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# On $p$-Valent Strongly Starlike and Strongly Convex Functions Associated with Generalized Linear Operator

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**Abstract**

In this paper, we define some new subclasses $\mathcal{S}^*(c,p,\lambda,m,\delta,\alpha,\beta)$ and $\mathcal{K}(c,p,\lambda,m,\delta,\alpha,\beta)$ of strongly starlike and strongly convex functions of order $\alpha$ and type $\beta$ by using the generalized linear operator $\mathcal{L}^{m,\delta}_{c,p,\lambda}$.We also derive some interesting properties, such as inclusion relationships of these classes and the integral operator $J_{\mu,p}$.

*Keywords:* Analytic functions, $p$-valent, Strongly starlike, Strongly convex.

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## 1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad (p \in \mathbb{N} = \{1,2,...\}), \tag{1.1}$$

which are analytic and $p-$valent in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f(z) \in \mathcal{A}_p$ is said to be in the class $S^*(p,\beta)$ of $p$-valently starlike function of order $\beta$, $0 \le \beta < p$, if $f(z) \neq 0$ and

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (z \in \Delta). \tag{1.2}$$

The class $S^*(p,\beta)$ was introduced and studied by Patel and Thakare [24] (see also [5, 14, 13]). Also, we note that $S^*(p,0) = S^*_p$, where $S^*_p$ is the class of $p$-valently starlike functions (see [11]).

A function $f(z) \in \mathcal{A}_p$ is said to be in the class $C(p,\beta)$ of $p$-valently convex functions of order $\beta$, $0 \le \beta < p$, if $f'(z) \neq 0$ and

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \quad (z \in \Delta). \tag{1.3}$$

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The class $C(p,\beta)$ was introduced and studied by Owa [23] ( see also [13, 30]). Also, we note that $C(p,0) = C_p$, where $C_p$ is the class of $p$-valently convex functions (see [11]).

A function $f(z) \in \mathcal{A}_p$ is said to be in the class $STS^*(p,\alpha,\beta)$ of strongly starlike functions of order $\alpha$ and type $\beta$, if it satisfies the following inequality (see [18]):

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \beta\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le p,\ 0 \le \beta < p,\ z \in \Delta). \tag{1.4}$$

Also, a function $f(z) \in \mathcal{A}_p$ is said to be in the class $STC(p,\alpha,\beta)$ of strongly convex function of order $\alpha$ and type $\beta$, if it satisfies the following inequality (see [18]):

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \beta\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le p,\ 0 \le \beta < p,\ z \in \Delta). \tag{1.5}$$

It is obvious that:

$$f(z) \in STC(p,\alpha,\beta) \Leftrightarrow \frac{zf'(z)}{p} \in STS^*(p,\alpha,\beta) \ \ (0 < \alpha \le p,\ 0 \le \beta < p,\ z \in \Delta). \tag{1.6}$$

*Remark* 1.1. We note that:
(i) $STS^*(p,\alpha,0) = STS^*(p,\alpha)$ (see Nunokawa et al.[20]);
(ii) $STS^*(1,\alpha,\beta) = S_s^*(\alpha,\beta)$ and $STC(1,\alpha,\beta) = C_s(\alpha,\beta)$ (see Nunokawa et al.[21] and Prajapat and Goyal [25]);
(iii) $STS^*(1,\alpha,0) = S_s^*(\alpha)$ and $STC(1,\alpha,0) = C_s(\alpha)$ (see Nunokawa [19] and Obradovic and Joshi [22]);
(iv) $STS^*(p,1,\beta) = S^*(p,\beta)$ (see Patel and Thakare [24]);
(v) $STC(p,1,\beta) = C(p,\beta)$ (see Owa [23]).

