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Generalizations and Applications of Srinivasa Ramanujan's Integral $\mathbf{R}_S(m,n)$ via Hypergeometric Approach and Integral Transforms

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Abstract

In this paper, we obtain analytical solution of an unsolved integral $\mathbf{R}_S(m,n)$ of Srinivasa Ramanujan, using hypergeometric approach, Mellin transforms, Infinite Fourier sine transforms, Infinite series decomposition identity and some algebraic properties of Pochhammer's symbol. Also we have given some generalizations of the Ramanujan's integral $\mathbf{R}_S(m,n)$ in the form of integrals $\mathbf{I}^*_S(\upsilon,b,c,\lambda,y)$, $\mathbf{J}_S(\upsilon,b,c,\lambda,y)$, $\mathbf{K}_S(\upsilon,b,c,\lambda,y)$, $\mathbf{I}_S(\upsilon,b,\lambda,y)$ and solved them in terms of ordinary hypergeometric functions ${}_2F_3$, with suitable convergence conditions. Moreover as applications of Ramanujan's integral $\mathbf{R}_S(m,n)$, the new three infinite summation formulas associated with hypergeometric functions ${}_1F_2$ and ${}_0F_1$ (or cosine, sine and Lommel functions) are obtained.

Keywords: Generalized hypergeometric function, Infinite Fourier sine transforms, Ramanujan's integrals, Fox-Wright psi hypergeometric function, Mellin transforms, Series decomposition identity, Bounded sequence

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1. Introduction and Preliminaries

In the literature of infinite Fourier sine transforms [10, 12, 20, 22, 23, 25, 31, 34, 36, 54, 55, 60, 61], the analytical solutions of $\int_0^\infty \frac{x^{\upsilon-1}\sin(xy)}{\{exp(bx)\pm 1\}}dx$, are available and expressed in terms of Riemann's zeta function, the Psi function (Digamma function), hyperbolic function and Beta function.
The analytical solution of the following integral of Ramanujan [48, p. 85, eq.(49) last line]:

$$\mathbf{R}_S(m,n) = \int_0^\infty x^m \frac{\sin(\pi n x)}{\{-1+\exp(2\pi\sqrt{x})\}}dx, \tag{1.1}$$

is not given for all positive rational values of n, and non-negative integral values of m.
A natural generalization of Gauss hypergeometric series ${}_2F_1$ is the general hypergeometric series ${}_pF_q$ [57, p.42, eq.(1)]

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and see also [11, 24] with p numerator parameters $\alpha_1, ..., \alpha_p$ and q denominator parameters $\beta_1, ..., \beta_q$. It is defined by

$$ {}_pF_q\left(\begin{array}{l} \alpha_1, ..., \alpha_p \ ; \\ \beta_1, ..., \beta_q \ ; \end{array} \ z \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_p)_n}{(\beta_1)_n ... (\beta_q)_n} \frac{z^n}{n!} \ , \tag{1.2} $$

where $\alpha_i \in \mathbb{C}$ $(i = 1, ..., p)$ and $\beta_j \in \mathbb{C} \backslash \mathbb{Z}_0^-$ $(j = 1, ..., q)$ $\left(\mathbb{Z}_0^- := \{0, -1, -2, ...\} \right)$ and $(\ p, \ q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, ...\})$. Also $(\lambda)_\upsilon$ $(\lambda, \upsilon \in \mathbb{C})$ denotes the Pochhammer's symbol (or the shifted factorial, since $(1)_n = n!$) is defined, in general, by

$$ (\lambda)_\upsilon := \frac{\Gamma(\lambda + \upsilon)}{\Gamma(\lambda)} = \begin{cases} 1, & (\upsilon = 0 \ ; \ \lambda \in \mathbb{C} \backslash \{0\}) \\ \lambda(\lambda + 1)...(\lambda + n - 1), & (\upsilon = n \in \mathbb{N} \ ; \ \lambda \in \mathbb{C}). \end{cases} \tag{1.3} $$

The ${}_pF_q$ series in the eq.(1.2) is convergent for $|z| < \infty$ if $p \leq q$, and for $|z| < 1$ if $p = q + 1$.
Furthermore, if we set

$$ \omega = \left(\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i \right), \tag{1.4} $$

it is known that the ${}_pF_q$ series, with $p = q + 1$, is
(i) absolutely convergent for $|z| = 1$ if $\Re(\omega) > 0$,
(ii) conditionally convergent for $|z| = 1, z \neq 1$, if $-1 < \Re(\omega) \leq 0$.
Also binomial function is given by

$$ (1 - z)^{-\alpha} = {}_1F_0 \, (\alpha \ ; \ - \ ; \ z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n \ , \tag{1.5} $$

where $|z| < 1; \ \alpha \neq 0, -1, -2, -3, ...$
The Fox-Wright psi function of one variable [27, 28, 62, 63] is given by

$$ {}_p\Psi_q\left[\begin{array}{l} (\alpha_1, A_1), ..., (\alpha_p, A_p); \\ (\beta_1, B_1), ..., (\beta_q, B_q); \end{array} \ z \right] = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1 + kA_1)...\Gamma(\alpha_p + kA_p)}{\Gamma(\beta_1 + kB_1)...\Gamma(\beta_q + kB_q)} \frac{z^k}{k!} = \frac{\Gamma(\alpha_1)...\Gamma(\alpha_p)}{\Gamma(\beta_1)...\Gamma(\beta_q)} \sum_{k=0}^{\infty} \frac{(\alpha_1)_{kA_1}...(\alpha_p)_{kA_p}}{(\beta_1)_{kB_1}...(\beta_q)_{kB_q}} \frac{z^k}{k!}, \tag{1.6} $$

