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Certain Generating Functions for Cigler's Polynomials

Sama Arjika [a]

[a]*Department of Mathematics and Informatics, University of Agadez, Post Box 199, Agadez, Niger. And International Chair of Mathematical Physics and Applications (ICMPA-UNESCO Chair) University of Abomey-Calavi, Post Box 072, Cotonou 50, Benin*

Abstract

In this paper, we use the homogeneous q-operators [J. Difference Equ. Appl. **20** (2014), 837–851.] to derive Rogers formulas, extended Rogers formulas and Srivastava-Agarwal type bilinear generating functions for Cigler's polynomials [J. Difference Equ. Appl. **24** (2018), 479–502.]. Finally, we also derive two interesting transformation formulas between ${}_2\Phi_1$, ${}_2\Phi_2$ and ${}_3\Phi_2$.

Keywords: Basic (or q-) hypergeometric series, Homogeneous q-difference operator, Cigler polynomials, Generating functions, Rogers type formulas, Extended Rogers type formulas, Srivastava-Agarwal type bilinear generating functions

2010 MSC: 05A30, 33D15, 33D45, 05A40, 11B65

1. Introduction

Wang and Cao [30] presented two extensions of Cigler's polynomials together with their several generating functions for these extended Cigler's polynomials by using the method of homogeneous q-difference equations. In this paper, by employing the homogeneous q-operators, we aim to establish more generalized generating functions for the extended Cigler's polynomials than those in Wang and Cao [30].

The q-Laguerre polynomials are defined by [19]

$$\mathcal{L}_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{q^{\binom{k}{2}}(q^{-n};q)_k}{(q^{\alpha+1},q;q)_k}(xq^{n+\alpha+1})^k \tag{1.1}$$

which belong to the Askey-scheme of basic hypergeometric orthogonal polynomials according to Koekoek and Swarttouw [19, Eq. (3.21.1)]. They appear in several branches of mathematics and physics, and their generalization arise in many applications (see for details, [2, 4, 3, 13, 15, 22]), such as (for example) quantum group, q-harmonic oscillator and coding theory, and so on.

Cigler studied q-Laguerre polynomials [14, Eq. (30)]

$$l_n^{(\alpha)}(x) = \frac{1}{q^{n^2+\alpha n}} \sum_{k=0}^{n} \begin{bmatrix} n+\alpha \\ n-k \end{bmatrix}_q \frac{(q;q)_n}{(q;q)_k}(-1)^k q^{\binom{n-k}{2}} x^{n-k} \tag{1.2}$$

and derived important results.

†*Article ID:* MTJPAM-D-20-00027

Email address: rjksama2008@gmail.com (Sama Arjika)

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⋆*Corresponding Author:* Sama Arjika

In 2013, Cao and Niu [11] introduced two extensions of Cigler's polynomials

$$C_n^{(\alpha)}(x,b)=\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix} n+\alpha \\ k \end{bmatrix}_q \frac{(q;q)_n}{(q;q)_k} b^k x^{n-k} \tag{1.3}$$

and

$$\mathcal{D}_n^{(\alpha)}(x,b)=\sum_{k=0}^{n} q^{k^2-nk}\begin{bmatrix} n+\alpha \\ k \end{bmatrix}_q \frac{(q;q)_n}{(q;q)_k} b^k x^{n-k} \tag{1.4}$$

as solution of q-difference equations and deduced their generating functions.

Recently, Wang and Cao [30] introduced two extensions of Cigler's polynomials

$$C_n^{(\alpha-n)}(x,y,b)=\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix} \alpha \\ k \end{bmatrix}_q b^k \frac{(q;q)_n}{(q;q)_{n-k}} p_{n-k}(x,y) \tag{1.5}$$

and

$$\mathcal{D}_n^{(\alpha)}(x,y,b)=\sum_{k=0}^{n} q^{\binom{k}{2}}\begin{bmatrix} \alpha \\ k \end{bmatrix}_q b^k \frac{(q;q)_n}{(q;q)_{n-k}} \left[(-1)^{n+k} q^{-\binom{n}{2}} p_{n-k}(y,x)\right], \tag{1.6}$$

and derived the following important results by the homogeneous q-difference equations.

Proposition 1.1. *[30, Theorem 5] For $\alpha\in\mathbb{R}$ and $s\in\mathbb{N}$, we have:*

$$\sum_{n=0}^{\infty} C_{n+s}^{(\alpha-n-s)}(x,y,b)\frac{t^{n+s}}{(q;q)_n}=(bt;q)_\alpha \frac{(yt;q)_\infty}{(xt;q)_\infty}{}_3\Phi_2\left[\begin{matrix} q^{-s}, xt, q^\alpha bt; \\ yt, bt; \end{matrix}\, q;q\right], \quad |xt|<1, \tag{1.7}$$

$$\sum_{n=0}^{\infty} \mathcal{D}_{n+s}^{(\alpha-n-s)}(x,y,b)(-1)^{n+s} q^{\binom{n+s}{2}}\frac{t^{n+s}}{(q;q)_n}=(bt;q)_\alpha \frac{(xt;q)_\infty}{(yt;q)_\infty}{}_3\Phi_2\left[\begin{matrix} q^{-s}, yt, q^\alpha bt; \\ xt, bt; \end{matrix}\, q;q\right], \quad |yt|<1. \tag{1.8}$$

Proposition 1.2. *[30, Corollary 6] For $\alpha\in\mathbb{R}$, we have:*

$$\sum_{n=0}^{\infty} C_n^{(\alpha-n)}(x,y,b)\frac{t^n}{(q;q)_n}=\frac{(yt,bt;q)_\infty}{(xt,q^\alpha bt;q)_\infty}, \quad \max\{|xt|,|q^\alpha bt|\}<1, \tag{1.9}$$

$$\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}} \mathcal{D}_n^{(\alpha-n)}(x,y,b)\frac{t^n}{(q;q)_n}=\frac{(xt,bt;q)_\infty}{(yt,q^\alpha bt;q)_\infty}, \quad \max\{|yt|,|q^\alpha bt|\}<1. \tag{1.10}$$

