



An Integral Formula Generated by Hurwitz-Lerch Zeta Function with Order 1

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Abstract

In this paper, we give some results associated with Hurwitz-Lerch zeta function with order 1 which is special case of Hurwitz-Lerch zeta function. One of the results is the integral formula including Hurwitz-Lerch zeta function with order 1. The other result is a corollary generated by integral representation of Hurwitz-Lerch zeta function with order 1.

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1. Introduction, definitions and preliminaries

Throughout this article, \mathbb{N} denotes the set of natural numbers, \mathbb{R} denotes the set of real numbers and \mathbb{C} denotes the set of complex numbers. Also,

$$\begin{aligned}\mathbb{N}_0 &= \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}, \\ \mathbb{Z}_0 &= \{-3, -2, -1, 0, 1, 2, 3, \dots\} = \mathbb{Z} \cup \{0\}\end{aligned}$$

and

$$\mathbb{Z}^- = \{-1, -2, -3, \dots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

The Hurwitz-Lerch zeta function $\Phi(z, s, w)$ is defined by (cf. [3], [6], [7])

$$\Phi(z, s, w) = \sum_{n=0}^{\infty} \frac{z^n}{(n+w)^s} \quad (1.1)$$

$$(w \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \operatorname{Re}(s) > 1 \text{ when } |z| = 1).$$

For $z = 1$, $\Phi(1, s, w) = \zeta(s, w)$ is the Hurwitz zeta function (cf. [5], [10]) defined by

$$\zeta(s, w) = \sum_{n=0}^{\infty} \frac{1}{(n+w)^s}, \quad (\operatorname{Re}(s) > 1, w \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

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For $z = 1$ and $w = 1$, $\Phi(1, s, 1) = \zeta(s)$ is the Riemann zeta function (cf. [5], [10]) defined by

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{n^s}, \quad (\operatorname{Re}(s) > 1, w \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

For $w = 1$, $z\Phi(z, s, 1) = Li_s(z)$ is the polylogarithmic function (cf. [9]) defined by

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \tag{1.2}$$

($s \in \mathbb{C}$ when $|z| < 1$; $\operatorname{Re}(s) > 1$ when $|z| = 1$).

For $z = e^{2\pi i n \xi}$ into (1.2), we arrive the Lerch zeta function (cf. [4]) defined by

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1)$$

($\xi \in \mathbb{R}; \operatorname{Re}(s) > 1$).

Hence, we conclude that the Lerch zeta function, the polylogarithmic function, the Hurwitz zeta function, the Riemann zeta function are special cases of the Hurwitz-Lerch zeta function.

Also, Hurwitz-Lerch zeta function (cf. [6], [8]) has an integral representation as follows:

$$\Phi(z, s, w) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-wt}}{1 - ze^{-t}} dt$$

where

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt.$$

The Hurwitz-Lerch zeta function with order 1 (cf. [1], [2]) is defined by

$$\Phi(z, 1, w) = \sum_{n=0}^{\infty} \frac{z^n}{n+w}, \quad (|z| < 1, w \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

In [1], Aygunes obtained some results associated to the Hurwitz-Lerch zeta function with order 1 as follows:

Theorem 1.1. (cf. [1]) Let $M \in \mathbb{N} \setminus \{0\}$ and $|z| < 1/2$. Then, we have

$$\frac{1}{1-z} \sum_{k=0}^{M-1} \binom{M-1}{k} \Phi\left(\frac{z}{1-z}, 1, k+1\right) = 2^M \Phi(2z, 1, M) - \Phi(z, 1, M).$$

Remark 1.2. (cf. [1]) In Theorem 1.1, by putting $M = 2$, we respectively have

$$\frac{1}{1-z} \left\{ \Phi\left(\frac{z}{1-z}, 1, 1\right) + \Phi\left(\frac{z}{1-z}, 1, 2\right) \right\} = 4\Phi(2z, 1, 2) - \Phi(z, 1, 2).$$

2. Main Results

In this section, by using the integral representation of Hurwitz-Lerch zeta function with order 1, we give an integral formula as corollary.

Suppose that $x \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $|\lambda| < 1$. We set

$$\frac{x^2 + 3x + 2}{(x+1)(x+2)} - \frac{2}{(x+1)(x+2)} + \sum_{m=2}^{\infty} \lambda^{m-1} = \frac{1}{1-\lambda} - \frac{2}{(x+1)(x+2)}$$

$$\begin{aligned}
 & \Updownarrow \\
 & \frac{x(x+3)}{(x+1)(x+2)} + \sum_{m=2}^{\infty} \lambda^{m-1} = \frac{1}{1-\lambda} - \frac{2}{(x+1)(x+2)} \\
 & \Updownarrow \\
 & \frac{x(x+3)\lambda}{(x+1)(x+2)} + \sum_{m=2}^{\infty} \lambda^m = \frac{\lambda}{1-\lambda} - \frac{2\lambda}{(x+1)(x+2)} \\
 & \Updownarrow \\
 & x \sum_{m=1}^{\infty} \left\{ \frac{x+3m}{(x+m)(x+2m)} \right\} \lambda^m + 2 \sum_{m=2}^{\infty} \left\{ \frac{m^2}{(x+m)(x+2m)} \right\} \lambda^m = \frac{\lambda}{1-\lambda} - \frac{2\lambda}{(x+1)(x+2)}. \tag{2.1}
 \end{aligned}$$