$$ = \frac{\Gamma(\alpha_1)...\Gamma(\alpha_p)}{\Gamma(\beta_1)...\Gamma(\beta_q)} {}_p\Psi_q^*\left[\begin{array}{l} (\alpha_1, A_1), ..., (\alpha_p, A_p); \\ (\beta_1, B_1), ..., (\beta_q, B_q); \end{array} \ z \right], \tag{1.7} $$

$$ = \frac{1}{2\pi\rho} \int_L \frac{\Gamma(\zeta) \prod_{i=1}^{p} \Gamma(\alpha_i - A_i\zeta)}{\prod_{j=1}^{q} \Gamma(\beta_j - B_j\zeta)} (-z)^{-\zeta} d\zeta \ , \tag{1.8} $$

where $\rho = \sqrt{-1}$, $z \in \mathbb{C}$; parameters $\alpha_i, \beta_j \in \mathbb{C}$; coefficients $A_i, B_j \in \mathbb{R} = (-\infty, +\infty)$ in case of series (1.6) (or $A_i, B_j \in \mathbb{R}_+ = (0, +\infty)$ in case of contour integral (1.8)), $A_i \neq 0$ $(i = 1, 2, ..., p)$, $B_j \neq 0$ $(j = 1, 2, ..., q)$. In eq. (1.6), the parameters α_i, β_j and coefficients A_i, B_j are adjusted in such a way that the product of Gamma functions in numerator and denominator should be well defined.
Suppose:

$$ \Delta^* = \left(\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i \right), \tag{1.9} $$

$$ \delta^* = \left(\prod_{i=1}^{p} |A_i|^{-A_i} \right) \left(\prod_{j=1}^{q} |B_j|^{B_j} \right), \tag{1.10} $$

$$\mu^* = \sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i + \left(\frac{p-q}{2}\right), \tag{1.11}$$

and

$$\sigma^* = (1 + A_1 + ... + A_p) - (B_1 + ... + B_q) = 1 - \Delta^*. \tag{1.12}$$

Then we have the following convergence conditions of (1.6) or (1.8).

Case(1): When contour (L) is a left loop beginning and ending at $-\infty$, then ${}_p\Psi_q[\cdot]$ given by (1.6)or (1.8) holds the following convergence conditions.

i) When $\Delta^* > -1$, $0 < |z| < \infty$, $z \neq 0$.

ii) When $\Delta^* = -1$, $0 < |z| < \delta^*$.

iii) When $\Delta^* = -1$,$|z| = \delta^*$, and $\Re(\mu^*) > \frac{1}{2}$.

Case(2): When contour (L) is a right loop beginning and ending at $+\infty$, then ${}_p\Psi_q[\cdot]$ given by (1.6)or (1.8) holds the following convergence conditions.

iv) When $\Delta^* < -1$, $0 < |z| < \infty$, $z \neq 0$.

v) When $\Delta^* = -1$, $|z| > \delta^*$.

vi) When $\Delta^* = -1$,$|z| = \delta^*$, and $\Re(\mu^*) > \frac{1}{2}$.

Case(3): When contour (L) is starting from $\gamma - i\infty$ and ending at $\gamma + i\infty$ where $\gamma \in \mathbb{R} = (-\infty, +\infty)$, then ${}_p\Psi_q[\cdot]$ is also convergent under the following conditions.

vii) When $\sigma^* > 0$, $|arg(-z)| < \frac{\pi}{2}\sigma^*$, $0 < |z| < \infty$, $z \neq 0$.

viii) When $\sigma^* = 0$, $arg(-z) = 0$, $0 < |z| < \infty$, $z \neq 0$ such that $-\gamma\Delta^* + \Re(\mu^*) > \frac{1}{2} + \gamma$.

ix) When $\gamma = 0$, $\sigma^* = 0, arg(-z) = 0$, $0 < |z| < \infty$, $z \neq 0$, such that $\Re(\mu^*) > \frac{1}{2}$.

The infinite Fourier sine transform of $g(x)$ over the interval $(0, \infty)$ is defined by

$$F_S\{g(x); y\} = \int_0^\infty g(x)\sin(xy)dx = G_S(y), \quad (y > 0), \tag{1.13}$$

then $g(x) = F_S^{-1}[G_S(y); x] = \dfrac{2}{\pi}\displaystyle\int_0^\infty G_S(y)\sin(xy)dy$.

Some authors have given the following definitions of infinite Fourier sine transforms and its inverse.

$$\mathcal{F}_S\{g(x); y\} = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\sin(xy)dx = \mathcal{G}_S(y), \quad (y > 0),$$

then $g(x) = \mathcal{F}_S^{-1}[\mathcal{G}_S(y); x] = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty \mathcal{G}_S(y)\sin(xy)dy$.