Proposition 1.3. *[30, Theorem 11] For $\alpha\in\mathbb{R}$, we have:*

$$\sum_{n=0}^{\infty} C_n^{(\alpha-n)}(x,y,b)(\lambda;q)_n\frac{t^n}{(q;q)_n}=\frac{(\lambda,yt,bt;q)_\infty}{(xt,q^\alpha bt;q)_\infty}{}_3\Phi_2\left[\begin{matrix} xt, q^\alpha bt, 0; \\ yt, bt; \end{matrix}\, q;\lambda\right], \tag{1.11}$$

where $\max\{|xt|,|\lambda|,|q^\alpha bt|\}<1$;

$$\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}} \mathcal{D}_n^{(\alpha-n)}(x,y,b)(\lambda;q)_n\frac{t^n}{(q;q)_n}=\frac{(\lambda,xt,bt;q)_\infty}{(yt,q^\alpha bt;q)_\infty}{}_3\Phi_2\left[\begin{matrix} yt, q^\alpha bt, 0; \\ xt, bt; \end{matrix}\, q;\lambda\right], \tag{1.12}$$

where $\max\{|yt|,|\lambda|,|q^\alpha bt|\}<1$.

Remark 1.4. For $s=0$, the assertions (1.7) and (1.8) of Proposition 1.1 reduce to the assertions (1.9) and (1.10).

Remark 1.5. For $\lambda=0$, the assertions (1.11) and (1.12) of Proposition 1.3 reduce to the assertions (1.9) and (1.10).

In this paper, motivated by Wang and Cao's results [30], we aim to establish more generalized generating functions for the extended Cigler's polynomials $\mathcal{C}_n^{(\alpha-n)}(x,y,b)$ and $\mathcal{D}_n^{(\alpha-n)}(x,y,b)$.

Our main results are stated as:

Theorem 1.6. [Rogers type formulas for $\mathcal{C}_n^{(\alpha-n)}(x,y,b)$ and $\mathcal{D}_n^{(\alpha-n)}(x,y,b)$] *For $\alpha \in \mathbb{R}$, the following Rogers type formulas hold true for Cigler's polynomials $\mathcal{C}_n^{(\alpha-n)}(x,y,b)$ and $\mathcal{D}_n^{(\alpha-n)}(x,y,b)$:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{C}_{n+m}^{(\alpha-n-m)}(x,y,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(ys,bs;q)_\infty}{(t/s,xs,q^\alpha bs;q)_\infty}{}_4\Phi_3\left[\begin{array}{c} xs,q^\alpha bs,0,0; \\ \\ qs/t,ys,bs; \end{array} q;q\right], \tag{1.13}$$

where $\max\{|t/s|,|xs|,|q^\alpha bs|\}<1$*;*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}q^{\binom{n+m}{2}}\mathcal{D}_{n+m}^{(\alpha-n-m)}(x,y,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(xs,bs;q)_\infty}{(t/s,ys,q^\alpha bs;q)_\infty}{}_4\Phi_3\left[\begin{array}{c} ys,q^\alpha bs,0,0; \\ \\ qs/t,xs,bs; \end{array} q;q\right], \tag{1.14}$$

where $\max\{|t/s|,|ys|,|q^\alpha bs|\}<1$.

Theorem 1.7. [Extended Rogers type formula for $\mathcal{C}_n^{(\alpha-n)}(x,y,b)$ and $\mathcal{D}_n^{(\alpha-n)}(x,y,b)$] *For $\alpha \in \mathbb{R}$, the following extended Rogers type formulas hold true for Cigler's polynomials $\mathcal{C}_n^{(\alpha-n)}(x,y,b)$ and $\mathcal{D}_n^{(\alpha-n)}(x,y,b)$:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{C}_{n+m+k}^{(\alpha-n-m-k)}(x,y,b)\frac{t^n s^m\omega^k}{(q;q)_{n+m}(q;q)_m(q;q)_k} = \frac{(y\omega,b\omega;q)_\infty}{(s/t,t/\omega,x\omega,q^\alpha b\omega;q)_\infty}{}_4\Phi_3\left[\begin{array}{c} x\omega,q^\alpha b\omega,0,0; \\ \\ y\omega,b\omega,q\omega/t; \end{array} q;q\right], \tag{1.15}$$

where $\max\{|s/t|,|t/\omega|,|x\omega|,|q^\alpha b\omega|\}<1$;

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{D}_{n+m+k}^{(\alpha-n-m-k)}(x,y,b)\frac{(-1)^{n+m+k}q^{\binom{n+m+k}{2}}t^n s^m\omega^k}{(q;q)_{n+m}(q;q)_m(q;q)_k} \\ = \;& \frac{(x\omega,b\omega;q)_\infty}{(s/t,t/\omega,y\omega,q^\alpha b\omega;q)_\infty}{}_4\Phi_3\left[\begin{array}{c} y\omega,q^\alpha b\omega,0,0; \\ \\ x\omega,b\omega,q\omega/t; \end{array} q;q\right],\end{aligned} \tag{1.16}$$

where $\max\{|s/t|,|t/\omega|,|y\omega|,|q^\alpha b\omega|\}<1$.

Theorem 1.8. [Srivastava-Agarwal type bilinear generating function for $\mathcal{C}_n^{(\alpha-n)}(x,y,b)$ and $\mathcal{D}_n^{(\alpha-n)}(x,y,b)$] *For $\alpha,\beta \in \mathbb{R}$, we have:*

$$\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)\mathcal{C}_n^{(\beta-n)}(u,v,b)\frac{t^n}{(q;q)_n} = \frac{(\alpha x,vt,bt;q)_\infty}{(x,ut,q^\beta bt;q)_\infty}{}_4\Phi_3\left[\begin{array}{c} \alpha,ut,q^\beta bt,0; \\ \\ q/x,vt,bt; \end{array} q;q\right], \tag{1.17}$$

where $\max\{|x|,|ut|,|q^\beta bt|\}<1$;

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}\psi_n^{(\alpha)}(x|q)\mathcal{D}_n^{(\beta-n)}(u,v,b)\frac{t^n}{(q;q)_n} \\ = \;& \frac{(q/x,uxtq,bxtq;q)_\infty}{(\alpha q,vxtq,bxtq^{1+\beta};q)_\infty}{}_3\Phi_3\left[\begin{array}{c} 1/(\alpha x),1/(uxt),1/(bxt); \\ \\ q/x,1/(vxt),q^{-\beta}/(bxt); \end{array} q;\frac{\alpha uq^{1-\beta}}{v}\right],\end{aligned} \tag{1.18}$$

where $\max\{|\alpha q|,|vxt|,|bxtq^{\beta+1}|\}<1$.