El-Ashwah et al. [10] defined the linear operator as follows:

$$\begin{aligned}\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) &= z^p + \sum_{k=1}^{\infty}\left(1 + \frac{\lambda k}{p}\right)^m \left(\frac{c+p}{c+k+p}\right)^{\delta} a_{k+p} z^{k+p} \\ &\quad (\delta \ge 0,\ c > -p,\ \lambda \ge 0,\ m \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\}).\end{aligned} \tag{1.7}$$

It is easily verified from (1.7) that (see [10])

$$z\left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)\right)' = (c+p)\,\mathcal{L}_{c,p,\lambda}^{m,\delta-1} f(z) - c\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \tag{1.8}$$

and

$$\lambda z\left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)\right)' = p\mathcal{L}_{c,p,\lambda}^{m+1,\delta} f(z) - p(1-\lambda)\,\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z). \tag{1.9}$$

*Remark* 1.2. By specializing the parameters $c,\ \lambda,\ \delta,\ p$ and $m$, we obtain the following operators:
(i) $\mathcal{L}_{c,p,0}^{1,\delta} f(z) = K_{c,p}^{\delta} f(z)$ (see [12]);
(ii) $\mathcal{L}_{a-1,1,\lambda}^{0,\delta} f(z) = L_a^{\delta} f(z)$ (see [2] and [3]);
(iii) $\mathcal{L}_{1,p,0}^{1,\delta} f(z) = \mathcal{I}_p^{\delta} f(z)$ (see [28] and [9]);
(iv) $\mathcal{L}_{c,p,\lambda}^{m,0} f(z) = D_{\lambda,p}^{m} f(z)$ (see [4]);
(v) $\mathcal{L}_{c,1,0}^{1,\delta} f(z) = \mathcal{P}_c^{\delta} f(z)$ (see [15] and [26]);
(vi) $\mathcal{L}_{c,1,0}^{1,1} f(z) = \mathcal{L}_c f(z)$ (see [6]);
(vii) $\mathcal{L}_{1,1,0}^{1,\delta} f(z) = \mathcal{I}^{\delta} f(z)$ (see [8]);
(viii) $\mathcal{L}_{c,1,\lambda}^{m,0} f(z) = \mathcal{D}_{\lambda}^{m} f(z)$ (see [1]);
(ix) $\mathcal{L}_{c,1,1}^{m,0} f(z) = \mathcal{D}^{m} f(z)$ (see [27]).

By using the linear operator $\mathcal{L}_{c,p,\lambda}^{m,\delta}$, we now define new subclasses of $\mathcal{A}_p$ as follows:

$$\mathcal{S}^*(c,p,\lambda,m,\delta,\alpha,\beta) = \left\{\begin{array}{l} f(z) \in \mathcal{A}_p :\ \mathcal{L}_{c,p,\lambda}^{m,\delta} f(z) \in STS^*(p,\alpha,\beta),\ \frac{z\left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)\right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} \ne \beta \\ (\delta \ge 0,\ c > -p,\ \lambda \ge 0,\ m \in \mathbb{Z},\ 0 < \alpha \le p,\ 0 \le \beta < p) \end{array}\right\} \tag{1.10}$$

and

$$\mathcal{K}(c,p,\lambda,m,\delta,\alpha,\beta)=\left\{\begin{array}{l} f(z)\in\mathcal{A}_p:\ \mathcal{L}^{m,\delta}_{c,p,\lambda}f(z)\in STC(p,\alpha,\beta),\ \frac{z\left(\mathcal{L}^{m,\delta}_{c,p,\lambda}f(z)\right)'}{\mathcal{L}^{m,\delta}_{c,p,\lambda}f(z)}\neq\beta \\ (\delta\geq 0,\ c>-p,\ \lambda\geq 0,\ m\in\mathbb{Z},\ 0<\alpha\leq p,\ 0\leq\beta<p) \end{array}\right\}. \tag{1.11}$$

It is obvious from the definitions (1.10) and (1.11) that:

$$f(z)\in\mathcal{K}(c,p,\lambda,m,\delta,\alpha,\beta)\Leftrightarrow\frac{zf'(z)}{p}\in\mathcal{S}^*(c,p,\lambda,m,\delta,\alpha,\beta). \tag{1.12}$$

## 2. Main Results

To establish our main results, we shall require the following lemma.