In this paper, we have applied the definition (1.13). If $b > 0$ and $-1 < \Re(s) < 1$, then the Mellin-transform of $\sin(bx)$ or infinite Fourier sine transform of x^{s-1}[25, p. 317, Entry 6.5(1), see also p.68, Entry 2.3(1)], [34, p.42, Entry(5.1)], [36, p.125,Entry(3.9)] and [25, p.72, 2.4(Entry 7)] is given by

$$\int_0^\infty x^{s-1}\sin(bx)dx = \frac{\Gamma(s)\sin(\frac{\pi s}{2})}{b^s}. \tag{1.14}$$

If $\Re(\mu) > -2$, $0 < \xi < 1$, $a > 0$ and $y > 0$, then we can prove the following integral by using Maclaurin's expansion of $\exp(-ax^\xi)$ and term by term integrating with the help of the result (1.14):

$$\int_0^\infty x^\mu exp(-ax^\xi)\sin(xy)dx = y^{-\mu-1}\sum_{\ell=0}^{\infty}\left(-\frac{a}{y^\xi}\right)^\ell \frac{1}{\ell!}\Gamma(\mu + 1 + \xi\ell)\ \cos\left\{\frac{\pi}{2}(\mu + \xi\ell)\right\}. \tag{1.15}$$

The condition $\Re(\mu) > -2$ stated in the integral (1.15) follows from the theory of analytic continuation [36, p.127, Entry(3.33)], [34, p.48, Entry(5.35)]. We have also verified the condition $\Re(\mu) > -2$, using Wolfram Mathematica. An infinite series decomposition identity [56, p.193,eq.(8)] and [21, 29, 30, 53] is given by

$$\sum_{\ell=0}^{\infty}\Omega(\ell) = \sum_{j=0}^{N-1}\left\{\sum_{\ell=0}^{\infty}\Omega(N\ell + j)\right\}, \tag{1.16}$$

where N is an arbitrary positive integer.
Put $N = 4$ in the above eq. (1.16), we get

$$\sum_{\ell=0}^{\infty}\Omega(\ell)=\sum_{j=0}^{3}\left\{\sum_{\ell=0}^{\infty}\Omega(4\ell+j)\right\}, \tag{1.17}$$

$$=\sum_{\ell=0}^{\infty}\Omega(4\ell)+\sum_{\ell=0}^{\infty}\Omega(4\ell+1)+\sum_{\ell=0}^{\infty}\Omega(4\ell+2)+\sum_{\ell=0}^{\infty}\Omega(4\ell+3), \tag{1.18}$$

provided that all involved infinite series are absolutely convergent.
For every positive integer m [57, p.22, eq.(26)], we have

$$(\lambda)_{mn}=m^{mn}\prod_{j=1}^{m}\left(\frac{\lambda+j-1}{m}\right)_n \qquad ; m\in\mathbb{N},\ n\in\mathbb{N}_0. \tag{1.19}$$

From the above result (1.19) with $\lambda = mz$, it can be proved that

$$\Gamma(mz)=(2\pi)^{\frac{(1-m)}{2}}m^{mz-\frac{1}{2}}\prod_{j=1}^{m}\Gamma\left(z+\frac{j-1}{m}\right), \tag{1.20}$$

where $z \neq 0, -\frac{1}{m}, -\frac{2}{m}, ...\ ; m \in \mathbb{N}$.

The equation (1.20) is known as Gauss-Legendre multiplication theorem for Gamma function.
Elementary trigonometric functions [57, p.44, eq.(9) and eq.(10)] are given by

$$\cos\ z={}_0F_1\left(\begin{array}{c}-;\\ \frac{1}{2};\end{array}\ \frac{-z^2}{4}\right), \tag{1.21}$$

$$\sin\ z=z\,{}_0F_1\left(\begin{array}{c}-;\\ \frac{3}{2};\end{array}\ \frac{-z^2}{4}\right). \tag{1.22}$$

The Lommel function [57, p.44, eq.(13)] is given by

$$s_{\mu,\upsilon}(z)=\frac{z^{\mu+1}}{(\mu-\upsilon+1)(\mu+\upsilon+1)}\,{}_1F_2\left(\begin{array}{c}1;\\ \frac{\mu-\upsilon+3}{2},\frac{\mu+\upsilon+3}{2};\end{array}\ \frac{-z^2}{4}\right), \tag{1.23}$$

where $\mu \pm \upsilon \in \mathbb{C} \setminus \{-1, -3, -5, -7, ...\}$.

In the available literature [2, 3, 4, 6, 7, 8, 9, 13, 14, 15, 16, 17, 19, 18, 38, 39, 40, 41, 42, 48, 49, 50, 51, 52, 26] on Ramanujan's Mathematics, the analytical solution of Ramanujan's integral $\mathbf{R}_S(m,n)$ is not given. Therefore, the main aim of this paper is to obtain the analytical solution of Ramanujan's integral in terms of ordinary hypergeometric functions. Also, our work on Ramanujan's Mathematics is motivated by the work given in references [32, 45, 46, 47, 43, 58]. Here in this paper, we generalize Ramanujan's integral $\mathbf{R}_S(m,n)$ in the following forms:

(i) $\mathbf{I}^*_S(\upsilon,b,c,\lambda,y)=\sum_{k=0}^{\infty}\Big[\frac{\Theta(k)}{k!}\int_0^{\infty}x^{\upsilon-1}e^{-(\lambda b+ck)\sqrt{x}}\sin(xy)dx\Big]$,

(ii) $\mathbf{J}_S(\upsilon,b,c,\lambda,y)=\int_0^{\infty}x^{\upsilon-1}e^{-b\lambda\sqrt{x}}\,{}_r\Psi_s\left[\begin{array}{c}(\alpha_1,A_1),...,(\alpha_r,A_r);\\ (\beta_1,B_1),...,(\beta_s,B_s);\end{array}\ e^{-c\sqrt{x}}\right]\sin(xy)dx$,