Remark 1.9. For $k=0$, Theorem 1.7 reduces to Theorem 1.6.

Corollary 1.10. *For* $\alpha,\ \beta \in \mathbb{R}$, *we have:*

$$\sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q) C_n^{(\beta-n)}(u,b) \frac{t^n}{(q;q)_n} = \frac{(\alpha x, bt;q)_\infty}{(x, ut, q^\beta bt;q)_\infty} {}_3\Phi_2\left[\begin{matrix} \alpha, ut, q^\beta bt; \\ q/x, bt; \end{matrix} q;q\right], \tag{1.19}$$

where $\max\{|x|, |ut|, |q^\beta bt|\} < 1$;

$$\sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}} \psi_n^{(\alpha)}(x|q) \mathcal{D}_n^{(\beta-n)}(u,b) \frac{t^n}{(q;q)_n}$$
$$= \frac{(q/x, uxtq, bxtq;q)_\infty}{(\alpha q, bxtq^{1+\beta};q)_\infty} {}_3\Phi_2\left[\begin{matrix} 1/(\alpha x), 1/(uxt), 1/(bxt); \\ q/x, q^{-\beta}/(bxt); \end{matrix} q; \alpha uxtq^{1-\beta}\right], \tag{1.20}$$

where $\max\{|\alpha q|, |bxtq^{\beta+1}|\} < 1$.

Remark 1.11. For $v = 0$, the assertions (1.17) and (1.18) of Theorem 1.8, reduce to the assertions (1.19) and (1.20) .

The rest of paper is organized as follows. In Section 2, we present notations and give some q-operator identities. In Section 3, we use the homogeneous q-operators to derive Rogers formulas and extended Rogers formulas. In Section 4, we give Srivastava-Agarwal type bilinear generating functions for generalized Cigler's polynomials. As an application of Srivastava-Agarwal type generating functions, we deduce two interesting transformation formulas between ${}_2\Phi_1$, ${}_2\Phi_2$ and ${}_3\Phi_2$ in Section 5. We end by the concluding remarks in Section 6.

2. Notations and lemmas

In this section, we adopt the common notation and terminology for basic hypergeometric series as in Refs. [16, 19]. Throughout this paper, we assume that q is a fixed nonzero real or complex number and $|q| < 1$. The q-shifted factorial and its compact factorial are defined [16, 19], respectively by:

$$(a;q)_0 := 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),\ (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) \tag{2.1}$$

and $(a_1, a_2, \ldots, a_r;q)_m = (a_1;q)_m(a_2;q)_m \cdots (a_r;q)_m,\ m \in \{0,1,2\cdots\}$.
We will use frequently the following relation

$$(aq^{-n};q)_n = (q/a;q)_n(-a)^n q^{-n-\binom{n}{2}}. \tag{2.2}$$

The generalized q-binomial coefficient is defined as [16]

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha};q)_k}{(q;q)_k}(-1)^k q^{\alpha k-\binom{k}{2}},\ \alpha \in \mathbb{C}. \tag{2.3}$$

Here, in our present investigation, we are mainly concerned with the Cauchy polynomials $p_n(x,y)$ as given below (see [12] and [16]):

$$p_n(x,y) := (x-y)(x-qy)\cdots(x-q^{n-1}y) = (y/x;q)_n\, x^n \tag{2.4}$$

which has the following generating function [12]

$$\sum_{n=0}^{\infty} p_n(x,y)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{2.5}$$

The generating function (2.5) is also the homogeneous version of the Cauchy identity or the q-binomial theorem given by [16]

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k = {}_1\Phi_0\left[\begin{matrix} a; \\ -; \end{matrix}\; q;z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|<1, \tag{2.6}$$

where the basic or q-hypergeometric function in the variable z (see Slater [25, Chap. 3], Srivastava and Karlsson [29, p. 347, Eq. (272)] for details) is defined as:

$${}_r\Phi_s\left[\begin{matrix} a_1,a_2,\ldots,a_r; \\ b_1,b_2,\ldots,b_s; \end{matrix}\; q;z\right] = \sum_{n=0}^{\infty}\Big[(-1)^n q^{\binom{n}{2}}\Big]^{1+s-r}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\frac{z^n}{(q;q)_n},$$

when $r>s+1$. Note that, for $r=s+1$, we have:

$${}_{r+1}\Phi_r\left[\begin{matrix} a_1,a_2,\ldots,a_{r+1}; \\ b_1,b_2,\ldots,b_r; \end{matrix}\; q;z\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{r+1};q)_n}{(b_1,b_2,\ldots,b_r;q)_n}\frac{z^n}{(q;q)_n}.$$

Putting $a=0$, the relation (2.6) becomes Euler's identity [16]

$$\sum_{k=0}^{\infty}\frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty}, \quad |z|<1 \tag{2.7}$$

and its inverse relation [16]

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}\, z^k}{(q;q)_k} = (z;q)_\infty. \tag{2.8}$$

We remark in passing that, in a recently-published survey-cum-expository review article, the so-called (p,q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [26, p. 340]).

Chen *et al.* [12] introduced homogeneous q-difference operator D_{xy}, Saad and Sukhi [24] introduced another homogeneous q-difference operator θ_{xy} as

$$D_{xy}\{f(x,y)\} := \frac{f(x,q^{-1}y)-f(qx,y)}{x-q^{-1}y},\ \ \theta_{xy}\{f(x,y)\} := \frac{f(q^{-1}x,y)-f(x,qy)}{q^{-1}x-y}, \tag{2.9}$$

which turn out to be suitable for dealing with the Cauchy polynomials.

