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Connections between Various Subclasses of Uniformly Harmonic Starlike Mappings and Poisson Distribution Series

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Abstract

In this paper, we use a power series with coefficients are the probabilities of Poisson distribution and obtain sufficient conditions for this power series and some related series to be in various subclasses of harmonic functions. Also, we investigate several mapping properties involving these subclasses.

Keywords: Harmonic starlike, Harmonic γ-uniformly starlike, Poisson distribution series, Univalent functions

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1. Introduction and Definition

Let $\tilde{H}$ be the family of all harmonic functions of the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{\kappa=2}^{\infty} A_\kappa z^\kappa, \qquad g(z) = \sum_{\kappa=1}^{\infty} B_\kappa z^\kappa, \qquad |B_1| < 1 \tag{1.1}$$

are analytic functions in $\Delta = \{z : z \in \mathbb{C}, |z| < 1\}$.

Let $\mathcal{S}_{\tilde{H}}$ be a subclass of $\tilde{H}$ which is univalent and sense-preserving in Δ. Also, we let $\mathcal{S}^0_{\tilde{H}}$ be a subclass of $\mathcal{S}_{\tilde{H}}$ as follows:

$$\mathcal{S}^0_{\tilde{H}} = \{f \in \mathcal{S}_{\tilde{H}} : f = h + \bar{g},\ g'(0) = B_1 = 0\}.$$

These classes are studied by Clunie and Sheil-Small [4]. Also, we let $\mathcal{K}^0_{\tilde{H}}$, $\mathcal{S}^{*,0}_{\tilde{H}}$ and $C^0_{\tilde{H}}$ denote the subclasses of $\mathcal{S}^0_{\tilde{H}}$ which are convex, starlike and close-to-convex in Δ (see [1, 4] and [5]).

Let $\mathcal{T}_{\tilde{H}}$ be the family of all harmonic functions of the form $f = h + \bar{g}$, where

$$h(z) = z - \sum_{\kappa=2}^{\infty} |A_\kappa| z^\kappa, \qquad g(z) = \sum_{\kappa=1}^{\infty} |B_\kappa| z^\kappa \qquad (z \in \Delta). \tag{1.2}$$

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This class is studied by Silverman [11]. For $0 \leq \alpha < 1$, $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$, let

$$\mathcal{N}_{\tilde{H}}(\alpha) = \left\{ f \in \tilde{H} : \operatorname{Re}\left(\frac{f'(z)}{z'}\right) \geq \alpha,\ z = re^{i\theta} \right\},$$

where

$$f'(z) = \frac{\partial}{\partial\theta} f(re^{i\theta}) = i(zh'(z) - \overline{zg'(z)}), \qquad z' = \frac{\partial}{\partial\theta}(re^{i\theta})$$

and define

$$\mathcal{T}\mathcal{N}_{\tilde{H}}(\alpha) = \mathcal{N}_{\tilde{H}}(\alpha) \cap \mathcal{T}_{\tilde{H}}.$$

These classes are studied by Ahuja and Jahangiri [3].

Definition 1.1. [2] A function $f = h + \bar{g}$ is said to be γ-uniformly harmonic starlike functions in Δ if satisfied the following condition:

$$\operatorname{Re}\left(\frac{zf'(z)}{z'[(1-\eta)z + \eta(h(z) + \overline{g(z)})]} - \delta\right) \geq \gamma \left| \frac{zf'(z)}{z'[(1-\eta)z + \eta(h(z) + \overline{g(z)})]} - 1 \right|,$$

where $0 \leq \eta \leq 1$, $0 \leq \delta < 1$, $0 \leq \gamma < \infty$ and

$$f'(z) = \frac{\partial}{\partial\theta} f(re^{i\theta}) = i(zh'(z) - \overline{zg'(z)}), \qquad z' = \frac{\partial}{\partial\theta}(re^{i\theta}) \qquad (0 \leq r < 1,\ 0 \leq \theta \leq 2\pi).$$

The family of this functions is denoted by $\mathfrak{G}_{\tilde{H}}(\gamma, \delta, \eta)$. Also, define $\mathfrak{T}\mathfrak{G}_{\tilde{H}}(\gamma, \delta, \eta) = \mathfrak{G}_{\tilde{H}}(\gamma, \delta, \eta) \cap \mathcal{T}_{\tilde{H}}$.