(iii) $\mathbf{K}_S(\upsilon,b,c,\lambda,y)=\int_0^{\infty}x^{\upsilon-1}e^{-b\lambda\sqrt{x}}\,{}_rF_s\Big(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s;\ e^{-c\sqrt{x}}\Big)\sin(xy)dx$,

(iv) $\mathbf{I}_S(\upsilon,b,\lambda,y)=\int_0^{\infty}x^{\upsilon-1}\{-1+\exp(b\sqrt{x})\}^{-\lambda}\sin(xy)dx$,

where $\{\Theta(k)\}_{k=0}^{\infty}$ is a bounded sequence and obtain the analytical solution. Moreover, we also show how the main general theorem given below, is applicable for obtaining new interesting results by suitable adjustment in parameters and variables (see in the sections 3,4,5). Also, our work is motivated by the work given in references [33, 59]

2. Main general theorem on infinite Fourier sine transform

Suppose $\{\Theta(k)\}_{k=0}^{\infty}$ is a bounded sequence of arbitrary real and complex numbers, and $\Re(\upsilon) > -1; c > 0, y > 0; \lambda > 0, b > 0 (or\ \lambda < 0, b < 0)$ then

$$\mathbf{I}_S^*(\upsilon, b, c, \lambda, y) = \sum_{k=0}^{\infty}\left[\frac{\Theta(k)}{k!}\int_0^{\infty} x^{\upsilon-1}e^{-(\lambda b+ck)\sqrt{x}}\sin(xy)dx\right], \tag{2.1}$$

$$= y^{-\upsilon}\sum_{k=0}^{\infty}\Big[\frac{\Theta(k)}{k!}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}(\lambda b+ck)^{\ell}\Gamma\left(\upsilon+\frac{\ell}{2}\right)}{y^{\frac{\ell}{2}}\ \ell!}\sin\left(\frac{\upsilon\pi}{2}+\frac{\ell\pi}{4}\right)\Big]. \tag{2.2}$$

Now replacing ℓ by $4\ell + j$, after simplification we get

$$\mathbf{I}_S^*(\upsilon, b, c, \lambda, y) = y^{-\upsilon}\sum_{k=0}^{\infty}\Big[\frac{\Theta(k)}{k!}\sum_{j=0}^{3}\frac{(-1)^j(\lambda b+ck)^j\,\Gamma\left(\upsilon+\frac{j}{2}\right)}{y^{\frac{j}{2}}\ j!}\sin\left(\frac{\upsilon\pi}{2}+\frac{j\pi}{4}\right){}_2F_3\left(\begin{array}{l}\Delta\left(2;\frac{2\upsilon+j}{2}\right);\\ \Delta^*(4;1+j);\end{array}\frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\Big], \tag{2.3}$$

$$= y^{-\upsilon}\sum_{k=0}^{\infty}\Big[\frac{\Theta(k)}{k!}\sum_{j=0}^{3}\frac{(-1)^j\Gamma\left(\upsilon+\frac{j}{2}\right)}{j!}\sin\left(\frac{\upsilon\pi}{2}+\frac{j\pi}{4}\right)\left(\frac{\lambda b}{\sqrt{y}}\right)^j\left\{\frac{(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^j{}_2F_3\left(\begin{array}{l}\Delta\left(2;\frac{2\upsilon+j}{2}\right);\\ \Delta^*(4;1+j);\end{array}\frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\Big], \tag{2.4}$$

$$= \frac{\Gamma(\upsilon)\sin\left(\frac{\upsilon\pi}{2}\right)}{y^{\upsilon}}\sum_{k=0}^{\infty}\Big[\frac{\Theta(k)}{k!}{}_2F_3\left(\begin{array}{l}\frac{\upsilon}{2},\frac{\upsilon+1}{2};\\ \frac{1}{4},\frac{1}{2},\frac{3}{4};\end{array}\frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\Big]-$$

$$-\frac{(\lambda b)\Gamma(\upsilon+\frac{1}{2})\sin\left(\frac{\upsilon\pi}{2}+\frac{\pi}{4}\right)}{y^{\upsilon+\frac{1}{2}}}\sum_{k=0}^{\infty}\Big[\frac{\Theta(k)}{k!}\left\{\frac{(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}{}_2F_3\left(\begin{array}{l}\frac{2\upsilon+1}{4},\frac{2\upsilon+3}{4};\\ \frac{1}{2},\frac{3}{4},\frac{5}{4}\ ;\end{array}\frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\Big]+$$

$$+\frac{(\lambda b)^2\Gamma(\upsilon+1)\cos\left(\frac{\upsilon\pi}{2}\right)}{2y^{\upsilon+1}}\sum_{k=0}^{\infty}\Big[\frac{\Theta(k)}{k!}\left\{\frac{(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^2{}_2F_3\left(\begin{array}{l}\frac{\upsilon+1}{2},\frac{\upsilon+2}{2};\\ \frac{3}{4},\frac{5}{4},\frac{3}{2}\ ;\end{array}\frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\Big]-$$

$$-\frac{(\lambda b)^3\Gamma(\upsilon+\frac{3}{2})\cos\left(\frac{\upsilon\pi}{2}+\frac{\pi}{4}\right)}{6y^{\upsilon+\frac{3}{2}}}\sum_{k=0}^{\infty}\Big[\frac{\Theta(k)}{k!}\left\{\frac{(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^3{}_2F_3\left(\begin{array}{l}\frac{2\upsilon+3}{4},\frac{2\upsilon+5}{4};\\ \frac{5}{4},\frac{3}{2},\frac{7}{4}\ ;\end{array}\frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\Big]. \tag{2.5}$$

Our result (2.3)or (2.4) or (2.5) is convergent in view of the convergence condition of ${}_pF_q(\cdot)$ series, when $p \leq q$, and $\forall\ |z| < \infty$.

