \max\{\delta_1, \delta_2\}$ .Then $\|d(y_{2k}, y_{2k+1})\| \le \delta\|d(y_{2k-1}, y_{2k})\|$ and $\|d(y_{2k+1}, y_{2k+2})\| \le \delta\|d(y_{2k}, y_{2k+1})\|$ for all $k = 0, 1, 2, \cdots$.

Hence

$$\begin{aligned}
&\|d(y_{n+1}, y_{n+2})\| \\
\le\ &\delta\|d(y_n, y_{n+1})\| \\
\le\ &\delta^2\|d(y_{n-1}, y_n)\| \le \cdots \le \delta^{n+1}\|d(y_0, y_1)\| \text{ for all } n = 0, 1, 2, \cdots.
\end{aligned} \tag{3.3}$$

Therefore for any two positive integers m, n with $m > n$ we get that

$$d(y_n, y_m) \precsim_{i_2} s[d(y_n, y_{n+1}) + d(y_{n+1}, y_m)].$$

Thus

$$\begin{aligned}
&\|d(y_n, y_m)\| \\
\le\ &s\|d(y_n, y_{n+1})\| + s\|d(y_{n+1}, y_m)\| \\
\le\ &s\|d(y_n, y_{n+1})\| + s^2\|d(y_{n+1}, y_{n+2})\| + s^2\|d(y_{n+2}, y_m)\| \\
\le\ &s\|d(y_n, y_{n+1})\| + s^2\|d(y_{n+1}, y_{n+2})\| + s^3\|d(y_{n+2}, y_{n+3})\| + s^3\|d(y_{n+3}, y_m)\| \\
&\ldots
\end{aligned}$$

$$\begin{aligned}
&i.e.,\ \|d(y_n, y_m)\| \\
\le\ &s\|d(y_n, y_{n+1})\| + s^2\|d(y_{n+1}, y_{n+2})\| + s^3\|d(y_{n+2}, y_{n+3})\| + \ldots \\
&\ldots + s^{m-n-1}\|d(y_{m-1}, y_m)\|.
\end{aligned}$$

Since $s \ge 0$, therefore we have

$$\begin{aligned}
\|d(y_n, y_m)\| &\le \|d(y_n, y_{n+1})\| + s^2\|d(y_{n+1}, y_{n+2})\| + s^3\|d(y_{n+2}, y_{n+3})\| + \\
&\quad \ldots + s^{m-n}\|d(y_{m-1}, y_m)\|
\end{aligned}$$

Hence by using (3.3)

$$\begin{aligned}\|d(y_n,y_m)\| &\le s\delta^n\|d(y_0,y_1)\|+s^2\delta^{n+1}\|d(y_0,y_1)\|\\ &\quad +s^3\delta^{n+2}\|d(y_0,y_1)\|+...+s^{m-n}\delta^{m-1}\|d(y_0,y_1)\|\\ i.e.,\ \|d(y_n,y_m)\| &\le \sum_{i=1}^{m-n}s^i\delta^{i+n-1}\|d(y_0,y_1)\|\\ i.e.,\ \|d(y_n,y_m)\| &\le s\delta^n\sum_{i=1}^{m-n}s^{i-1}\delta^{i-1}\|d(y_0,y_1)\|\\ i.e.,\ \|d(y_n,y_m)\| &\le s\delta^n\sum_{i=1}^{\infty}s^{i-1}\delta^{i-1}\|d(y_0,y_1)\|\\ i.e.,\ \|d(y_n,y_m)\| &\le \frac{s\delta^n}{1-s\delta}\|d(y_0,y_1)\|.\end{aligned}$$

Again since $\frac{s\delta^n}{1-s\delta}\to 0$ as $n\to\infty$, therefore for any $\varepsilon>0$ there exists a positive integer n_0 such that $\|d(y_n,y_m)\|<\varepsilon$ for all $m,n>n_0$. Hence $\{y_n\}$ is Cauchy in X. Also X is a complete bicomplex valued b-metric space. Therefore there exists $z\in X$ such that $\lim_{n\to\infty}Sx_{2n}=\lim_{n\to\infty}gx_{2n+1}=\lim_{n\to\infty}Tx_{2n+1}=\lim_{n\to\infty}fx_{2n+2}=z$. If $f(X)$ is a complete subspace of X then there exists some $u\in X$ such that $z=fu$.

Now we show that $Su=z$. If not then $0\prec_{i_2}d(Su,z)\in\mathbb{C}_2$.