The above-defined class includes several simpler subclasses. We point out here some of these special cases as follows:

(a) Putting $\gamma = 0$ and $\eta = 0$, we obtain $N_{\tilde{H}}(\delta)$, which was studied by Ahuja and Jahangiri [3];

(b) Putting $\gamma = 0$ and $\eta = 1$, we obtain $S^*_{\tilde{H}}(\delta)$, which was studied by Jahangiri [6];

(c) Putting $\eta = 1$ and $\delta = 0$, we obtain $G^*_{\tilde{H}}(\gamma)$, which was studied by Rosy et al. [10];

(d) Putting $\gamma = 1$, $\delta = 0$, $\eta = 1$ and $g(z) \equiv 0$, we obtain US^*, which was studied by Rønning [9];

(e) Putting $\eta = 1$, we obtain $HUS^*(\gamma, \delta)$, which was defined by Porwal and Srivastava [8].

Recently, Porwal [7] defined a Poisson distribution series as

$$\mathcal{M}(s, z) = z + \sum_{\kappa=2}^{\infty} \frac{s^{\kappa-1}}{(\kappa-1)!} e^{-s} z^{\kappa} \qquad (s > 0,\ z \in \Delta).$$

Now, for $s_1,\ s_2 > 0$ and $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ given by (1.1), we define the linear operator

$$\mathcal{R}(s_1, s_2)(f) = \mathcal{M}(s_1, z) \star h(z) + \overline{\mathcal{M}(s_2, z) \star g(z)} = \mathcal{H}(z) + \overline{\mathcal{Q}(z)},$$

the symbol $\star$ denote the convolution of two functions and

$$\mathcal{H}(z) = z + \sum_{\kappa=2}^{\infty} \frac{s_1^{\kappa-1}}{(\kappa-1)!} e^{-s_1} A_{\kappa} z^{\kappa}, \qquad \mathcal{Q}(z) = B_1 z + \sum_{\kappa=2}^{\infty} \frac{s_2^{\kappa-1}}{(\kappa-1)!} e^{-s_2} B_{\kappa} z^{\kappa} \qquad (z \in \Delta). \tag{1.3}$$

In this work, the authors obtain sufficient conditions for the power series with coefficients are the probabilities of Poisson distribution and some related series to be in some subclasses of γ–uniformly harmonic starlike functions. In order to establish sufficient conditions on the parameters for the inclusion relations, we need the following lemmas:

Lemma 1.2. *[4] If $f \in \mathcal{K}^0_{\tilde{H}}$ and $f = h + \bar{g}$ where h and g are given by (1.1) with $B_1 = 0$, then*

$$|A_\kappa| \le \frac{\kappa+1}{2}, \qquad |B_\kappa| \le \frac{\kappa-1}{2} \qquad (\kappa \ge 1).$$

Lemma 1.3. *[1] If $f \in C^0_{\tilde{H}}$ or $\mathcal{S}^{*,0}_{\tilde{H}}$ and $f = h + \bar{g}$ where h and g are given by (1.1) with $B_1 = 0$, then*

$$|A_\kappa| \le \frac{(2\kappa+1)(\kappa+1)}{6}, \qquad |B_\kappa| \le \frac{(2\kappa-1)(\kappa-1)}{6} \qquad (\kappa \ge 1).$$