Proof. The result (2.2) is obtained by the application of the integral (1.15) [with substitutions $\mu = \upsilon-1, a = \lambda b+ck, \xi = \frac{1}{2}$] in the R.H.S. of eq.(2.1). The results (2.3),(2.4) and (2.5) are obtained by using the infinite series decomposition formulas (1.17),(1.18), Pochhammer's identity (1.19) and other algebraic properties of Pochhammer's symbols. □

3. Infinite Fourier sine transforms of $x^{\upsilon-1}e^{-b\lambda\sqrt{x}}{}_r\Psi_s[\cdot]$ and $x^{\upsilon-1}e^{-b\lambda\sqrt{x}}{}_rF_s(\cdot)$

If we put $\Theta(k) = \dfrac{\Gamma(\alpha_1+kA_1)...\Gamma(\alpha_r+kA_r)}{\Gamma(\beta_1+kB_1)...\Gamma(\beta_s+kB_s)}$, $(k = 0, 1, 2, 3, ...)$ in the equations (2.1) and (2.3), then after simplification we get (3.1) and (3.2)

$$\mathbf{J}_S(\upsilon, b, c, \lambda, y) = \int_0^{\infty} x^{\upsilon-1}e^{-b\lambda\sqrt{x}}{}_r\Psi_s\left[\begin{array}{l}(\alpha_1, A_1), ..., (\alpha_r, A_r);\\ (\beta_1, B_1), ..., (\beta_s, B_s);\end{array} e^{-c\sqrt{x}}\right]\sin(xy)dx, \tag{3.1}$$

$$= y^{-\upsilon}\sum_{k=0}^{\infty}\Big[\frac{\Gamma(\alpha_1+kA_1)...\Gamma(\alpha_r+kA_r)}{\Gamma(\beta_1+kB_1)...\Gamma(\beta_s+kB_s)k!}\sum_{j=0}^{3}\frac{(-1)^j(\lambda b+ck)^j\,\Gamma\left(\upsilon+\frac{j}{2}\right)}{y^{\frac{j}{2}}\;j!}\times$$

$$\times\sin\left(\frac{\upsilon\pi}{2}+\frac{j\pi}{4}\right){}_2F_3\left(\begin{array}{l}\Delta\left(2;\frac{2\upsilon+j}{2}\right);\\ \Delta^*(4;1+j);\end{array}\;\frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\Big], \quad (3.2)$$

where $\Re(\upsilon) > -1;\ c > 0, y > 0; \lambda > 0, b > 0\ (or\ \lambda < 0, b < 0)$; parameters $\alpha_i, \beta_j \in \mathbb{C}$; coefficients $A_i, B_j \in \mathbb{R} = (-\infty, +\infty); A_i \neq 0\ (i = 1, 2, ..., r), B_j \neq 0\ (j = 1, 2, ..., s)$ and ${}_r\Psi_s[\cdot]$ is Fox-Wright psi function of one variable subject to suitable convergence conditions derived from convergence conditions discussed in **case(1)** or **case(2)** or **case(3)** of the function ${}_p\Psi_q[\cdot]$ given by (1.6),(1.7) and (1.8).

When N is positive integer then $\Delta(N;\lambda)$ denotes the array of N parameters given by $\frac{\lambda}{N}, \frac{\lambda+1}{N}, ..., \frac{\lambda+N-1}{N}$. When N and j are independent variables then the notation $\Delta(N; j+1)$ denotes the set of N parameters given by $\frac{j+1}{N}, \frac{j+2}{N}, ..., \frac{j+N}{N}$. When j is dependent variable that is $j = 0, 1, 2, 3, ..., N-1$, then the asterisk in $\Delta^*(N; j+1)$ represents the fact that the (denominator) parameters $\frac{N}{N}$ is always omitted (due to the need of factorial in denominator in the power series form of hypergeometric function) so that the set $\Delta^*(N; j+1)$ obviously contains only $(N-1)$ parameters [57, Chap.3, p.214].

Remark 3.1. When $A_1 = ... = A_r = B_1 = ... = B_s = 1$ in (3.1), (3.2) then we get

$$\begin{aligned}\mathbf{K}_S(\upsilon, b, c, \lambda, y) &= \int_0^{\infty} x^{\upsilon-1}e^{-b\lambda\sqrt{x}}{}_rF_s\left(\begin{array}{l}\alpha_1, ..., \alpha_r;\\ \beta_1, ..., \beta_s;\end{array}\; e^{-c\sqrt{x}}\right)\sin(xy)dx,\\ &= y^{-\upsilon}\sum_{k=0}^{\infty}\Big[\frac{(\alpha_1)_k...(\alpha_r)_k}{(\beta_1)_k...(\beta_s)_k\;k!}\sum_{j=0}^{3}\frac{(-1)^j(\lambda b+ck)^j\,\Gamma\left(\upsilon+\frac{j}{2}\right)}{y^{\frac{j}{2}}\;j!}\times\\ &\times\ \sin\left(\frac{\upsilon\pi}{2}+\frac{j\pi}{4}\right){}_2F_3\left(\begin{array}{l}\Delta\left(2;\frac{2\upsilon+j}{2}\right);\\ \Delta^*(4;1+j);\end{array}\;\frac{-1}{64y^2}\left\{\frac{(\lambda b)(\frac{\lambda b+c}{c})_k}{(\frac{\lambda b}{c})_k}\right\}^4\right)\Big],\end{aligned} \quad (3.3)$$

where $\Re(\upsilon) > -1;\ c > 0, y > 0; \lambda > 0, b > 0\ (or\ \lambda < 0, b < 0)$; parameters $\alpha_i, \beta_j \in \mathbb{C}\ (i = 1, 2, ..., r), (j = 1, 2, ..., s)$ and $r \leq s + 1$.