**Lemma 2.1.** *[19]. Let a function $h(z)=1+c_1z+c_2z^2+...$ be analytic in $\Delta$ and $h(z)\neq 0$, $z\in\Delta$. If there exists a point $z_0\in\Delta$ such that*

$$\left|\arg h(z)\right|<\frac{\pi}{2}\alpha\quad(|z|<|z_0|)\quad\text{and}\quad\left|\arg h(z_0)\right|=\frac{\pi}{2}\alpha\quad(0\leq\alpha<1), \tag{2.1}$$

*we have*

$$\frac{z_0h'(z_0)}{h(z_0)}=ik\alpha, \tag{2.2}$$

*where*

$$k\geq\frac{1}{2}\left(a+\frac{1}{a}\right)\quad\text{when}\quad\arg h(z_0)=\frac{\pi}{2}\alpha \tag{2.3}$$

$$k\leq-\frac{1}{2}\left(a+\frac{1}{a}\right)\quad\text{when}\quad\arg h(z_0)=-\frac{\pi}{2}\alpha \tag{2.4}$$

*and*

$$(h(z_0))^{\frac{1}{\alpha}}=\pm ia\quad(a>0). \tag{2.5}$$

**Theorem 2.2.** *Let $f(z)\in\mathcal{A}_p$,then we have*

$$\mathcal{S}^*(c,p,\lambda,m+1,\delta,\alpha,\beta)\subset\mathcal{S}^*(c,p,\lambda,m,\delta,\alpha,\beta)\subset\mathcal{S}^*(c,p,\lambda,m,\delta+1,\alpha,\beta). \tag{2.6}$$

*Proof.* We first prove that

$$\mathcal{S}^*(c,p,\lambda,m+1,\delta,\alpha,\beta)\subset\mathcal{S}^*(c,p,\lambda,m,\delta,\alpha,\beta). \tag{2.7}$$

Let $f\in\mathcal{S}^*(c,p,\lambda,m+1,\delta,\alpha,\beta)$ and suppose that

$$h(z)=\frac{1}{p-\beta}\left(\frac{z\left(\mathcal{L}^{m,\delta}_{c,p,\lambda}f(z)\right)'}{\mathcal{L}^{m,\delta}_{c,p,\lambda}f(z)}-\beta\right)\quad(z\in\Delta), \tag{2.8}$$

where $h$ is analytic *in* $\Delta$ *with* $h(0)=1$. Combining (1.9) and (2.8), we find that

$$\frac{p}{\lambda}\frac{\mathcal{L}^{m+1,\delta}_{c,p,\lambda}f(z)}{\mathcal{L}^{m,\delta}_{c,p,\lambda}f(z)}=(p-\beta)h(z)+\frac{p(1-\lambda)}{\lambda}+\beta. \tag{2.9}$$

Differentiating (2.9) logarithmically with respect to $z$, we have

$$\frac{z\left(\mathcal{L}^{m+1,\delta}_{c,p,\lambda}f(z)\right)'}{\mathcal{L}^{m+1,\delta}_{c,p,\lambda}f(z)}-\beta=(p-\beta)h(z)+\frac{(p-\beta)zh'(z)}{(p-\beta)h(z)+\frac{p(1-\lambda)}{\lambda}+\beta}. \tag{2.10}$$

Suppose now that there exists a point $z_0 \in \Delta$ such that the conditions (2.1) to (2.4) of Lemma 2.1 are satisfied. Thus, if $\arg(h(z_0)) = \frac{\pi}{2}\alpha$ for $z_0 \in \Delta$, then

$$\begin{aligned}\frac{z_0\left(\mathcal{L}_{c,p,\lambda}^{m+1,\delta} f(z_0)\right)'}{\mathcal{L}_{c,p,\lambda}^{m+1,\delta} f(z_0)} - \beta &= (p-\beta)\,h(z_0)\left[1 + \frac{\frac{z_0 h'(z_0)}{h(z_0)}}{(p-\beta)\,h(z_0) + \frac{p(1-\lambda)}{\lambda} + \beta}\right] \\ &= (p-\beta)\,a^{\alpha} e^{-\frac{i\pi\alpha}{2}}\left[1 + \frac{ik\alpha}{(p-\beta)\,a^{\alpha} e^{-\frac{i\pi\alpha}{2}} + \frac{p(1-\lambda)}{\lambda} + \beta}\right].\end{aligned}$$

