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About a Subclass of Analytic Functions Defined by a Fractional Integral Operator

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Abstract

In this paper we have introduced and studied the subclass $\mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$ using the fractional integral operator associated with a linear differential operator. The main object is to investigate several properties such as coefficient estimates, distortion theorems, closure theorems, neighborhoods and the radii of starlikeness, convexity and close-to-convexity of functions belonging to the class $\mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$.

Keywords: Analytic functions, Univalent functions, Radii of starlikeness and convexity, Neighborhood property

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1. Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}(p,t) = \{f \in \mathcal{H}(U) : f(z) = z^p + \sum_{j=p+t}^{\infty} a_j z^j, \ z \in U\}$, with $\mathcal{A}(1,1) = \mathcal{A}$ and $\mathcal{H}[a,t] = \{f \in \mathcal{H}(U) : f(z) = a + a_t z^t + a_{t+1} z^{t+1} + \ldots, \ z \in U\}$, where $p, t \in \mathbb{N}$, $a \in \mathbb{C}$.

Using the convolution product studied in [1], we define the differential operator $D^m_{n,\delta,g} f$ and considering $g = f$ we obtain the operator $D^m_{n,\delta} f$, for which we find its fractional integral $D_z^{-\lambda} D^m_{n,\delta} f$

Definition 1.1. ([5]) Let $m, n \in \mathbb{N}$. Denote by $D^m_{n,\delta,g} : \mathcal{A} \to \mathcal{A}$ the linear differential operator defined by

$$\begin{aligned}
D^0_{n,\delta,g} f(z) &= (f * g)(z), \\
D^1_{n,\delta,g} f(z) &= [1 - (1-\delta)^n](f * g)(z) + (1-\delta)^n z (f * g)'(z), \\
&\ldots \\
D^m_{n,\delta,g} f(z) &= [1 - (1-\delta)^n] D^{m-1}_{n,\delta,g} f(z) + (1-\delta)^n z \left(D^{m-1}_{n,\delta,g} f(z)\right)', \quad z \in U,
\end{aligned}$$

for any $z \in U$ and each nonnegative integers m, n.

Denote by $D^m_{n,\delta} : \mathcal{A} \to \mathcal{A}$,

$$D^m_{n,\delta} f(z) = D^m_{n,\delta,f} f(z).$$

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If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then

$$D_{n,\delta}^m f(z) = z + \sum_{j=2}^{\infty} \left[1 + (j-1)(1-\delta)^n\right]^m a_j^2 z^j, \quad z \in U.$$

Definition 1.2. ([4]) The fractional integral of order λ ($\lambda > 0$) is defined for a function f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{1.1}$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.

From Definition 1.1 and Definition 1.2, we get the fractional integral associated with the linear differential operator,

$$D_z^{-\lambda} D_{n,\delta}^m f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{D_{n,\delta}^m f(t)}{(z-t)^{1-\lambda}} dt = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{t}{(z-t)^{1-\lambda}} dt + \sum_{j=2}^{\infty} \frac{\left[1 + (j-1)(1-\delta)^n\right]^m}{\Gamma(\lambda)} a_j^2 \int_0^z \frac{t^j}{(z-t)^{1-\lambda}} dt,$$

which has the following form, after a simple calculation,

$$D_z^{-\lambda} D_{n,\delta}^m f(z) = \frac{1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{\left[1 + (j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} a_j^2 z^{j+\lambda},$$

for the function $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A}$. We note that $D_z^{-\lambda} D_{n,\delta}^m f(z) \in \mathcal{A}(\lambda+1, 1)$.

We follow the works from [2] and [3].