Therefore,

$$d(Su,z)\precsim_{i_2}s[d(Su,Tx_{2n+1})+d(Tx_{2n+1},z)]$$

$$\begin{aligned}\Rightarrow \frac{1}{s}d(Su,z)\precsim_{i_2}&\ Ad(fu,gx_{2n+1})+Bd(fu,Su)+Cd(gx_{2n+1},Tx_{2n+1})\\ &+D[d(gx_{2n+1},Su)+d(fu,Tx_{2n+1})]+d(Tx_{2n+1},z)\\ \precsim_{i_2}&\ d(z,y_{2n})+Bd(z,Su)+Cd(y_{2n},y_{2n+1})\\ &+D[d(y_{2n},Su)+d(z,y_{2n+1})]+d(y_{2n+1},z)\end{aligned}$$

Taking limit as $n\to\infty$ we get that

$$\frac{1}{s}d(Su,z)\precsim_{i_2}d(z,z)+Bd(z,Su)+Cd(z,z)+D[d(z,Su)+d(z,z)]+d(z,z)$$

$$\begin{aligned}i.e.,\ d(Su,z)&\precsim_{i_2}s(B+D)d(Su,z)\\ i.e.,\ \|d(Su,z)\|&\le s(B+D)\|d(z,Su)\|\\ i.e.,\ (1-sB-sD)\|d(Su,z)\|&\le 0.\end{aligned}$$

Using

$$A+sB+C+2sD<1 \text{ and } s\ge 1 \tag{3.4}$$

we obtain that $1-sB-sD>0$. Therefore,

$$\|d(Su,z)\|=0$$

which is a contradiction. Hence $Su=fu=z$.

Also we have $S(X)\subseteq g(X)$. Therefore there exists some $v\in X$ such that $z=gv$.

Now we show that $Tv=z$. If not then $0\prec_{i_2}d(Tv,z)\in\mathbb{C}_2$.

Therefore,

$$\begin{aligned}d(Tv,z)&=d(Tv,Su)=d(Su,Tv)\\ &\precsim_{i_2}Ad(fu,gv)+Bd(fu,Su)+Cd(gv,Tv)+D[d(gv,Su)+d(fu,Tv)]\\ &\precsim_{i_2}Ad(z,z)+Bd(z,z)+Cd(z,Tv)+D[d(z,z)+d(z,Tv)]\end{aligned}$$

$$\begin{aligned} i.e.,\ d(Tv,z) &\precsim_{i_2} (C+D)\, d(z,Tv) \\ i.e.,\ \|d(Tv,z)\| &\leq (C+D)\, \|d(Tv,z)\| \\ i.e.,\ (1-C-D)\, \|d(Tv,z)\| &\leq 0. \end{aligned}$$

Using (3.4) we have $1-C-D>0$. Therefore,

$$\|d(Tv,z)\| = 0,$$

which is a contradiction. Thus $Tv = gv = z$ and so $Su = fu = Tv = gv = z$.

Also f and S are weakly compatible mappings, therefore $Sfu = fSu \Rightarrow Sz = fz$. Now we show that $Sz = z$. If not then $0 \prec_{i_2} d(Sz,z) \in \mathbb{C}_2$.

Therefore,

$$\begin{aligned} d(Sz,z) &= d(Sz,Tv) \\ &\precsim_{i_2} Ad(fz,gv) + Bd(fz,Sz) + Cd(gv,Tv) + D[d(gv,Sz) + d(fz,Tv)] \\ &\precsim_{i_2} Ad(Sz,z) + Bd(Sz,Sz) + Cd(z,z) + D[d(z,Sz) + d(Sz,z)] \\ &\precsim_{i_2} (A+2D)\, d(Sz,z) \end{aligned}$$

$$\begin{aligned} i.e., \|d(Sz,z)\| &\leq (A+2D)\, \|d(Sz,z)\| \\ i.e., (1-A-2D)\, \|d(Sz,z)\| &\leq 0. \end{aligned}$$

Using (3.4) we have $1-A-2D>0$. Hence

$$\|d(Sz,z)\| = 0,$$

which is a contradiction. Therefore $Sz = fz = z$.

Again since g and T are weakly compatible mappings, therefore $Tgv = gTv \Rightarrow Tz = gz$. Now we show that $Tz = z$. If not then $0 \prec_{i_2} d(Tz,z) \in \mathbb{C}_2$.

Therefore,

$$\begin{aligned} d(Tz,z) &= d(Tz,Sz) = d(Sz,Tz) \\ &\precsim_{i_2} Ad(fz,gz) + Bd(fz,Sz) + Cd(gz,Tz) + D[d(gz,Sz) + d(fz,Tz)] \\ &\precsim_{i_2} Ad(z,Tz) + Bd(z,z) + Cd(Tz,Tz) + D[d(Tz,z) + d(z,Tz)] \\ &\precsim_{i_2} (A+2D)\, d(Tz,z) \end{aligned}$$

$$\begin{aligned} i.e., \|d(Tz,z)\| &\leq (A+2D)\, \|d(Tz,z)\| \\ i.e., (1-A-2D)\, \|d(Tz,z)\| &\leq 0. \end{aligned}$$

Using (3.4) we have $1-A-2D>0$, implying that

$$\|d(Tz,z)\| = 0,$$

which is a contradiction. Therefore $Tz = gz = z$ and so $Sz = Tz = fz = gz = z$. This shows that z is a common fixed point of f, g, S and T.