Cao [10] defined another homogeneous q-difference opeators

$$\mathbb{T}(a,zD_{xy}) = \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}\Big(z\,D_{xy}\Big)^k, \quad \mathbb{E}(a,z\theta_{xy}) = \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}\Big(-z\,\theta_{xy}\Big)^k, \tag{2.10}$$

and obtain some results from the perspective of q-difference equations (see [10], for more details).

In order to reach our goals in this paper, we need the following Lemmas.

Lemma 2.1.

$$C_n^{(\alpha-n)}(x,y,b) = \mathbb{T}(q^{-\alpha},bq^{\alpha}D_{xy})\{p_n(x,y)\}, \tag{2.11}$$

$$\mathcal{D}_n^{(\alpha-n)}(x,y,b) = \mathbb{E}(q^{-\alpha},bq^{\alpha}\theta_{xy})\Big\{(-1)^n q^{-\binom{n}{2}}p_n(y,x)\Big\}. \tag{2.12}$$

We now state the q-identities asserted by Lemma 2.2 below.

Lemma 2.2. *It is asserted that*

$$\mathbb{T}(q^{-\alpha},zD_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = \frac{(yt,q^{-\alpha}zt;q)_\infty}{(xt,zt;q)_\infty}, \quad \max\{|xt|,|zt|\}<1 \tag{2.13}$$

and

$$\mathbb{E}(q^{-\alpha},z\theta_{xy})\left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\} = \frac{(xt,q^{-\alpha}zt;q)_\infty}{(yt,zt;q)_\infty}, \quad \max\{|yt|,|zt|\}<1. \tag{2.14}$$

3. Proof of Theorems 1.6 and 1.7

In this section, we use the homogeneous q-operators defined in (2.10) to prove Theorems 1.6 and 1.7.

First, we give the identities (3.1) and (3.2) below, which will be used later in order to derive the Rogers type formulas and extended Rogers type formula for the Cigler's polynomials $C_n^{(\alpha-n)}(x,y,b)$ and $\mathcal{D}_n^{(\alpha-n)}(x,y,b)$.

Lemma 3.1. *It is asserted that*

$$\mathbb{T}(q^{-\alpha}, bD_{xy})\left\{\frac{p_n(x,y)\,(ys;q)_\infty}{(ys;q)_n(xs;q)_\infty}\right\} = s^{-n}\frac{(ys, q^{-\alpha}bs;q)_\infty}{(xs, bs;q)_\infty}\,{}_3\Phi_2\left[\begin{matrix} q^{-n}, xs, bs; \\ ys, q^{-\alpha}bs; \end{matrix}\; q;q\right], \tag{3.1}$$

where $\max\{|xs|, |bs|\} < 1$;

$$\mathbb{E}(q^{-\alpha}, b\theta_{xy})\left\{\frac{p_n(y,x)\,(xs;q)_\infty}{(xs;q)_n(ys;q)_\infty}\right\} = s^{-n}\frac{(xs, q^{-\alpha}bs;q)_\infty}{(ys, bs;q)_\infty}\,{}_3\Phi_2\left[\begin{matrix} q^{-n}, ys, bs; \\ xs, q^{-\alpha}bs; \end{matrix}\; q;q\right], \tag{3.2}$$

where $\max\{|ys|, |bs|\} < 1$.

We are in position to prove Theorems 1.6 and 1.7.

Proof of Theorem 1.6. In light of (2.11), we have:

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{n+m}^{(\alpha-n-m)}(x,y,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathbb{T}(q^{-\alpha}, q^{\alpha}bD_{xy})\{p_{n+m}(x,y)\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} \\
&= \mathbb{T}(q^{-\alpha}, q^{\alpha}bD_{xy})\left\{\sum_{n=0}^{\infty} p_n(x,y)\frac{t^n}{(q;q)_n}\sum_{m=0}^{\infty} p_m(x, yq^n)\frac{s^m}{(q;q)_m}\right\} \\
&= \mathbb{T}(q^{-\alpha}, q^{\alpha}bD_{xy})\left\{\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\frac{p_n(x,y)}{(ys;q)_n}\frac{(ys;q)_\infty}{(xs;q)_\infty}\right\} \\
&= \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\mathbb{T}(q^{-\alpha}, q^{\alpha}bD_{xy})\left\{\frac{p_n(x,y)}{(ys;q)_n}\frac{(ys;q)_\infty}{(xs;q)_\infty}\right\}.
\end{aligned}$$

By using (3.1), we obtain:

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{n+m}^{(\alpha-n-m)}(x,y,b)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} &= \frac{(ys,bs;q)_\infty}{(xs,q^\alpha bs;q)_\infty}\sum_{n=0}^{\infty}\frac{(t/s)^n}{(q;q)_n}{}_3\Phi_2\left[\begin{matrix} q^{-n},xs,q^\alpha bs; \\ ys,bs; \end{matrix} q;q\right] \\
&= \frac{(ys,bs;q)_\infty}{(xs,q^\alpha bs;q)_\infty}\sum_{n=0}^{\infty}\frac{(t/s)^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(q^{-n},xs,q^\alpha bs;q)_k\,q^k}{(ys,bs,q;q)_k} \\
&= \frac{(ys,bs;q)_\infty}{(xs,q^\alpha bs;q)_\infty}\sum_{k=0}^{\infty}\frac{(xs,q^\alpha bs;q)_k\,q^k}{(ys,bs,q;q)_k}\sum_{n=0}^{\infty}\frac{(q^{-n};q)_k(t/s)^n}{(q;q)_n} \\
&= \frac{(ys,bs;q)_\infty}{(xs,q^\alpha bs;q)_\infty}\sum_{k=0}^{\infty}\frac{(xs,q^\alpha bs;q)_k\,q^k}{(ys,bs,q;q)_k}\sum_{n=k}^{\infty}\frac{(t/s)^n(-1)^k q^{\binom{k}{2}-nk}}{(q;q)_{n-k}} \\
&= \frac{(ys,bs;q)_\infty}{(xs,q^\alpha bs;q)_\infty}\sum_{k=0}^{\infty}\frac{(xs,q^\alpha bs;q)_k\,q^k}{(ys,bs,q;q)_k}(-t/s)^k q^{-\binom{k}{2}-k}\sum_{n=0}^{\infty}\frac{(tq^{-k}/s)^n}{(q;q)_n} \\
&= \frac{(ys,bs;q)_\infty}{(t/s,xs,q^\alpha bs;q)_\infty}\sum_{k=0}^{\infty}\frac{(xs,q^\alpha bs;q)_k\,q^k}{(ys,bs,q;q)_k}\frac{(-t/s)^k q^{-\binom{k}{2}-k}}{(tq^{-k}/s;q)_k} \quad \text{by (2.2)} \\
&= \frac{(ys,bs;q)_\infty}{(t/s,xs,q^\alpha bs;q)_\infty}\sum_{k=0}^{\infty}\frac{(xs,q^\alpha bs;q)_k\,q^k}{(qs/t,ys,bs,q;q)_k} \\
&= \frac{(ys,bs;q)_\infty}{(t/s,xs,q^\alpha bs;q)_\infty}{}_4\Phi_3\left[\begin{matrix} xs,q^\alpha bs,0,0; \\ qs/t,ys,bs; \end{matrix} q;q\right].
\end{aligned}$$