Definition 1.3. Let the function $f \in \mathcal{A}$. Then f is said to be in the class $\mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$ if it satisfies the following criterion:

$$\left| \frac{1}{d} \left(\frac{z(D_z^{-\lambda} D_{n,\delta}^m f(z))' + \gamma z^2 (D_z^{-\lambda} D_{n,\delta}^m f(z))''}{(1-\gamma) D_z^{-\lambda} D_{n,\delta}^m f(z) + \gamma z (D_z^{-\lambda} D_{n,\delta}^m f(z))'} - 1 \right) \right| < \beta, \tag{1.2}$$

where $\lambda, \delta > 0$, $d \in \mathbb{C} - \{0\}$, $0 \le \gamma \le 1$, $0 < \beta \le 1$, $m, n \in \mathbb{N}$, $z \in U$.

In this paper we shall first deduce a necessary and sufficient condition for a function f to be in the class $\mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$. Then obtain the distortion and growth theorems, closure theorems, neighborhood and radii of univalent starlikeness, convexity and close-to-convexity of order k, $0 \le k < 1$, for these functions.

2. Coefficient Inequality

Theorem 2.1. *Let the function $f \in \mathcal{A}$. Then f is said to be in the class $\mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$ if and only if*

$$\sum_{j=2}^{\infty} \frac{\left[1 + (j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{ \lambda j^2 + \left[\gamma(2\lambda - 2 + \beta|d|) + 1\right] j + \left[\gamma(\lambda-1) + 1\right](\lambda - 1 + \beta|d|) \right\} a_j^2$$
$$\le \frac{(\gamma\lambda + 1)(\beta|d| - \lambda)}{\Gamma(\lambda+2)}, \tag{2.1}$$

where $\lambda, \delta > 0$, $d \in \mathbb{C} - \{0\}$, $0 \le \alpha \le 1$, $0 < \beta \le 1$, $m, n \in \mathbb{N}$, $z \in U$.

Proof. Let $f(z) \in \mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$. Assume that inequality (2.1) holds true. Then we find that

$$\left| \frac{z(D_z^{-\lambda} D_{n,\delta}^m f(z))' + \gamma z^2 (D_z^{-\lambda} D_{n,\delta}^m f(z))''}{(1-\gamma) D_z^{-\lambda} D_{n,\delta}^m f(z) + \gamma z (D_z^{-\lambda} D_{n,\delta}^m f(z))'} - 1 \right| =$$

$$\left| \frac{\frac{\lambda(\gamma\lambda+1)}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{[1+(j-1)(1-\delta)^n]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [2\gamma(\lambda-1)+1] j + (\lambda-1)[\gamma(\lambda-1)+1] \right\} a_j^2 z^{j+\lambda}}{\frac{\gamma\lambda+1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{[1+(j-1)(1-\delta)^n]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} [\gamma j + \gamma(\lambda-1)+1] a_j^2 z^{j+\lambda}} \right| \leq$$

$$\frac{\frac{\lambda(\gamma\lambda+1)}{\Gamma(\lambda+2)} + \sum_{j=2}^{\infty} \frac{[1+(j-1)(1-\delta)^n]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [2\gamma(\lambda-1)+1] j + (\lambda-1)[\gamma(\lambda-1)+1] \right\} a_j^2 \left|z^{j-1}\right|}{\frac{\gamma\lambda+1}{\Gamma(\lambda+2)} - \sum_{j=2}^{\infty} \frac{[1+(j-1)(1-\delta)^n]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} [\gamma j + \gamma(\lambda-1)+1] a_j^2 \left|z^{j-1}\right|} \leq \beta|d|.$$

Choosing values of z on real axis and letting $z \to 1^-$, we have

$$\sum_{j=2}^{\infty} \frac{[1+(j-1)(1-\delta)^n]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [\gamma(2\lambda-2+\beta|d|)+1] j + [\gamma(\lambda-1)+1](\lambda-1+\beta|d|) \right\} a_j^2$$

$$\leq \frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}.$$

Conversely, assume that $f(z) \in \mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$, then we get the following inequality

$$Re\left\{ \frac{z(D_z^{-\lambda} D_{n,\delta}^m f(z))' + \gamma z^2 (D_z^{-\lambda} D_{n,\delta}^m f(z))''}{(1-\gamma) D_z^{-\lambda} D_{n,\delta}^m f(z) + \gamma z (D_z^{-\lambda} D_{n,\delta}^m f(z))'} - 1 \right\} > -\beta|d|$$