4. Infinite Fourier sine transform of $x^{\upsilon-1}\{-1 + \exp(b\sqrt{x})\}^{-\lambda}$

The following generalization $\mathbf{I}_S(\upsilon, b, \lambda, y)$ of the Ramanujan's integral $\mathbf{R}_S(m, n)$ in terms of ordinary hypergeometric functions ${}_2F_3$ holds true:

$$\mathbf{I}_S(\upsilon, b, \lambda, y) = \int_0^{\infty} x^{\upsilon-1}\frac{\sin(xy)}{\{-1+\exp(b\sqrt{x})\}^{\lambda}}dx, \quad (4.1)$$

$$= y^{-\upsilon}\sum_{k=0}^{\infty}\Big[\frac{(\lambda)_k}{k!}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}(\lambda b+bk)^{\ell}\Gamma\left(\upsilon+\frac{\ell}{2}\right)}{y^{\frac{\ell}{2}}\;\ell!}\sin\left(\frac{\upsilon\pi}{2}+\frac{\ell\pi}{4}\right)\Big], \quad (4.2)$$

$$= y^{-\upsilon}\sum_{k=0}^{\infty}\Big[\frac{(\lambda)_k}{k!}\sum_{j=0}^{3}\frac{(-1)^j(\lambda b+bk)^j\,\Gamma\left(\upsilon+\frac{j}{2}\right)}{y^{\frac{j}{2}}\;j!}\sin\left(\frac{\upsilon\pi}{2}+\frac{j\pi}{4}\right){}_2F_3\left(\begin{array}{c}\Delta\left(2;\frac{2\upsilon+j}{2}\right);\\ \Delta^*(4;1+j);\end{array}\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\lambda+1)_k}{(\lambda)_k}\right\}^4\right)\Big], \tag{4.3}$$

$$= \frac{\Gamma(\upsilon)\sin\left(\frac{\upsilon\pi}{2}\right)}{y^{\upsilon}}\sum_{k=0}^{\infty}\Big[\frac{(\lambda)_k}{k!}{}_2F_3\left(\begin{array}{c}\frac{\upsilon}{2},\frac{\upsilon+1}{2};\\ \frac{1}{4},\frac{1}{2},\frac{3}{4};\end{array}\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\lambda+1)_k}{(\lambda)_k}\right\}^4\right)\Big]-$$

$$-\frac{(\lambda b)\Gamma(\upsilon+\frac{1}{2})\sin\left(\frac{\upsilon\pi}{2}+\frac{\pi}{4}\right)}{y^{\upsilon+\frac{1}{2}}}\sum_{k=0}^{\infty}\Big[\frac{(\lambda+1)_k}{k!}{}_2F_3\left(\begin{array}{c}\frac{2\upsilon+1}{4},\frac{2\upsilon+3}{4};\\ \frac{1}{2},\frac{3}{4},\frac{5}{4}\end{array};\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\lambda+1)_k}{(\lambda)_k}\right\}^4\right)\Big]+$$

$$+\frac{(\lambda b)^2\Gamma(\upsilon+1)\cos\left(\frac{\upsilon\pi}{2}\right)}{2y^{\upsilon+1}}\sum_{k=0}^{\infty}\Big[\frac{\{(\lambda+1)_k\}^2}{(\lambda)_k\,k!}{}_2F_3\left(\begin{array}{c}\frac{\upsilon+1}{2},\frac{\upsilon+2}{2};\\ \frac{3}{4},\frac{5}{4},\frac{3}{2}\end{array};\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\lambda+1)_k}{(\lambda)_k}\right\}^4\right)\Big]-$$

$$-\frac{(\lambda b)^3\Gamma(\upsilon+\frac{3}{2})\cos\left(\frac{\upsilon\pi}{2}+\frac{\pi}{4}\right)}{6y^{\upsilon+\frac{3}{2}}}\sum_{k=0}^{\infty}\Big[\frac{\{(\lambda+1)_k\}^3}{k!\,\{(\lambda)_k\}^2}{}_2F_3\left(\begin{array}{c}\frac{2\upsilon+3}{4},\frac{2\upsilon+5}{4};\\ \frac{5}{4},\frac{3}{2},\frac{7}{4}\end{array};\ \frac{-1}{64y^2}\left\{\frac{(\lambda b)(\lambda+1)_k}{(\lambda)_k}\right\}^4\right)\Big], \tag{4.4}$$

where $\Re(\upsilon) > -1;\ y > 0; \lambda > 0, b > 0$.