This implies that

$$\begin{aligned}\arg\left(\frac{z_0\left(\mathcal{L}_{c,p,\lambda}^{m+1,\delta} f(z_0)\right)'}{\mathcal{L}_{c,p,\lambda}^{m+1,\delta} f(z_0)} - \beta\right) &= -\frac{\pi\alpha}{2} + \arg\left(1 + \frac{ik\alpha}{(p-\beta)\,a^{\alpha} e^{-\frac{i\pi\alpha}{2}} + \frac{p(1-\lambda)}{\lambda} + \beta}\right) \\ &= -\frac{\pi\alpha}{2} + \tan^{-1}\left(\frac{k\alpha\left[\left(\frac{p(1-\lambda)}{\lambda}+\beta\right)+(p-\beta)a^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)\right]}{\left(\frac{p(1-\lambda)}{\lambda}+\beta\right)^2+2\left(\frac{p(1-\lambda)}{\lambda}+\beta\right)(p-\beta)a^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)+(p-\beta)^2a^{2\alpha}-k\alpha(p-\beta)a^{\alpha}\sin\left(\frac{\pi\alpha}{2}\right)}\right).\end{aligned}$$

This gives that

$$\arg\left(\frac{z_0\left(\mathcal{L}_{c,p,\lambda}^{m+1,\delta} f(z_0)\right)'}{\mathcal{L}_{c,p,\lambda}^{m+1,\delta} f(z_0)} - \beta\right) \leq -\frac{\pi\alpha}{2},$$

since

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \text{ and } z_0 \in \Delta,$$

this contradicts the condition $f(z) \in \mathcal{S}^*(c, p, \lambda, m+1, \delta, \alpha, \beta)$.
On the other hand if we set $\arg(h(z_0)) = \frac{\pi}{2}\alpha$ for $z_0 \in \Delta$, then it can similarly be shown that

$$\arg\left(\frac{z_0\left(\mathcal{L}_{c,p,\lambda}^{m+1,\delta} f(z_0)\right)'}{\mathcal{L}_{c,p,\lambda}^{m+1,\delta} f(z_0)} - \beta\right) \geq \frac{\pi\alpha}{2},$$

since

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \text{ and } z_0 \in \Delta,$$

which again contradicts the hypothesis that $f(z) \in \mathcal{S}^*(c, p, \lambda, m+1, \delta, \alpha, \beta)$.
Thus, the function defined by (2.8) has to satisfy the following inequality

$$\arg(h(z)) \leq \frac{\pi}{2}\alpha \ (z \in \Delta),$$

which implies that

$$\left|\arg\left(\frac{z\left(\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)\right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta} f(z)} - \beta\right)\right| \leq \frac{\pi}{2}\alpha \quad (z \in \Delta).$$

For the second inclusion relationship asserted by Theorem 2.2, using arguments similar to those detailed above with (1.8), we obtain

$$\mathcal{S}^*(c, p, \lambda, m, \delta, \alpha, \beta) \subset \mathcal{S}^*(c, p, \lambda, m, \delta+1, \alpha, \beta). \tag{2.11}$$

We thus complete the proof of Theorem 2.2. □

**Theorem 2.3.** *Let $f(z) \in \mathcal{A}_p$, then we have*

$$\mathcal{K}(c,p,\lambda,m+1,\delta,\alpha,\beta) \subset \mathcal{K}(c,p,\lambda,m,\delta,\alpha,\beta) \subset \mathcal{K}(c,p,\lambda,m,\delta+1,\alpha,\beta). \tag{2.12}$$

*Proof.*

$$\begin{aligned} f(z) &\in \mathcal{K}(c,p,\lambda,m+1,\delta,\alpha,\beta) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*(c,p,\lambda,m+1,\delta,\alpha,\beta) \\ &\Rightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*(c,p,\lambda,m,\delta,\alpha,\beta) \\ &\Leftrightarrow f(z) \in \mathcal{K}(c,p,\lambda,m,\delta,\alpha,\beta) \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} f(z) &\in \mathcal{K}(c,p,\lambda,m,\delta,\alpha,\beta) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*(c,p,\lambda,m,\delta,\alpha,\beta) \\ &\Rightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*(c,p,\lambda,m,\delta+1,\alpha,\beta) \\ &\Leftrightarrow f(z) \in \mathcal{K}(c,p,\lambda,m,\delta+1,\alpha,\beta). \end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14), we deduce that the assertion of Theorem 2.3 holds.
For a function $f(z) \in \mathcal{A}_p$ the integral operator $J_{\mu,p}$ defined by (see Choi et al. [7]):

$$\begin{aligned} J_{\mu,p}(f)(z) &= \frac{\mu+p}{z^\mu} \int_0^z t^{\mu-1} f(t)dt \\ &= \left( z^p + \sum_{k=1}^{\infty} \frac{\mu+p}{\mu+p+k} z^{k+p} \right) * f(z) \ (\mu > -p, \ p \in \mathbb{N}), \end{aligned} \tag{2.15}$$

which satisfies the following relationship:

$$z\left(\mathcal{L}^{m,\delta}_{c,p,\lambda} J_{\mu,p}(f)(z)\right)' = (\mu+p)\,\mathcal{L}^{m,\delta}_{c,p,\lambda} f(z) - \mu \mathcal{L}^{m,\delta}_{c,p,\lambda} J_{\mu,p}(f)(z). \tag{2.16}$$