Now we show that S, T, f and g have a unique common fixed point. If possible suppose $z^* \in X$ be another common fixed point of S, T, f and g, i.e. $Sz^* = Tz^* = fz^* = gz^* = z^*$.

Then

$$\begin{aligned} d(z,z^*) &= d(Sz,Tz^*) \precsim_{i_2} Ad(fz,gz^*) + Bd(fz,Sz) + Cd(gz^*,Tz^*) \\ &\quad + D[d(gz^*,Sz) + d(fz,Tz^*)] \\ &\precsim_{i_2} Ad(z,z^*) + Bd(z,z) + Cd(z^*,z^*) + D[d(z^*,z) + d(z,z^*)] \\ &\precsim_{i_2} (A+2D)\, d(z,z^*) \end{aligned}$$

$$\begin{aligned} i.e., \|d(z,z^*)\| &\leq (A+2D)\|d(z,z^*)\| \\ i.e., (1-A-2D)\|d(z,z^*)\| &\leq 0. \end{aligned}$$

Using (3.4) we have $1 - A - 2D > 0$. Therefore

$$\begin{aligned} \|d(z,z^*)\| &= 0 \\ i.e., z &= z^*. \end{aligned}$$

This shows that z is the unique common fixed point of S, T, f and g. □

Taking $f = g = I$ (identity mapping) in Theorem 3.1 we get the following corollary:

Corollary 3.2. *Let (X, d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$. Let $S, T : X \to X$ be mappings satisfying the condition*

$$d(Sx,Ty) \precsim_{i_2} Ad(x,y) + Bd(x,Sx) + Cd(y,Ty) + D\left[d(y,Sx) + d(x,Ty)\right]$$

for all $x, y \in X$, where A, B, C and D are non negative real numbers such that $A + sB + C + 2sD < 1$.Then S and T have a unique common fixed point in X.

Putting $T = S$ in the Corollary 3.1 we get the following corollary:

Corollary 3.3. *Let (X, d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$. Let $S, f, g : X \to X$ be mappings such that*

$$S(X) \subseteq g(X) \text{ and } S(X) \subseteq f(X)$$

also satisfying the condition

$$d(Sx,Sy) \precsim_{i_2} Ad(fx,gy) + Bd(fx,Sx) + Cd(gy,Sy) + D\left[d(gy,Sx) + d(fx,Sy)\right]$$

for all x, $y \in X$, where A, B, C and D are non negative real numbers such that $A + sB + C + 2sD < 1$. Suppose the pairs $\{f, S\}$ and $\{g, S\}$ be weakly compatible and $f(X)$ or $g(X)$ be a complete subspace of X. Then S, f and g have a unique common fixed point in X.

4. Future Prospect

In the line of the works as carried out in the paper one may think of the deduction of fixed point theorems using fuzzy metric, quasi metric, partial metric, probabilistic metric, p-adic metric (where p is a prime number), cone metric, quasi semi metric, convex metric, D-metric and other different types of metrics under the flavour of bicomplex analysis. This may be regarded as an active area of research to the future workers in this branch.

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References

[1] A. Azam, B. Fisher and M. Khan, *Common fixed point theorems in complex valued metric spaces*, Numer. Funct. Anal. Optim. **32 (3)**, 243-253, 2011.
[2] J. Ahmad, A. Azam and S. Saejung, *Common fixed point results for contractive mappings in complex valued metric spaces*, Fixed point Theory and App. **2014**, 1-11, 2014.
[3] D. Alpay, M. E. Luna-Elizarrarás, M. Shapiro and D. C. Struppa, *Basics of Functional Analysis with Bicomplex Scalars, and Bicomplex Schur Analysis*, Springer Briefs, 2014.
[4] S. Banach, *Sur les operations dans les ensembles abs traits et leur application aux equations integrales*, Fund. Math. **3**, 133-181, 1922.
[5] S. Bhatt, S. Chaukiyal and R. C. Dimri, *A common fixed point theorem for wealky compatible maps in complex valued metric spaces*, Internat. J. Math. Sci. Appl. **1 (3)**, 1385-1389, 2011.