The proof of the assertion (1.14) of Theorem 1.6 is the same to that of the first assertion (1.13) by using the representation (2.12). The details involved are, therefore, being omitted here. □

Proof of Theorem 1.7. In view of the formula (2.11), we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} C_{n+m+k}^{(\alpha-n-m-k)}(x,y,b)\frac{t^n s^m\omega^k}{(q;q)_m(q;q)_{n+m}(q;q)_k} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\mathbb{T}(q^{-\alpha},q^\alpha bD_{xy})\{p_{n+m+k}(x,y)\}\;\frac{t^n s^m\omega^k}{(q;q)_m(q;q)_{n+m}(q;q)_k} \\
&= \mathbb{T}(q^{-\alpha},q^\alpha bD_{xy})\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{p_{n+m}(x,y)\,t^n s^m}{(q;q)_m(q;q)_{n+m}}\sum_{k=0}^{\infty}p_k(x,q^{n+m}y)\frac{\omega^k}{(q;q)_k}\right\} \\
&= \mathbb{T}(q^{-\alpha},q^\alpha bD_{xy})\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{p_{n+m}(x,y)t^n s^m}{(q;q)_m(q;q)_{n+m}}\frac{(y\omega q^{n+m};q)_\infty}{(x\omega;q)_\infty}\right\} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n s^m}{(q;q)_m(q;q)_{n+m}}\mathbb{T}(q^{-\alpha},q^\alpha bD_{xy})\left\{\frac{p_{n+m}(x,y)}{(y\omega;q)_{n+m}}\frac{(y\omega;q)_\infty}{(x\omega;q)_\infty}\right\}.
\end{aligned}$$

Setting n by $n+m$ in (3.1), we then obtain

$$\mathbb{T}(q^{-\alpha},q^\alpha bD_{xy})\left\{\frac{p_{n+m}(x,y)}{(y\omega;q)_{n+m}}\frac{(y\omega;q)_\infty}{(x\omega;q)_\infty}\right\} = \omega^{-n-m}\frac{(y\omega,b\omega;q)_\infty}{(x\omega,q^\alpha b\omega;q)_\infty}{}_3\Phi_2\left[\begin{matrix} q^{-n-m},x\omega,q^\alpha b\omega; \\ y\omega,b\omega; \end{matrix} q;q\right]$$

which, in conjunction with (3.3), gives

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} C_{n+m+k}^{(\alpha-n-m-k)}(x,y,b)\frac{t^n\, s^m \omega^k}{(q;q)_m(q;q)_{n+m}(q;q)_k} &= \frac{(y\omega, b\omega;q)_\infty}{(x\omega, q^\alpha b\omega;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(t/\omega)^n(s/\omega)^m}{(q;q)_m(q;q)_{n+m}} \\
&\times {}_3\Phi_2\left[\begin{matrix} q^{-n-m}, x\omega, q^\alpha b\omega; \\ y\omega, b\omega; \end{matrix}\, q;q\right] \\
&= \frac{(y\omega, b\omega;q)_\infty}{(x\omega, q^\alpha b\omega;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(t/\omega)^n(s/\omega)^m}{(q;q)_m(q;q)_{n+m}} \\
&\times \sum_{k=0}^{n+m}\frac{(q^{-n-m}, x\omega, q^\alpha b\omega;q)_k\, q^k}{(q, y\omega, b\omega;q)_k} \\
&= \frac{(y\omega, b\omega;q)_\infty}{(x\omega, q^\alpha b\omega;q)_\infty} \\
&\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n+m=k}^{\infty}\frac{(t/\omega)^n(s/\omega)^m(-1)^k q^{\binom{k}{2}-k(m+n)}}{(q;q)_m(q;q)_{n+m-k}}\frac{(x\omega, q^\alpha b\omega;q)_k\, q^k}{(q, y\omega, b\omega;q)_k} \\
&= \frac{(y\omega, b\omega;q)_\infty}{(x\omega, q^\alpha b\omega;q)_\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(s/t)^m}{(q;q)_m}\frac{(t/\omega)^{j+k}}{(q;q)_j}\frac{(-1)^k q^{\binom{k}{2}-k(k+j)}}{(q;q)_k} \\
&\times \frac{(x\omega, q^\alpha b\omega;q)_k\, q^k}{(y\omega, b\omega;q)_k} \\
&= \frac{(y\omega, b\omega;q)_\infty}{(x\omega, q^\alpha b\omega;q)_\infty}\sum_{m=0}^{\infty}\frac{(s/t)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(x\omega, q^\alpha z\omega;q)_k(-t/\omega)^k q^{-\binom{k}{2}}}{(y\omega, b\omega, q;q)_k} \\
&\times \sum_{j=0}^{\infty}\frac{(tq^{-k}/\omega)^j}{(q;q)_j} \\
&= \frac{(y\omega, b\omega;q)_\infty}{(s/t, x\omega, q^\alpha b\omega;q)_\infty}\sum_{k=0}^{\infty}\frac{(x\omega, q^\alpha b\omega;q)_k}{(y\omega, b\omega, q;q)_k}\frac{(-t/\omega)^k q^{-\binom{k}{2}}}{(tq^{-k}/\omega;q)_\infty} \quad \text{by (2.2)} \\
&= \frac{(y\omega, b\omega;q)_\infty}{(s/t, t/\omega, x\omega, q^\alpha b\omega;q)_\infty}\sum_{k=0}^{\infty}\frac{(x\omega, q^\alpha b\omega;q)_k\, q^k}{(y\omega, b\omega, q\omega/t, q;q)_k} \\
&= \frac{(y\omega, b\omega;q)_\infty}{(s/t, t/\omega, x\omega, q^\alpha b\omega;q)_\infty}{}_4\Phi_3\left[\begin{matrix} x\omega, q^\alpha b\omega, 0, 0; \\ y\omega, b\omega, q\omega/t; \end{matrix}\, q;q\right].
\end{aligned}$$