Lemma 1.4. *[3] If $f \in \mathcal{TN}_{\tilde{H}}(\alpha)$ and $f = h + \bar{g}$ where h and g are given by (1.2), then*

$$|A_\kappa| \le \frac{1-\alpha}{\kappa} \quad (\kappa \ge 2), \qquad |B_\kappa| \le \frac{1-\alpha}{\kappa} \quad (\kappa \ge 1,\ 0 \le \alpha < 1).$$

Lemma 1.5. *[2] For $0 \le \eta \le 1$, $0 \le \delta < 1$, $0 \le \gamma < \infty$ and $f = h + \bar{g}$, where h and g are given by (1.1). If the following condition*

$$\sum_{\kappa=2}^{\infty} \frac{\kappa(\gamma+1) - \eta(\gamma+\delta)}{1-\delta}|A_\kappa| + \sum_{\kappa=1}^{\infty} \frac{\kappa(\gamma+1) + \eta(\gamma+\delta)}{1-\delta}|B_\kappa| \le 1 \tag{1.4}$$

is true, then f is sense-preserving and harmonic mapping in Δ and $f \in \mathfrak{G}_{\tilde{H}}(\gamma, \delta, \eta)$.

Lemma 1.6. *[2] For $0 \le \eta \le 1$, $0 \le \delta < 1$, $0 \le \gamma < \infty$ and $f = h + \bar{g}$, where h and g are given by (1.2). A function $f(z) \in \mathfrak{T}\mathfrak{G}_{\tilde{H}}(\gamma, \delta, \eta)$ if and only if the condition (1.4) holds. Moreover, if $f(z) \in \mathfrak{T}\mathfrak{G}_{\tilde{H}}(\gamma, \delta, \eta)$, then*

$$\begin{aligned} |A_\kappa| &\le \frac{1-\delta}{\kappa(\gamma+1) - \eta(\gamma+\delta)} \qquad (\kappa \ge 2), \\ |B_\kappa| &\le \frac{1-\delta}{\kappa(\gamma+1) + \eta(\gamma+\delta)} \qquad (\kappa \ge 1). \end{aligned}$$

2. Main Results

Unless otherwise mentioned, we suppose that $s_1, s_2 > 0$, $0 \le \eta \le 1$, $0 \le \delta < 1$ and $0 \le \gamma < \infty$.

Theorem 2.1. *If the following inequality*

$$(\gamma+1)(s_1^2 + s_2^2 + 4s_1 + 2s_2 + 2(1-e^{-s_1})) - \eta(\gamma+\delta)(s_1 - s_2 + 2(1-e^{-s_1})) \le 2(1-\delta)$$

is holds, then $\mathcal{R}(s_1, s_2)(\mathcal{K}^0_{\tilde{H}}) \subset \mathfrak{G}_{\tilde{H}}(\gamma, \delta, \eta)$.

Proof. Let $f \in \mathcal{K}^0_{\tilde{H}}$ with $B_1 = 0$, then we will show that $\mathcal{R}(s_1, s_2)(f) \in \mathfrak{G}_{\tilde{H}}(\gamma, \delta, \eta)$. Now, from Lemma 1.5, we prove that

$$T_1 = \sum_{\kappa=2}^{\infty}[\kappa(\gamma+1) - \eta(\gamma+\delta)]\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1}|A_\kappa| + \sum_{\kappa=1}^{\infty}[\kappa(\gamma+1) + \eta(\gamma+\delta)]\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2}|B_\kappa| \le 1-\delta.$$