Proof. In eq. (2.1), put $\Theta(k) = (\lambda)_k$ and $c = b$, we obtain

$$\mathbf{I}_S(\upsilon, b, \lambda, y) = \int_0^{\infty} x^{\upsilon-1}e^{-(\lambda b)\sqrt{x}}\left\{\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!}e^{-(bk)\sqrt{x}}\right\}\sin(xy)dx. \tag{4.5}$$

Using binomial expansion (1.5) in the above eq. (4.5), after simplification, we get the equation (4.1). The equations (4.2), (4.3) and (4.4) are obtained from (2.2), (2.3) and (2.5) by putting $\Theta(k) = (\lambda)_k$ and $c = b$. □

5. Ramanujan's integral $\mathbf{R}_S(m, n)$

The analytical solution of the integral $\mathbf{R}_S(m, n)$ is given by

$$\mathbf{R}_S(m,n) = \int_0^{\infty} x^m\frac{\sin(\pi nx)}{\{-1+\exp(2\pi\sqrt{x})\}}dx, \tag{5.1}$$

$$= (n\pi)^{-m-1}\sum_{k=0}^{\infty}\Big[\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left\{\frac{-(2\pi+2\pi k)}{\sqrt{n\pi}}\right\}^{\ell}\Gamma\left(m+1+\frac{\ell}{2}\right)\cos\left(\frac{m\pi}{2}+\frac{\ell\pi}{4}\right)\Big], \tag{5.2}$$

$$= (n\pi)^{-m-1}\sum_{k=0}^{\infty}\Big[\sum_{j=0}^{3}\frac{1}{j!}\left\{\frac{-(2\pi+2\pi k)}{\sqrt{n\pi}}\right\}^{j}\Gamma\left(m+1+\frac{j}{2}\right)\times$$

$$\times\ \cos\left(\frac{m\pi}{2}+\frac{j\pi}{4}\right){}_2F_3\left(\begin{array}{c}\Delta\left(2;\frac{2m+j+2}{2}\right);\\ \Delta^*(4;1+j);\end{array}\ \frac{-\pi^2}{4n^2}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\Big], \tag{5.3}$$

$$= \frac{m!\,\cos\left(\frac{m\pi}{2}\right)}{(n\pi)^{m+1}}\sum_{k=0}^{\infty}\Big[{}_2F_3\left(\begin{array}{c}\frac{m+1}{2},\frac{m+2}{2};\\ \frac{1}{4},\frac{1}{2},\frac{3}{4}\end{array};\ -\frac{\pi^2}{4n^2}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\Big]-$$

$$-\ \frac{\left(\frac{3}{2}\right)_m\cos\left(\frac{m\pi}{2}+\frac{\pi}{4}\right)}{(\pi)^m(n)^{m+\frac{3}{2}}}\sum_{k=0}^{\infty}\Big[\left\{\frac{(2)_k}{(1)_k}\right\}{}_2F_3\left(\begin{array}{c}\frac{2m+3}{4},\frac{2m+5}{4};\\ \frac{1}{2},\frac{3}{4},\frac{5}{4}\end{array};\ \frac{-\pi^2}{4n^2}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\Big]-$$

$$-\ \frac{(2)(m+1)!\sin\left(\frac{m\pi}{2}\right)}{(\pi)^m(n)^{m+2}}\sum_{k=0}^{\infty}\Big[\left\{\frac{(2)_k}{(1)_k}\right\}^2{}_2F_3\left(\begin{array}{c}\frac{m+2}{2},\frac{m+3}{2};\\ \frac{3}{4},\frac{5}{4},\frac{3}{2}\end{array};\ \frac{-\pi^2}{4n^2}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\Big]+$$

$$+\ \frac{\left(\frac{5}{2}\right)_m\sin\left(\frac{m\pi}{2}+\frac{\pi}{4}\right)}{(\pi)^{m-1}(n)^{m+\frac{5}{2}}}\sum_{k=0}^{\infty}\Big[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_2F_3\left(\begin{array}{c}\frac{2m+5}{4},\frac{2m+7}{4};\\ \frac{5}{4},\frac{3}{2},\frac{7}{4}\end{array};\ \frac{-\pi^2}{4n^2}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\Big], \tag{5.4}$$

where m is a non-negative integer and n is positive rational number.

Proof. The results (5.1), (5.2), (5.3) and (5.4) are obtained from (4.1), (4.2), (4.3) and (4.4) by putting $\upsilon = m+1$, $b = 2\pi$, $\lambda = 1$ and $y = n\pi$. □

6. Applications of Ramanujan's integrals

In this section we have established the following three infinite new summation formulas associated with hypergeometric series ${}_1F_2$ and ${}_0F_1$:

$$\sum_{k=0}^{\infty}\left[{}_1F_2\left(\begin{array}{c}1\ ;\\ \frac{1}{4},\frac{3}{4};\end{array}\frac{-\pi^2}{4}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]-\frac{\pi}{\sqrt{2}}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}{}_0F_1\left(\begin{array}{c}-;\\ \frac{1}{2};\end{array}\frac{-\pi^2}{4}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]+$$
$$+\frac{\pi^2}{\sqrt{2}}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_0F_1\left(\begin{array}{c}-;\\ \frac{3}{2};\end{array}\frac{-\pi^2}{4}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]=\frac{\pi\sqrt{2}-4}{8}, \tag{6.1}$$