The proof of the assertion (1.16) of Theorem 1.7 is the same to that of the first assertion (1.15) by using the representation (2.12). The details involved are, therefore, being omitted here. □

4. Proof of Theorem 1.8

The Hahn polynomials [17, 18] (or Al-Salam and Carlitz polynomials [1, 7]) are defined as

$$\phi_n^{(a)}(x|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (a;q)_k\, x^k, \quad \psi_n^{(a)}(x|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)}(aq^{1-k};q)_k\, x^k. \tag{4.1}$$

Srivastava and Agarwal [27] utilized the method of transformation theory to deduce the following results, while Cao [7] used the technique of exponential operator decomposition. For more information, please refer to [17, 18, 1, 27, 5, 6, 7].

Lemma 4.1. *([27, Eq. (3.20)] and [7, Eq. (5.4)])*

$$\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)(\lambda;q)_n\frac{t^n}{(q;q)_n}=\frac{(\lambda t;q)_\infty}{(t;q)_\infty}{}_2\Phi_1\left[\begin{matrix}\lambda,\alpha;\\ \\ \lambda t;\end{matrix}\;q;xt\right],\quad \max\{|t|,|xt|\}<1, \tag{4.2}$$

$$\sum_{n=0}^{\infty}\psi_n^{(\alpha)}(x|q)(1/\lambda;q)_n\frac{(\lambda tq)^n}{(q;q)_n}=\frac{(xtq;q)_\infty}{(\lambda xtq;q)_\infty}{}_2\Phi_1\left[\begin{matrix}1/\lambda,1/(\alpha x);\\ \\ 1/(\lambda xt);\end{matrix}\;q;\alpha q\right], \tag{4.3}$$

where $\max\{|\lambda xtq|,|\alpha q|\}<1$.

In the proof of Theorem 1.8, the following q-Chu-Vandermonde formula will be needed.

Lemma 4.2. *(q-Chu-Vandermonde [16, Eq. (II.7)])*

$${}_2\Phi_1\left[\begin{matrix}q^{-n},a;\\ \\ c;\end{matrix}\;q;\frac{cq^n}{a}\right]=\frac{(c/a;q)_n}{(c;q)_n}. \tag{4.4}$$

Proof of Theorem 1.8. We observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)C_n^{(\beta-n)}(u,v,b)\frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)\mathbb{T}(q^{-\beta},q^{\beta}bD_{uv})\{p_n(u,v)\}\frac{t^n}{(q;q)_n}\\
&= \mathbb{T}(q^{-\beta},q^{\beta}bD_{uv})\left\{\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)p_n(u,v)\frac{t^n}{(q;q)_n}\right\}\\
&= \mathbb{T}(q^{-\beta},q^{\beta}bD_{uv})\left\{\sum_{n=0}^{\infty}\sum_{n=k}^{\infty}p_n(u,v)\frac{(\alpha;q)_k\,x^kt^n}{(q;q)_k(q;q)_{n-k}}\right\}\\
&= \mathbb{T}(q^{-\beta},q^{\beta}bD_{uv})\left\{\sum_{k=0}^{\infty}p_k(u,v)\frac{(\alpha;q)_k\,(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}p_n(u,vq^k)\frac{t^n}{(q;q)_k}\right\}\\
&= \mathbb{T}(q^{-\beta},q^{\beta}bD_{uv})\left\{\sum_{k=0}^{\infty}p_k(u,v)\frac{(\alpha;q)_k\,(xt)^k}{(q;q)_k}\frac{(vtq^k;q)_\infty}{(ut;q)_\infty}\right\}\\
&= \sum_{k=0}^{\infty}\frac{(\alpha;q)_k\,(xt)^k}{(q;q)_k}\mathbb{T}(q^{-\beta},q^{\beta}bD_{uv})\left\{\frac{p_k(u,v)}{(vt;q)_k}\frac{(vt;q)_\infty}{(ut;q)_\infty}\right\}.\end{aligned}$$

By using (3.1), we have:

$$\begin{aligned}
\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)C_n^{(\beta-n)}(u,v,b)\frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty}\frac{(\alpha;q)_n\,(xt)^n}{(q;q)_n}t^{-n}\frac{(vt,bt;q)_\infty}{(ut,q^\beta bt;q)_\infty}{}_3\Phi_2\left[\begin{matrix} q^{-n},ut,q^\beta bt; \\ vt,bt; \end{matrix}\; q;q\right] \\
&= \frac{(vt,bt;q)_\infty}{(ut,q^\beta bt;q)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha;q)_n\,x^n}{(q;q)_n}{}_3\Phi_2\left[\begin{matrix} q^{-n},ut,q^\beta bt; \\ vt,bt; \end{matrix}\; q;q\right] \\
&= \frac{(vt,bt;q)_\infty}{(ut,q^\beta bt;q)_\infty}\sum_{k=0}^{n}\frac{(ut,q^\beta bt;q)_k\,q^k}{(vt,bt,q;q)_k}\sum_{n=0}^{\infty}\frac{(q^{-n};q)_k(\alpha;q)_n\,x^n}{(q;q)_n} \\
&= \frac{(vt,bt;q)_\infty}{(ut,q^\beta bt;q)_\infty}\sum_{k=0}^{n}\frac{(ut,q^\beta bt;q)_k\,q^k}{(vt,bt,q;q)_k}\sum_{n=k}^{\infty}\frac{(-1)^k q^{\binom{k}{2}-nk}\,(\alpha;q)_n\,x^n}{(q;q)_{n-k}} \\
&= \frac{(vt,bt;q)_\infty}{(ut,q^\beta bt;q)_\infty}\sum_{k=0}^{n}\frac{(\alpha,ut,q^\beta bt;q)_k(-x)^k q^{-\binom{k}{2}}}{(vt,bt,q;q)_k}\sum_{n=0}^{\infty}\frac{(\alpha q^k;q)_n\,(xq^{-k})^n}{(q;q)_n} \\
&= \frac{(vt,bt;q)_\infty}{(ut,q^\beta bt;q)_\infty}\sum_{k=0}^{n}\frac{(\alpha,ut,q^\beta bt;q)_k}{(vt,bt,q;q)_k}(-x)^k q^{-\binom{k}{2}}\frac{(\alpha x;q)_\infty}{(q^{-k}x;q)_\infty} \\
&= \frac{(\alpha x,vt,bt;q)_\infty}{(x,ut,q^\beta bt;q)_\infty}\sum_{k=0}^{n}\frac{(\alpha,ut,q^\beta bt;q)_k}{(vt,bt,q;q)_k}\frac{(-x)^k q^{-\binom{k}{2}}}{(xq^{-k};q)_k}\;\text{ by (2.2)} \\
&= \frac{(\alpha x,vt,bs;q)_\infty}{(x,ut,q^\beta bt;q)_\infty}\sum_{k=0}^{n}\frac{(\alpha,\mu t,q^\beta bt;q)_k\,q^k}{(q/x,vt,bt,q;q)_k} \\
&= \frac{(\alpha x,vt,bt;q)_\infty}{(x,ut,q^\beta bt;q)_\infty}{}_4\Phi_3\left[\begin{matrix} \alpha,ut,q^\beta bt,0; \\ q/x,vt,bt; \end{matrix}\; q;q\right].
\end{aligned}$$

The proof of the assertion (1.17) is thus completed.
Letting $(\lambda, t) = (v/u, tu)$ in equation (4.3), we then obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\psi_n^{(\alpha)}(x|q)p_n(v,u)\frac{(qt)^n}{(q;q)_n} &= \frac{(uxtq;q)_\infty}{(vxtq;q)_\infty}{}_2\Phi_1\left[\begin{matrix} 1/(\alpha x),u/v; \\ 1/(vxt); \end{matrix}\; q;\alpha q\right] \\
&= \frac{(uxtq;q)_\infty}{(vxtq;q)_\infty}\sum_{k=0}^{\infty}\frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k}\frac{(u/v;q)_k}{(1/(vxt);q)_k}\;\text{ by (4.4)} \\
&= \frac{(uxtq;q)_\infty}{(vxtq;q)_\infty}\sum_{k=0}^{\infty}\frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k}{}_2\Phi_1\left[\begin{matrix} q^{-k},1/(uxt); \\ 1/(vxt); \end{matrix}\; q;\frac{uq^k}{v}\right] \\
&= \sum_{k=0}^{\infty}\frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k}\sum_{n=0}^{\infty}\frac{(q^{-k};q)_n\,q^{nk}}{(q;q)_n}\frac{(1/(uxt);q)_n(uxtq;q)_\infty}{(1/(vxt);q)_n(vxtq;q)_\infty}\left(\frac{u}{v}\right)^n \\
&= \sum_{k=0}^{\infty}\frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k}\sum_{n=0}^{\infty}\frac{(q^{-k};q)_n\,q^{nk}}{(q;q)_n}\frac{(uxtq^{1-n};q)_\infty}{(vxtq^{1-n};q)_\infty}.
\end{aligned}$$

We observe that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\psi_n^{(\alpha)}(x|q)\mathcal{D}_n^{(\beta-n)}(u,v,b)\frac{(-1)^n q^{\binom{n+1}{2}}\,t^n}{(q;q)_n} &= \sum_{n=0}^{\infty}\psi_n^{(\alpha)}(x|q)\mathbb{E}(q^{-\beta},q^{\beta}b\theta_{uv})\,\{p_n(v,u)\}\,\frac{(qt)^n}{(q;q)_n}\\
&= \mathbb{E}(q^{-\beta},q^{\beta}b\theta_{uv})\left\{\sum_{n=0}^{\infty}\psi_n^{(\alpha)}(x|q)p_n(v,u)\frac{(qt)^n}{(q;q)_n}\right\} \text{ by (4.5)}\\
&= \mathbb{E}(q^{-\beta},q^{\beta}b\theta_{uv})\left\{\sum_{k=0}^{\infty}\frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k}\sum_{n=0}^{\infty}\frac{(q^{-k};q)_n\,q^{nk}}{(q;q)_n}\frac{(uxtq^{1-n};q)_\infty}{(vxtq^{1-n};q)_\infty}\right\}\\
&= \sum_{k=0}^{\infty}\frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k}\sum_{n=0}^{\infty}\frac{(q^{-k};q)_n\,q^{nk}}{(q;q)_n}\mathbb{E}(q^{-\beta},q^{\beta}b\theta_{uv})\left\{\frac{(uxtq^{1-n};q)_\infty}{(vxtq^{1-n};q)_\infty}\right\}\\
&= \sum_{k=0}^{\infty}\frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k}\sum_{n=0}^{\infty}\frac{(q^{-k};q)_n\,q^{nk}}{(q;q)_n}\frac{(uxtq^{1-n},bxtq^{1-n};q)_\infty}{(vxtq^{1-n},bxtq^{\beta+1-n};q)_\infty}\\
&= \frac{(uxtq,bxtq;q)_\infty}{(vxtq,bxtq^{1+\beta};q)_\infty}\sum_{k=0}^{\infty}\frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k}\\
&\times \sum_{n=0}^{\infty}\frac{(q^{-k},uxtq^{1-n},bxtq^{1-n};q)_n\,q^{nk}}{(vxtq^{1-n},bxtq^{\beta+1-n},q;q)_n}\\
&= \frac{(uxtq,bxtq;q)_\infty}{(vxtq,bxtq^{1+\beta};q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(1/(uxt),1/(bxt);q)_n}{(1/(vxt),q^{-\beta}/(bxt),q;q)_n}\left(\frac{uq^{-\beta}}{v}\right)^n\\
&\times \sum_{k=n}^{\infty}\frac{(1/(\alpha x);q)_k(\alpha q)^k}{(q;q)_{k-n}}\\
&= \frac{(uxtq,bxtq;q)_\infty}{(vxtq,bxtq^{1+\beta};q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(1/(\alpha x),1/(uxt),1/(bxt);q)_n}{(1/(vxt),q^{-\beta}/(bxt),q;q)_n}\\
&\times \left(\frac{\alpha uq^{1-\beta}}{v}\right)^n\sum_{k=0}^{\infty}\frac{(q^n/(\alpha x);q)_k(\alpha q)^k}{(q;q)_k}\\
&= \frac{(uxtq,bxtq;q)_\infty}{(vxtq,bxtq^{1+\beta};q)_\infty}\\
&\times \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(1/(\alpha x),1/(uxt),1/(bxt);q)_n}{(1/(vxt),q^{-\beta}/(bxt),q;q)_n}\left(\frac{\alpha uq^{1-\beta}}{v}\right)^n\frac{(q^{1+n}/x;q)_\infty}{(\alpha q;q)_\infty}\\
&= \frac{(q/x,uxtq,bxtq;q)_\infty}{(\alpha q,vxtq,bxtq^{1+\beta};q)_\infty}\\
&\times \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(1/(\alpha x),1/(uxt),1/(bxt);q)_n}{(q/x,1/(vxt),q^{-\beta}/(bxt),q;q)_n}\left(\frac{\alpha uq^{1-\beta}}{v}\right)^n\\
&= \frac{(q/x,uxtq,bxtq;q)_\infty}{(\alpha q,vxtq,bxtq^{1+\beta};q)_\infty}\,{}_3\Phi_3\left[\begin{matrix}1/(\alpha x),1/(uxt),1/(bxt);\\ q/x,1/(vxt),q^{-\beta}/(bxt);\end{matrix}\;q;\frac{\alpha uq^{1-\beta}}{v}\right],
\end{aligned}
$$