As a consequence of Lemma 1.2, it follows that

$$\begin{aligned} T_1 &\le \sum_{\kappa=2}^{\infty}[\kappa(\gamma+1) - \eta(\gamma+\delta)]\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)\left(\frac{\kappa+1}{2}\right)e^{-s_1} + \sum_{\kappa=1}^{\infty}[\kappa(\gamma+1) + \eta(\gamma+\delta)]\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)\left(\frac{\kappa-1}{2}\right)e^{-s_2} \\ &= \frac{1}{2}\left[\sum_{\kappa=2}^{\infty}\kappa(\kappa+1)(\gamma+1)\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1} - \sum_{\kappa=2}^{\infty}\eta(\kappa+1)(\gamma+\delta)\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1}\right. \\ &+ \left.\sum_{\kappa=1}^{\infty}\kappa(\kappa-1)(\gamma+1)\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2} + \sum_{\kappa=1}^{\infty}\eta(\kappa-1)(\gamma+\delta)\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2}\right]. \end{aligned}$$

Using the following equations

$$\begin{aligned}
\sum_{\kappa=2}^{\infty}\kappa(\kappa+1)\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1} &= \sum_{\kappa=3}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-3)!}\right)e^{-s_1}+4\sum_{\kappa=2}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-2)!}\right)e^{-s_1}+2\sum_{\kappa=2}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1}\\
&= \sum_{\kappa=0}^{\infty}s_1^2\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1}+4\sum_{\kappa=0}^{\infty}s_1\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1}+2\sum_{\kappa=1}^{\infty}\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1}\\
&= s_1^2+4s_1+2(1-e^{-s_1}),\\
\sum_{\kappa=2}^{\infty}(\kappa+1)\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1} &= \sum_{\kappa=2}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-2)!}\right)e^{-s_1}+2\sum_{\kappa=2}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1}\\
&= \sum_{\kappa=0}^{\infty}s_1\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1}+2\sum_{\kappa=1}^{\infty}\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1}\\
&= s_1+2(1-e^{-s_1}),\\
\sum_{\kappa=1}^{\infty}\kappa(\kappa-1)\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2} &= \sum_{\kappa=3}^{\infty}\left(\frac{s_2^{\kappa-1}}{(\kappa-3)!}\right)e^{-s_2}+2\sum_{\kappa=2}^{\infty}\left(\frac{s_2^{\kappa-1}}{(\kappa-2)!}\right)e^{-s_2}\\
&= \sum_{\kappa=0}^{\infty}s_2^2\left(\frac{s_2^{\kappa}}{\kappa!}\right)e^{-s_2}+2\sum_{\kappa=0}^{\infty}s_2\left(\frac{s_2^{\kappa}}{\kappa!}\right)e^{-s_2}\\
&= s_2^2+2s_2,
\end{aligned}$$

we obtain

$$\begin{aligned}
T_1 &\leq \frac{1}{2}\Big[(\gamma+1)(s_1^2+4s_1+2(1-e^{-s_1}))-\eta(\gamma+\delta)(s_1+2(1-e^{-s_1}))+(\gamma+1)(s_2^2+2s_2)+\eta(\gamma+\delta)s_2\Big]\\
&\leq 1-\delta.
\end{aligned}$$

Therefore the proof of Theorem 2.1 is completed. □

Remark 2.2. Putting $\eta = 1$ in Theorem 2.1, we improve the result obtained in [8, Theorem 3.2].

Theorem 2.3. *If the following inequality*

$$(\gamma+1)(2s_1^3+2s_2^3+15s_1^2+9s_2^2+24s_1+6s_2+6(1-e^{-s_1}))-\eta(\gamma+\delta)(2s_1^2-2s_2^2+9s_1-3s_2+6(1-e^{-s_1}))\leq 6(1-\delta)$$

is holds, then $\mathcal{R}(s_1,s_2)(C^0_{\tilde{H}})\subset\mathfrak{G}_{\tilde{H}}(\gamma,\delta,\eta)$ *or* $\mathcal{R}(s_1,s_2)(\mathcal{S}^{*,0}_{\tilde{H}})\subset\mathfrak{G}_{\tilde{H}}(\gamma,\delta,\eta)$.