Suppose that

$$\frac{1}{(x+m)(x+2m)} = \frac{A}{x+m} + \frac{B}{x+2m}$$

or

$$1 = A(x+2m) + B(x+m).$$

Then, we get $A = 1/m$ and $B = -1/m$. Therefore,

$$\frac{1}{(x+m)(x+2m)} = \frac{1}{m} \left\{ \frac{1}{x+m} - \frac{1}{x+2m} \right\}. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\begin{aligned}
 & x \sum_{m=1}^{\infty} \left\{ \frac{(x+3m)}{m} \left(\frac{1}{x+m} - \frac{1}{x+2m} \right) \right\} \lambda^m + 2 \sum_{m=2}^{\infty} m \left(\frac{1}{x+m} - \frac{1}{x+2m} \right) \lambda^m \\
 & = \frac{\lambda}{1-\lambda} - \frac{2\lambda}{(x+1)(x+2)} \\
 & \Updownarrow \\
 & x^2 \sum_{m=1}^{\infty} \left\{ \frac{1}{x+m} - \frac{1}{2\left(\frac{x}{2}+m\right)} \right\} \frac{\lambda^m}{m} + 3x \sum_{m=1}^{\infty} \left\{ \frac{1}{x+m} - \frac{1}{2\left(\frac{x}{2}+m\right)} \right\} \lambda^m \\
 & = \frac{\lambda}{1-\lambda} - \frac{2\lambda}{(x+1)(x+2)} - 2 \sum_{m=2}^{\infty} m \left\{ \frac{1}{x+m} - \frac{1}{2\left(\frac{x}{2}+m\right)} \right\} \lambda^m \\
 & \Updownarrow \\
 & x^2 \sum_{m=0}^{\infty} \left\{ \frac{1}{x+m+1} - \frac{1}{2\left(\frac{x}{2}+m+1\right)} \right\} \frac{\lambda^{m+1}}{m+1} + 3x \sum_{m=0}^{\infty} \left\{ \frac{1}{x+m+1} - \frac{1}{2\left(\frac{x}{2}+m+1\right)} \right\} \lambda^{m+1} \\
 & = \frac{\lambda}{1-\lambda} - \frac{2\lambda}{(x+1)(x+2)} - 2 \sum_{m=0}^{\infty} (m+2) \left\{ \frac{1}{x+m+2} - \frac{1}{2\left(\frac{x}{2}+m+2\right)} \right\} \lambda^{m+2} \\
 & \Updownarrow
 \end{aligned}$$

$$\begin{aligned}
 & x^2 \left\{ \int_0^\lambda \left(\Phi(t, 1, x+1) - \frac{\Phi(t, 1, \frac{x}{2} + 1)}{2} \right) dt \right\} + 3\lambda x \left\{ \Phi(\lambda, 1, x+1) - \frac{\Phi(\lambda, 1, \frac{x}{2} + 1)}{2} \right\} \\
 &= \frac{\lambda}{1-\lambda} - \frac{2\lambda}{(x+1)(x+2)} - 2\lambda \frac{\partial}{\partial \lambda} \left\{ \lambda^2 \left(\Phi(\lambda, 1, x+2) - \frac{\Phi(\lambda, 1, \frac{x}{2} + 2)}{2} \right) \right\}.
 \end{aligned}$$

Then, we obtain the following theorem:

Theorem 2.1. *Let $x \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $|\lambda| < 1$. Then, we have*

$$\begin{aligned}
 & \frac{1}{1-\lambda} - \frac{2}{(x+1)(x+2)} - \frac{x^2}{\lambda} \left\{ \int_0^\lambda \left(\Phi(t, 1, x+1) - \frac{\Phi(t, 1, \frac{x}{2} + 1)}{2} \right) dt \right\} \\
 &= 3x \left\{ \Phi(\lambda, 1, x+1) - \frac{\Phi(\lambda, 1, \frac{x}{2} + 1)}{2} \right\} + 4\lambda \left\{ \Phi(\lambda, 1, x+2) - \frac{\Phi(\lambda, 1, \frac{x}{2} + 2)}{2} \right\} \\
 &+ 2\lambda^2 \frac{\partial}{\partial \lambda} \left\{ \Phi(\lambda, 1, x+2) - \frac{\Phi(\lambda, 1, \frac{x}{2} + 2)}{2} \right\}.
 \end{aligned}$$