$$\sum_{k=0}^{\infty}\left[{}_1F_2\left(\begin{array}{c}1\ ;\\ \frac{1}{4},\frac{3}{4};\end{array}\frac{-\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]-\frac{\pi}{2}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}{}_0F_1\left(\begin{array}{c}-;\\ \frac{1}{2};\end{array}\frac{-\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]+$$
$$+\frac{\pi^2}{4}\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_0F_1\left(\begin{array}{c}-;\\ \frac{3}{2};\end{array}\frac{-\pi^2}{16}\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]=\frac{\pi-2}{8}, \tag{6.2}$$

$$\sum_{k=0}^{\infty}\left[{}_1F_2\left(\begin{array}{c}1\ ;\\ \frac{1}{4},\frac{3}{4};\end{array}-\pi^2\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]-\pi\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}{}_0F_1\left(\begin{array}{c}-;\\ \frac{1}{2};\end{array}-\pi^2\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]+$$
$$+2\pi^2\sum_{k=0}^{\infty}\left[\left\{\frac{(2)_k}{(1)_k}\right\}^3{}_0F_1\left(\begin{array}{c}-;\\ \frac{3}{2};\end{array}-\pi^2\left\{\frac{(2)_k}{(1)_k}\right\}^4\right)\right]=\frac{\pi-3}{8}. \tag{6.3}$$

Proof. The following theorem is proved by Ramanujan [48, p.76-77, eqns(10 and 10′)]:
If
$$\Phi(n)=\int_0^{\infty}\frac{\cos(\pi n x)}{\{-1+\exp(2\pi\sqrt{x})\}}dx, \tag{6.4}$$
and
$$\Upsilon(n)-\frac{1}{2\pi n}=\int_0^{\infty}\frac{\sin(\pi n x)}{\{-1+\exp(2\pi\sqrt{x})\}}dx=R_S(0,n), \tag{6.5}$$
then
$$\Phi(n)=\frac{1}{n}\sqrt{\left(\frac{2}{n}\right)}\,\Upsilon\left(\frac{1}{n}\right)-\Upsilon(n), \tag{6.6}$$
and
$$\Upsilon(n)=\frac{1}{n}\sqrt{\left(\frac{2}{n}\right)}\,\Phi\left(\frac{1}{n}\right)+\Phi(n), \tag{6.7}$$
where n is positive rational number.
For particular values of n, some values of Ramanujan's integral [48, p.85 (eq. 48)] are given below
$$\Phi(1)=\int_0^{\infty}\frac{\cos(\pi x)}{\{-1+\exp(2\pi\sqrt{x})\}}dx=\frac{2-\sqrt{2}}{8}, \tag{6.8}$$
$$\Phi(2)=\int_0^{\infty}\frac{\cos(2\pi x)}{\{-1+\exp(2\pi\sqrt{x})\}}dx=\frac{1}{16}, \tag{6.9}$$

$$\Phi\left(\frac{1}{2}\right) = \int_0^\infty \frac{\cos\left(\frac{\pi x}{2}\right)}{\{-1 + \exp(2\pi\sqrt{x})\}} dx = \frac{1}{4\pi}. \tag{6.10}$$

From eqns(6.5) and (6.7), we obtain

$$\int_0^\infty \frac{\sin(\pi n x)}{\{-1 + \exp(2\pi\sqrt{x})\}} dx = \frac{1}{n}\sqrt{\left(\frac{2}{n}\right)}\,\Phi\left(\frac{1}{n}\right) + \Phi(n) - \frac{1}{2\pi n}. \tag{6.11}$$

Setting $n = 1, 2, \frac{1}{2}$ in the above eq. (6.11), using values of $\Phi(1), \Phi(2)$ and $\Phi\left(\frac{1}{2}\right)$, after simplification we get the following three results:

$$\mathbf{R}_S(0,1) = \int_0^\infty \frac{\sin(\pi x)}{\{-1 + \exp(2\pi\sqrt{x})\}} dx = \frac{\pi\sqrt{2} - 4}{8\pi}, \tag{6.12}$$

$$\mathbf{R}_S(0,2) = \int_0^\infty \frac{\sin(2\pi x)}{\{-1 + \exp(2\pi\sqrt{x})\}} dx = \frac{\pi - 2}{16\pi}, \tag{6.13}$$

$$\mathbf{R}_S(0,1/2) = \int_0^\infty \frac{\sin\left(\frac{\pi x}{2}\right)}{\{-1 + \exp(2\pi\sqrt{x})\}} dx = \frac{\pi - 3}{4\pi}. \tag{6.14}$$

When $m = 0$ with $n = 1,\ 2,\ \frac{1}{2}$ in the equations (5.1),(5.4) and comparing with equations (6.12), (6.13), and (6.14), we get the results (6.1), (6.2) and (6.3) respectively. In view of the hypergeometric functions (1.21), (1.22) and (1.23), we can express the above results (6.1) to (6.3) in terms of cosine, sine and Lommel functions. Our results (6.1) to (6.3) are convergent in view of the convergence condition of ${}_pF_q(\cdot)$ series, when $p \leq q$, and for all $|z| < \infty$. □

7. Conclusion

Here, we have described some infinite Fourier sine transforms of Ramanujan. Thus certain Ramanujan's integrals, which may be different from those of presented here, can also be evaluated in a similar way. The results established above may be of significant in nature. We conclude our observation by remarking that various new results can be obtained from our general theorem by appropriate choice of parameters $\upsilon, \lambda, b, c, y$ and bounded sequence $\{\Theta(k)\}_{k=0}^\infty$ in $\mathbf{I}_S^*(\upsilon, b, c, \lambda, y)$. This work is in continuation to our earlier work [44] on infinite Fourier cosine transforms.

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