Proof. Let $f\in C^0_{\tilde{H}}$ or $f\in\mathcal{S}^{*,0}_{\tilde{H}}$ with $B_1=0$, then we will show that $\mathcal{R}(s_1,s_2)(f)\in\mathfrak{G}_{\tilde{H}}(\gamma,\delta,\eta)$. Now, from Lemma 1.5, we prove that

$$T_2=\sum_{\kappa=2}^{\infty}[\kappa(\gamma+1)-\eta(\gamma+\delta)]\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1}|A_\kappa|+\sum_{\kappa=1}^{\infty}[\kappa(\gamma+1)+\eta(\gamma+\delta)]\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2}|B_\kappa|\leq 1-\delta.$$

As a consequence of Lemma 1.3, it follows that

$$\begin{aligned}
T_2 &\leq \sum_{\kappa=2}^{\infty}[\kappa(\gamma+1)-\eta(\gamma+\delta)]\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)\left(\frac{(2\kappa+1)(\kappa+1)}{6}\right)e^{-s_1}\\
&+ \sum_{\kappa=1}^{\infty}[\kappa(\gamma+1)+\eta(\gamma+\delta)]\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)\left(\frac{(2\kappa-1)(\kappa-1)}{6}\right)e^{-s_2}\\
&= \frac{1}{6}\Bigg[\sum_{\kappa=2}^{\infty}\kappa(2\kappa+1)(\kappa+1)(\gamma+1)\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1}-\sum_{\kappa=2}^{\infty}\eta(2\kappa+1)(\kappa+1)(\gamma+\delta)\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1}\\
&+ \sum_{\kappa=1}^{\infty}\kappa(2\kappa-1)(\kappa-1)(\gamma+1)\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2}+\sum_{\kappa=1}^{\infty}\eta(2\kappa-1)(\kappa-1)(\gamma+\delta)\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2}\Bigg].
\end{aligned}$$

Using the following equations

$$\begin{aligned}\sum_{\kappa=2}^{\infty}\kappa(2\kappa+1)(\kappa+1)\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1} &= 2\sum_{\kappa=4}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-4)!}\right)e^{-s_1}+15\sum_{\kappa=3}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-3)!}\right)e^{-s_1}+24\sum_{\kappa=2}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-2)!}\right)e^{-s_1}\\ &+ 6\sum_{\kappa=2}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1}\\ &= 2\sum_{\kappa=0}^{\infty}s_1^3\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1}+15\sum_{\kappa=0}^{\infty}s_1^2\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1}+24\sum_{\kappa=0}^{\infty}s_1\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1}+6\sum_{\kappa=1}^{\infty}\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1}\\ &= 2s_1^3+15s_1^2+24s_1+6(1-e^{-s_1}),\end{aligned}$$

$$\begin{aligned}\sum_{\kappa=2}^{\infty}(2\kappa+1)(\kappa+1)\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1} &= 2\sum_{\kappa=3}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-3)!}\right)e^{-s_1}+9\sum_{\kappa=2}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-2)!}\right)e^{-s_1}+6\sum_{\kappa=2}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1}\\ &= 2\sum_{\kappa=0}^{\infty}s_1^2\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1}+9\sum_{\kappa=0}^{\infty}s_1\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1}+6\sum_{\kappa=1}^{\infty}\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1}\\ &= 2s_1^2+9s_1+6(1-e^{-s_1}),\end{aligned}$$

$$\begin{aligned}\sum_{\kappa=1}^{\infty}\kappa(2\kappa-1)(\kappa-1)\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2} &= 2\sum_{\kappa=4}^{\infty}\left(\frac{s_2^{\kappa-1}}{(\kappa-4)!}\right)e^{-s_2}+9\sum_{\kappa=3}^{\infty}\left(\frac{s_2^{\kappa-1}}{(\kappa-3)!}\right)e^{-s_2}+6\sum_{\kappa=2}^{\infty}\left(\frac{s_2^{\kappa-1}}{(\kappa-2)!}\right)e^{-s_2}\\ &= 2\sum_{\kappa=0}^{\infty}s_2^3\left(\frac{s_2^{\kappa}}{\kappa!}\right)e^{-s_2}+9\sum_{\kappa=0}^{\infty}s_2^2\left(\frac{s_2^{\kappa}}{\kappa!}\right)e^{-s_2}+6\sum_{\kappa=0}^{\infty}s_2\left(\frac{s_2^{\kappa}}{\kappa!}\right)e^{-s_2}\\ &= 2s_2^3+9s_2^2+6s_2,\end{aligned}$$