For $|z| < 1$ and $w \in \mathbb{C} \setminus \mathbb{Z}_0^-$, the integral representation of the Hurwitz-Lerch zeta function with order 1 is given by

$$\Phi(\lambda, 1, x) = \int_0^1 \frac{u^{x-1}}{1-\lambda u} du. \tag{2.3}$$

Also, by using partial derivative, we get

$$\begin{aligned}
 \frac{\partial}{\partial \lambda} \left\{ \lambda^2 \Phi(\lambda, 1, x) \right\} &= \frac{\partial}{\partial \lambda} \left\{ \int_0^1 \frac{\lambda^2 u^{x-1}}{1-\lambda u} du \right\} \\
 &= \int_0^1 \frac{(2\lambda - u\lambda^2)u^{x-1}}{(1-\lambda u)^2} du.
 \end{aligned} \tag{2.4}$$

By using (2.3) and (2.4) into Theorem 2.1, we have

$$\begin{aligned}
 & \frac{1}{1-\lambda} - \frac{2}{(x+1)(x+2)} - \frac{x^2}{\lambda} \int_0^\lambda \left\{ \int_0^1 \frac{u^x}{1-tu} du - \frac{1}{2} \int_0^1 \frac{u^{x/2}}{1-tu} du \right\} dt \\
 &= 3x \left\{ \int_0^1 \frac{u^x}{1-\lambda u} du - \frac{1}{2} \int_0^1 \frac{u^{x/2}}{1-\lambda u} du \right\} + 4\lambda \left\{ \int_0^1 \frac{u^{x+1}}{1-\lambda u} du - \frac{1}{2} \int_0^1 \frac{u^{(x/2)+1}}{1-\lambda u} du \right\} \\
 &+ 2\lambda^2 \frac{\partial}{\partial \lambda} \left\{ \int_0^1 \frac{u^{x+1}}{1-\lambda u} du - \frac{1}{2} \int_0^1 \frac{u^{(x/2)+1}}{1-\lambda u} du \right\} \\
 &\quad \Downarrow
 \end{aligned}$$

$$\begin{aligned} & \frac{1}{1-\lambda} - \frac{2}{(x+1)(x+2)} - \frac{x^2}{\lambda} \int_0^1 \left(u^x - \frac{u^{x/2}}{2}\right) \left\{ \int_0^\lambda \frac{dt}{1-tu} \right\} du \\ = & 3x \int_0^1 \left(u^x - \frac{u^{x/2}}{2}\right) \frac{1}{1-\lambda u} du + 2\lambda \int_0^1 \left(u^{x+1} - \frac{u^{(x/2)+1}}{2}\right) \frac{(2-\lambda u)}{(1-\lambda u)^2} du. \end{aligned}$$

We note that

$$\begin{aligned} \int_0^\lambda \frac{dt}{1-tu} &= -\frac{1}{u} \int_0^\lambda \frac{(-u)}{1-tu} dt \\ &= -\frac{\ln(1-\lambda u)}{u}. \end{aligned}$$

Consequently, we arrive at the following corollary:

Corollary 2.2. *Let $x \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $|\lambda| < 1$. Then, we have*

$$\begin{aligned} & \frac{1}{1-\lambda} - \frac{2}{(x+1)(x+2)} - \frac{x^2}{\lambda} \int_0^1 \left(u^x - \frac{u^{x/2}}{2}\right) \ln(1-\lambda u) du \\ = & 3x \int_0^1 \left(u^x - \frac{u^{x/2}}{2}\right) \frac{1}{1-\lambda u} du + 2\lambda \int_0^1 \left(u^{x+1} - \frac{u^{(x/2)+1}}{2}\right) \frac{(2-\lambda u)}{(1-\lambda u)^2} du. \end{aligned}$$

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References

- [1] A. A. Aygunes, *Some identities for a special case of Hurwitz-Lerch zeta function*, Numerical Anal. Appl. Math. AIP Conf. Proc. **1863** (1), 300018, 2016.
- [2] A. A. Aygunes, *Hurwitz-Lerch zeta function with order 1*, J. Math. Sci. **1**, 16-24, 2017.
- [3] A. Erdélyi, *Higher Transcendental Functions, vol. 1*, McGraw-Hill, New York, 1955.
- [4] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions, vol. 1*, McGraw-Hill Book Company, New York, Toronto and London, 1953.
- [5] E. C. Titchmarsh, *The Theory of the Riemann-zeta function*, Oxford University (Clarendon) Press, Oxford and London, 1951; Second Edition (Revised by D. R. Heath-Brown), 1986.
- [6] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.
- [7] H. M. Srivastava, *Some properties and results involving the zeta and associated functions*, Functional Analysis, Approximation and Computation **7** (2), 89-133, 2015.
- [8] J. Choi, *Remark on the Hurwitz-Lerch zeta function*, Fixed Point Theory and Applications **70**, 2-10, 2013.
- [9] L. Lewin, *Polylogarithms and Associated Functions*, North-Holland, New York, 1981.
- [10] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, Heidelberg and Berlin, 1976.