□

**Theorem 2.4.** *Let $f(z) \in \mathcal{A}_p$. If $f(z) \in \mathcal{S}^*(c,p,\lambda,m,\delta,\alpha,\beta)$ with*

$$\frac{z\left(\mathcal{L}^{m,\delta}_{c,p,\lambda} J_{\mu,p}(f)(z)\right)'}{\mathcal{L}^{m,\delta}_{c,p,\lambda} J_{\mu,p}(f)(z)} \neq \beta \quad (z \in \Delta),$$

*we have*

$$J_{\mu,p}(f)(z) \in \mathcal{S}^*(c,p,\lambda,m,\delta,\alpha,\beta).$$

*Proof.* We begin by setting

$$\frac{z\left(\mathcal{L}^{m,\delta}_{c,p,\lambda} J_{\mu,p}(f)(z)\right)'}{\mathcal{L}^{m,\delta}_{c,p,\lambda} J_{\mu,p}(f)(z)} = \beta + (p-\beta)\,h(z) \quad (z \in \Delta), \tag{2.17}$$

where $h$ is analytic *in* $\Delta$ *with* $h(0) = 1$. Combining (2.16) and (2.17), we find that

$$\frac{(\mu+p)\,\mathcal{L}^{m,\delta}_{c,p,\lambda} f(z)}{\mathcal{L}^{m,\delta}_{c,p,\lambda} J_{\mu,p}(f)(z)} = (\beta+\mu) + (p-\beta)\,h(z). \tag{2.18}$$

Differentiating (2.18) logarithmically with respect to $z$, we have

$$\frac{z\left(\mathcal{L}_{c,p,\lambda}^{m,\delta}f(z)\right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta}f(z)}-\beta=(p-\beta)\,h(z)+\frac{(p-\beta)\,zh'(z)}{(p-\beta)\,h(z)+(\beta+\mu)}. \tag{2.19}$$

Suppose now that there exists a point $z_0\in\Delta$ such that the conditions (2.1) to (2.4) of Lemma 2.1 are satisfied. Thus, if $\arg(h(z_0))=\frac{\pi}{2}\alpha$ for $z_0\in\Delta$, then

$$\begin{aligned}\frac{z_0\left(\mathcal{L}_{c,p,\lambda}^{m,\delta}f(z_0)\right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta}f(z_0)}-\beta &= (p-\beta)\,h(z_0)\left[1+\frac{\frac{z_0h'(z_0)}{h(z_0)}}{(p-\beta)\,h(z_0)+(\beta+\mu)}\right]\\ &= (p-\beta)\,a^{\alpha}e^{-\frac{i\pi\alpha}{2}}\left[1+\frac{ik\alpha}{(p-\beta)\,a^{\alpha}e^{-\frac{i\pi\alpha}{2}}+(\beta+\mu)}\right].\end{aligned}$$

This implies that

$$\begin{aligned}\arg\left(\frac{z_0\left(\mathcal{L}_{c,p,\lambda}^{m,\delta}f(z_0)\right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta}f(z_0)}-\beta\right) &= -\frac{\pi\alpha}{2}+\arg\left(1+\frac{ik\alpha}{(p-\beta)\,a^{\alpha}e^{-\frac{i\pi\alpha}{2}}+(\beta+\mu)}\right)\\ &= -\frac{\pi\alpha}{2}+\tan^{-1}\left(\frac{k\alpha\left[(\beta+\mu)+(p-\beta)a^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)\right]}{(\beta+\mu)^2+2(\beta+\mu)(p-\beta)a^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)+(p-\beta)^2a^{2\alpha}-k\alpha(p-\beta)a^{\alpha}\sin\left(\frac{\pi\alpha}{2}\right)}\right).\end{aligned}$$