$$\begin{aligned}\sum_{\kappa=1}^{\infty}(2\kappa-1)(\kappa-1)\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2} &= 2\sum_{\kappa=3}^{\infty}\left(\frac{s_2^{\kappa-1}}{(\kappa-3)!}\right)e^{-s_2}+3\sum_{\kappa=2}^{\infty}\left(\frac{s_2^{\kappa-1}}{(\kappa-2)!}\right)e^{-s_2}\\ &= 2\sum_{\kappa=0}^{\infty}s_2^2\left(\frac{s_2^{\kappa}}{\kappa!}\right)e^{-s_2}+3\sum_{\kappa=0}^{\infty}s_2\left(\frac{s_2^{\kappa}}{\kappa!}\right)e^{-s_2}\\ &= 2s_2^2+3s_2,\end{aligned}$$

we obtain

$$\begin{aligned}T_2 &\le \frac{1}{6}\Big[(\gamma+1)(2s_1^3+15s_1^2+24s_1+6(1-e^{-s_1}))-\eta(\gamma+\delta)(2s_1^2+9s_1+6(1-e^{-s_1}))\\ &+(\gamma+1)(2s_2^3+9s_2^2+6s_2)+\eta(\gamma+\delta)(2s_2^2+3s_2)\Big]\\ &\le 1-\delta.\end{aligned}$$

Therefore the proof of Theorem 2.3 is completed. □

Remark 2.4. Putting $\eta = 1$ in Theorem 2.3, we improve the result obtained in [8, Theorem 3.4].

Theorem 2.5. *If the following inequality*

$$\begin{aligned}(1-\alpha)\Bigg[(\gamma+1)(2-e^{-s_1}-e^{-s_2})-\frac{\eta(\gamma+\delta)}{s_1}[1-e^{-s_1}(1+s_1)]\\ +\frac{\eta(\gamma+\delta)}{s_2}[1-e^{-s_2}(1+s_2)]\Bigg] \le 1-\delta-[\gamma+1+\eta(\gamma+\delta)]|B_1|\end{aligned}$$

is holds, then $\mathcal{R}(s_1,s_2)(\mathcal{T}\mathcal{N}_{\tilde{H}}) \subset \mathfrak{G}_{\tilde{H}}(\gamma,\delta,\eta)$.

Proof. Let $f \in \mathcal{TN}_{\tilde{H}}$, then we will show that $\mathcal{R}(s_1, s_2)(f) \in \mathfrak{G}_{\tilde{H}}(\gamma, \delta, \eta)$. Now, from Lemma 1.5, we prove that

$$\begin{aligned} T_3 &= \sum_{\kappa=2}^{\infty}[\kappa(\gamma+1)-\eta(\gamma+\delta)]\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1}|A_\kappa| + \sum_{\kappa=2}^{\infty}[\kappa(\gamma+1)+\eta(\gamma+\delta)]\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2}|B_\kappa| \\ &+ [\gamma+1+\eta(\gamma+\delta)]|B_1| \leq 1-\delta. \end{aligned}$$