which achieves the proof of the assertion (1.18) of Theorem 1.8. □

5. Another Srivastava-Agarwal type generating functions for the Al-Salam-Carlitz polynomials

In this section, we derive another Srivastava-Agarwal type generating functions for the Al-Salam-Carlitz polynomials (4.1). As an application of Srivastava-Agarwal type generating functions, we deduce two interesting transformation formulas between ${}_2\Phi_1$, ${}_2\Phi_2$ and ${}_3\Phi_2$.

Theorem 5.1. *For $\alpha \in \mathbb{R}$, we have:*

$$\sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q)(\lambda;q)_n \frac{t^n}{(q;q)_n} = \frac{(\alpha x, \lambda t;q)_\infty}{(x,t;q)_\infty} {}_3\Phi_2 \left[\begin{array}{c} \alpha, t, 0; \\ \\ q/x, \lambda t; \end{array} q;q \right], \ \max\{|t|,|x|\} < 1, \tag{5.1}$$

$$\sum_{n=0}^{\infty} \psi_n^{(\alpha)}(x|q)(1/\lambda;q)_n \frac{(\lambda tq)^n}{(q;q)_n} = \frac{(q/x, xtq;q)_\infty}{(\alpha q, \lambda xtq;q)_\infty} {}_2\Phi_2 \left[\begin{array}{c} 1/(\alpha x), 1/(xt); \\ \\ q/x, 1/(\lambda xt); \end{array} q; \frac{\alpha q}{\lambda} \right], \tag{5.2}$$

where $\max\{|\alpha q|, |xt|\} < 1$.

We now derive two transformation formulas for q-series.

Theorem 5.2. *We have:*

$${}_2\Phi_1 \left[\begin{array}{c} a, b; \\ \\ c; \end{array} q;z \right] = \frac{(abz/c;q)_\infty}{(az/c;q)_\infty} {}_3\Phi_2 \left[\begin{array}{c} b, c/a, 0; \\ \\ qc/(az), c; \end{array} q;q \right]. \tag{5.3}$$

Theorem 5.3. *[16, Eq. (III.4)] We have:*

$${}_2\Phi_1 \left[\begin{array}{c} a, b; \\ \\ c; \end{array} q;z \right] = \frac{(bz;q)_\infty}{(z;q)_\infty} {}_2\Phi_2 \left[\begin{array}{c} b, c/a; \\ \\ bz, c; \end{array} q;az \right]. \tag{5.4}$$

Remark 5.4. For $b = 0$ and $u = 1$, the assertion (1.17) of Theorem 1.8 reduces to (5.1). For $b = 0$, $u = 1/\lambda$, $v = 1$ and $t = \lambda t$, the assertion (1.18) of Theorem 1.8 reduces to (5.2).

Remark 5.5. Comparing the assertion (4.2) of Lemma 4.1 and the assertion (5.1) of Theorem 5.1 and upon setting $\lambda = a$, $\alpha = b$, $\lambda t = c$ and $xt = z$, we obtain (5.3).

Remark 5.6. Comparing the assertion (4.3) of Lemma 4.1 and the assertion (5.2) of Theorem 5.1 and upon setting $1/\lambda = a$, $1/(\alpha x) = b$, $1/(\lambda xt) = c$ and $\alpha q = z$, we then obtain (5.4).

6. Concluding remarks and observations

In our present investigation, we have used two q-operators $\mathbb{T}(a, zD_{xy})$ and $\mathbb{E}(a, z\theta_{xy})$ to derive Rogers formulas, extended Rogers formulas and Srivastava-Agarwal type bilinear generating functions for Cigler's polynomials by means of the q-difference equations. We have also briefly described relevant connections of various special cases and consequences of our main results with several known results.

It is believed that the q-series identities, which we have presented in this paper, as well as the various related recent works cited here, see also [28], will provide encouragement and motivation for further researches on the topics that are dealt with and investigated in this paper.

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