[6] S. Bhatt, S. Chaukiyal and R. C. Dimri, *Common fixed point of mappings satisfying rational inequality in complex valued metric space*, Internat. J. Pure. App. Math. **73 (2)**, 159-164, 2011.
[7] H. De. Bie, D. C. Struppa, A. Vajiac and M. Vajiac, *The Cauchy-Kowalevski product for bicomplex holomorphic functions*, Math. Nachr **285 (10)**, 1230-1242, 2012.
[8] F. Colombo, I. Sabadini, D.C. Struppa, A. Vajiac and M. Vajiac, *Singularities of functions of one and several bicomplex variables*, Ark. Math., 2010.
[9] B. S. Choudhury, N. Metiya and P. Konar, *Fixed point results in partially ordered complex valued metric spaces for rational type expressions*, Bangmod Int. J. Math. & Comp. Sci. **1 (1)**, 55-62, 2015.
[10] B. S. Choudhury, N. Metiya and P. Konar, *Fixed point results for rational type contruction in partially ordered complex valued metric spaces*, Bulletin Internat. Math. Virtual Institute **5**, 73-80, 2015.
[11] J. Choi, S. K. Datta, T. Biswas and Md. N. Islam, *Some common fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces*, Honam Mathematical J. **39 (1)**, 115-126, 2017.
[12] G. Scorza Dragoni, *Sulle funzioni olomorfe di una variabile bicomplessa*, Reale Accad. d'Italia, Mem. Classe Sci. Nat. Fis. Mat. **5**, 597-665, 1934.
[13] A. K. Dubey, R. Shukla and P. P. Dubey, *Some common fixed point theorems for contractive mappings in complex-valued b-metric spaces*, Asian J. Math. App. **1**, Article ID ama0266, 2015.
[14] S. K. Datta, D. Pal, N. Biswas and S. Sarkar, *On the study of fixed point theorems in bicomplex valued metric spaces*, Journal of the Calcutta Mathematical Society **16 (1)**, 73-94, 2020.
[15] S. K. Datta, D. Pal, R. Sarkar and J. Saha, *Some common fixed point theorems for contracting mappings in bicomplex valued b-metric spaces*, Bull. Cal. Math. Soc. **112 (4)**, 329-354, 2020.
[16] S. K. Datta, D. Pal, R. Sarkar and J. Saha, *Common fixed point theorems in bicomplex valued metric spaces b-metric spaces or rational contraction*, Jnbha **50 (2)**, 93-105, 2020.
[17] Jaishree, *On Conjugates and modulii of bicomplex numbers*, Internat. J. Engrg. Sci. Tech. **4 (6)**, 2567-2575, 2012.
[18] I. H. Jebril, S. K. Datta, R. Sarkar and N. Biswas, *Common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces*, Journal of Interdisciplinary Mathematics **22 (7)**, 1071-1082, 2019.
[19] A. Kumar, P. Kumar and P. Dixit, *Maximum and minimum modulus principle for bicomplex holomorphic functions*, Internat. J. Engrg. Tech. **3 (2)**, 1484-1491, 2011.
[20] A. A. Mukheimer, *Some common fixed point theorems in complex-valued b-metric spaces*, Sci. World J. 2014, Article ID 587825, 2014.
[21] T. Mitra, *A common fixed point theorem in complex valued b-metric spaces*, Internat. J. Adv. Sci. Tech. Res. **4 (5)**, 504-548, 2015.
[22] G. B. Price, *An introduction to multicomplex spaces and functions*, Marcel Dekker, New York, 1991.
[23] D. Rochon and M. Shapiro, *On algebraic properties of bicomplex and hyperbolic numbers*, An. Univ. Oradea Fasc. Mat. **11**, 71-110, 2004.
[24] F. Rouzkard and M. Imdad, *Some common fixed point theorems on complex valued metric spaces*, Computers and Math. App. **64**, 1866-1874, 2012.
[25] K. P. R. Rao, P. Ramga Swamy and J. Rajendra Prasad, *A common fixed point theorem in complex-valued b-metric spaces*, Bull. Math. & Stat. Res. **1 (1)**, 1-8, 2013.
[26] C. Segre, *Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici*, Math. Ann. **40**, 413-467, 1892.
[27] D. Singh, O. P. Chauhan, N. Singh and V. Joshi, *Common fixed point theorems in complex-valued b-metric spaces*, J. Math. Comput. Sci. **5 (3)**, 412-429, 2015.
[28] R. Tiwari and D. P. Shukla, *Six maps with a common fixed point in complex valued metric spaces*, Res. J. Pure Algebra **2 (12)**, 365-369, 2012.
[29] R. K. Verma and H. K. Pathak, *Common fixed point theorems for a pair of mappings in complex-valued metric spaces*, J. Math. Sci. & Comp. Sci. **6**, 18-26, 2013.