This gives that

$$\arg\left(\frac{z_0\left(\mathcal{L}_{c,p,\lambda}^{m+1,\delta}f(z_0)\right)'}{\mathcal{L}_{c,p,\lambda}^{m+1,\delta}f(z_0)}-\beta\right)\leq-\frac{\pi\alpha}{2},$$

since

$$k\leq-\frac{1}{2}\left(a+\frac{1}{a}\right)\leq-1\ \text{ and }\ z_0\in\Delta,$$

this contradicts the condition $f(z)\in\mathcal{S}^*(c,p,\lambda,m,\delta,\alpha,\beta)$.

On the other hand if we set $\arg(h(z_0))=\frac{\pi}{2}\alpha$ for $z_0\in\Delta$, then it can similarly be shown that

$$\arg\left(\frac{z_0\left(\mathcal{L}_{c,p,\lambda}^{m,\delta}f(z_0)\right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta}f(z_0)}-\beta\right)\geq\frac{\pi\alpha}{2},$$

since

$$k\geq\frac{1}{2}\left(a+\frac{1}{a}\right)\geq1\ \text{ and }\ z_0\in\Delta,$$

which again contradicts the hypothesis that $f(z)\in\mathcal{S}^*(c,p,\lambda,m,\delta,\alpha,\beta)$.

Thus, the function defined by (2.17) has to satisfy the following inequality:.

$$\arg(h(z))\leq\frac{\pi}{2}\alpha\ \ (z\in\Delta).$$

This shows that

$$\left|\arg\left(\frac{z\left(\mathcal{L}_{c,p,\lambda}^{m,\delta}J_{\mu,p}(f)(z)\right)'}{\mathcal{L}_{c,p,\lambda}^{m,\delta}J_{\mu,p}(f)(z)}-\beta\right)\right|\leq\frac{\pi}{2}\alpha\quad(z\in\Delta).$$

This completes the proof of Theorem 2.4. □

**Theorem 2.5.** *Let $f(z) \in \mathcal{A}_p$. If $f(z) \in \mathcal{K}(c,p,\lambda,m,\delta,\alpha,\beta)$ with*

$$1+\frac{z\left(\mathcal{L}^{m,\delta}_{c,p,\lambda}J_{\mu,p}(f)(z)\right)''}{\left(\mathcal{L}^{m,\delta}_{c,p,\lambda}J_{\mu,p}(f)(z)\right)'} \neq \beta \quad (z \in \Delta),$$

*we get*

$$J_{\mu,p}(f)(z) \in \mathcal{K}(c,p,\lambda,m,\delta,\alpha,\beta).$$

*Proof.* In view of (1.12) and Theorem 2.4, we find that

$$\begin{aligned} f(z) &\in \mathcal{K}(c,p,\lambda,m,\delta,\alpha,\beta) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*(c,p,\lambda,m,\delta,\alpha,\beta) \\ &\Rightarrow \frac{z(J_{\mu,p}(f)(z))'}{p} \in \mathcal{S}^*(c,p,\lambda,m,\delta,\alpha,\beta) \\ &\Leftrightarrow J_{\mu,p}(f)(z) \in \mathcal{K}(c,p,\lambda,m,\delta,\alpha,\beta), \end{aligned} \tag{2.20}$$

we deduce that the assertion of Theorem 2.5 holds.

□

*Remark* 2.6. By specializing the parameters $c$, $\lambda$, $\delta$, $p$ and $m$, we obtain various results for different operators defined in the introduction.

## 3. Conclusions

In his recently-published review-cum-expository review article, in addition to applying the $q$-analysis to Geometric Function Theory of Complex Analysis, Srivastava [29] pointed out the fact that the results for the $q$-analogues can easily (and possibly trivially) be translated into the corresponding results for the $(p,q)$-analogues (with $0 < q < p \leq 1$) by applying some obvious parametric and argument variations, the additional parameter $p$ being redundant. Of course, this exposition and observation of Srivastava [29, p. 340] (see also [31, 32]) would apply also to the results which we have considered in our present investigation for $0 < q < 1$, we will make use of the concept of $q$-derivative operator with a view to introducing a new classes of strongly starlike functions of order $\alpha$ and type $\beta$ and strongly convex functions of order $\alpha$ and type $\beta$.

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