As a consequence of Lemma 1.4, it follows that

$$\begin{aligned} T_3 &\leq \sum_{\kappa=2}^{\infty}[\kappa(\gamma+1)-\eta(\gamma+\delta)]\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)\left(\frac{1-\alpha}{\kappa}\right)e^{-s_1} + \sum_{\kappa=2}^{\infty}[\kappa(\gamma+1)+\eta(\gamma+\delta)]\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)\left(\frac{1-\alpha}{\kappa}\right)e^{-s_2} \\ &+ [\gamma+1+\eta(\gamma+\delta)]|B_1| \\ &= (1-\alpha)\left[\sum_{\kappa=2}^{\infty}[\kappa(\gamma+1)-\eta(\gamma+\delta)]\left(\frac{s_1^{\kappa-1}}{\kappa!}\right)e^{-s_1} + \sum_{\kappa=2}^{\infty}[\kappa(\gamma+1)+\eta(\gamma+\delta)]\left(\frac{s_2^{\kappa-1}}{\kappa!}\right)e^{-s_2}\right] \\ &+ [\gamma+1+\eta(\gamma+\delta)]|B_1| \\ &= (1-\alpha)\left[\sum_{\kappa=2}^{\infty}(\gamma+1)\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1} - \sum_{\kappa=2}^{\infty}\eta(\gamma+\delta)\left(\frac{s_1^{\kappa-1}}{\kappa!}\right)e^{-s_1} + \sum_{\kappa=2}^{\infty}(\gamma+1)\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2}\right. \\ &+ \left.\sum_{\kappa=2}^{\infty}\eta(\gamma+\delta)\left(\frac{s_2^{\kappa-1}}{\kappa!}\right)e^{-s_2}\right] + [\gamma+1+\eta(\gamma+\delta)]|B_1|. \end{aligned}$$

Using the following equations

$$\begin{aligned} \sum_{\kappa=2}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1} &= \sum_{\kappa=1}^{\infty}\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1} = 1-e^{-s_1}, \\ \sum_{\kappa=2}^{\infty}\left(\frac{s_1^{\kappa-1}}{\kappa!}\right)e^{-s_1} &= \sum_{\kappa=2}^{\infty}\left(\frac{1}{s_1}\right)\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1} = \left(\frac{1}{s_1}\right)(1-e^{-s_1}(1+s_1)), \\ \sum_{\kappa=2}^{\infty}\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2} &= \sum_{\kappa=1}^{\infty}\left(\frac{s_2^{\kappa}}{\kappa!}\right)e^{-s_2} = 1-e^{-s_2}, \\ \sum_{\kappa=2}^{\infty}\left(\frac{s_2^{\kappa-1}}{\kappa!}\right)e^{-s_2} &= \sum_{\kappa=2}^{\infty}\left(\frac{1}{s_2}\right)\left(\frac{s_2^{\kappa}}{\kappa!}\right)e^{-s_2} = \left(\frac{1}{s_2}\right)(1-e^{-s_2}(1+s_2)), \end{aligned}$$

we obtain

$$\begin{aligned} T_3 &\leq (1-\alpha)\left[(\gamma+1)(2-e^{-s_1}-e^{-s_2}) - \frac{\eta(\gamma+\delta)}{s_1}[1-e^{-s_1}(1+s_1)] + \frac{\eta(\gamma+\delta)}{s_2}[1-e^{-s_2}(1+s_2)]\right] \\ &+ [\gamma+1+\eta(\gamma+\delta)]|B_1| \leq 1-\delta. \end{aligned}$$

Therefore the proof of Theorem 2.5 is completed. □

Remark 2.6. Putting $\eta = 1$ in Theorem 2.5, we obtain the result obtained in [8, Theorem 2.7].

Theorem 2.7. *If the following inequality*

$$e^{-s_1} + e^{-s_2} \geq 1 + \frac{\gamma+1+\eta(\gamma+\delta)}{1-\delta}|B_1|$$

is holds, then $\mathcal{R}(s_1, s_2)(\mathfrak{T}\mathfrak{G}_{\tilde{H}}(\gamma, \delta, \eta)) \subset \mathfrak{G}_{\tilde{H}}(\gamma, \delta, \eta)$.

Proof. Let $f \in \mathfrak{T}\mathfrak{G}_{\tilde{H}}(\gamma,\delta,\eta)$, then we will show that $\mathcal{R}(s_1,s_2)(f) \in \mathfrak{G}_{\tilde{H}}(\gamma,\delta,\eta)$. Now, from Lemma 1.5, we prove that

$$\begin{aligned} T_4 &= \sum_{\kappa=2}^{\infty}[\kappa(\gamma+1)-\eta(\gamma+\delta)]\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1}|A_\kappa| + \sum_{\kappa=2}^{\infty}[\kappa(\gamma+1)+\eta(\gamma+\delta)]\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2}|B_\kappa| \\ &+ [\gamma+1+\eta(\gamma+\delta)]|B_1| \le 1-\delta. \end{aligned}$$

As a consequence of Lemma 1.6, it follows that

$$\begin{aligned} T_4 &\le \sum_{\kappa=2}^{\infty}[\kappa(\gamma+1)-\eta(\gamma+\delta)]\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)\left(\frac{1-\delta}{\kappa(\gamma+1)-\eta(\gamma+\delta)}\right)e^{-s_1} \\ &+ \sum_{\kappa=2}^{\infty}[\kappa(\gamma+1)+\eta(\gamma+\delta)]\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)\left(\frac{1-\delta}{[\kappa(\gamma+1)+\eta(\gamma+\delta)]}\right)e^{-s_2} + [\gamma+1+\eta(\gamma+\delta)]|B_1| \\ &= (1-\delta)\left[\sum_{\kappa=2}^{\infty}\left(\frac{s_1^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_1} + \sum_{\kappa=2}^{\infty}\left(\frac{s_2^{\kappa-1}}{(\kappa-1)!}\right)e^{-s_2}\right] + [\gamma+1+\eta(\gamma+\delta)]|B_1| \\ &= (1-\delta)\left[\sum_{\kappa=1}^{\infty}\left(\frac{s_1^{\kappa}}{\kappa!}\right)e^{-s_1} + \sum_{\kappa=1}^{\infty}\left(\frac{s_2^{\kappa}}{\kappa!}\right)e^{-s_2}\right] + [\gamma+1+\eta(\gamma+\delta)]|B_1| \\ &= (1-\delta)[2-e^{-s_1}-e^{-s_2}] + [\gamma+1+\eta(\gamma+\delta)]|B_1| \le 1-\delta. \end{aligned}$$

Therefore the proof of Theorem 2.7 is completed. □

Remark 2.8. Putting $\eta = 1$ in Theorem 2.7, we obtain the result obtained in [8, Theorem 2.8].

Theorem 2.9. *The following inequality*

$$e^{-s_1} + e^{-s_2} \ge 1 + \frac{\gamma+1+\eta(\gamma+\delta)}{1-\delta}|B_1|$$

is satisfied if and only if $\mathcal{R}(s_1,s_2)(\mathfrak{T}\mathfrak{G}_{\tilde{H}}(\gamma,\delta,\eta)) \subset \mathfrak{T}\mathfrak{G}_{\tilde{H}}(\gamma,\delta,\eta)$.

Proof. The proof of the Theorem 2.9 is similar to that of Theorem 2.7 and so we omit it. □

3. Conclusion

In view of the numerous applications of Poisson distributions, we use the modified series $\mathcal{M}(s_1,z)$ and $\mathcal{M}(s_2,z)$ and the convolution to define the functions $\mathcal{H}(z)$ and $\mathcal{Q}(z)$ to establish sufficient conditions on the parameters s_1 and s_2 for which some subclasses of harmonic function included in the classes $\mathfrak{G}_{\tilde{H}}(\gamma,\delta,\eta)$ and $\mathfrak{T}\mathfrak{G}_{\tilde{H}}(\gamma,\delta,\eta)$.

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