$$Re\left\{ \frac{\frac{(\lambda+1)(\gamma\lambda+1)}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{[1+(j-1)(1-\delta)^n]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left[\gamma j^2 + (2\gamma\lambda-\gamma+1) j + \lambda(\gamma\lambda-\gamma+1) \right] a_j^2 z^{j+\lambda}}{\frac{\gamma\lambda+1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{[1+(j-1)(1-\delta)^n]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} [\gamma j + \gamma(\lambda-1)+1] a_j^2 z^{j+\lambda}} - 1 + \beta|d| \right\} > 0$$

$$Re \frac{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{[1+(j-1)(1-\delta)^n]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [\gamma(2\lambda-2+\beta|d|)+1] j + [\gamma(\lambda-1)+1](\lambda-1+\beta|d|) \right\} a_j^2 z^{j+\lambda}}{\frac{\gamma\lambda+1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{[1+(j-1)(1-\delta)^n]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} [\gamma j + \gamma(\lambda-1)+1] a_j^2 z^{j+\lambda}} > 0.$$

Since $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to

$$\frac{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)} r^{\lambda+1} - \sum_{j=2}^{\infty} \frac{[1+(j-1)(1-\delta)^n]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [\gamma(2\lambda-2+\beta|d|)+1] j + [\gamma(\lambda-1)+1](\lambda-1+\beta|d|) \right\} a_j^2 r^{j+\lambda}}{\frac{\gamma\lambda+1}{\Gamma(\lambda+2)} r^{\lambda+1} - \sum_{j=2}^{\infty} \frac{[1+(j-1)(1-\delta)^n]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} [\gamma j + \gamma(\lambda-1)+1] a_j^2 r^{j+\lambda}} > 0.$$

Letting $r \to 1^-$ and by the mean value theorem we have desired inequality (2.1).

This completes the proof of Theorem 2.1 □

Corollary 2.2. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$. Then*

$$a_j \leq \sqrt{\frac{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}{\frac{[1+(j-1)(1-\delta)^n]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{ \gamma j^2 + [\gamma(2\lambda-2+\beta|d|)+1] j + [\gamma(\lambda-1)+1](\lambda-1+\beta|d|) \right\}}}, \quad j \geq 2.$$

3. Distortion Theorems

Theorem 3.1. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$. Then for $|z| = r < 1$, we have*

$$r - \sqrt{\frac{(\gamma\lambda + 1)(\lambda + 2)(\beta|d| - \lambda)\Gamma(\lambda)}{2\left[1 + (j-1)(1-\delta)^n\right]^m \{4\lambda - \beta|d| + (\lambda+1)\left[1 + \gamma(\lambda - 3) - \gamma\beta|d|\right]\}}}\, r^2 \le |f(z)|$$

$$\le r + \sqrt{\frac{(\gamma\lambda + 1)(\lambda + 2)(\beta|d| - \lambda)\Gamma(\lambda)}{2\left[1 + (j-1)(1-\delta)^n\right]^m \{4\lambda - \beta|d| + (\lambda+1)\left[1 + \gamma(\lambda - 3) - \gamma\beta|d|\right]\}}}\, r^2.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z + \sqrt{\frac{(\gamma\lambda + 1)(\lambda + 2)(\beta|d| - \lambda)\Gamma(\lambda)}{2\left[1 + (j-1)(1-\delta)^n\right]^m \{4\lambda - \beta|d| + (\lambda+1)\left[1 + \gamma(\lambda - 3) - \gamma\beta|d|\right]\}}}\, z^2.$$

Proof. Given that $f(z) \in \mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$, from the equation (2.1) and since $2\left[1 + (j-1)(1-\delta)^n\right]^m \{4\lambda - \beta|d| + (\lambda+1)\left[1 + \gamma(\lambda - 3) - \gamma\beta|d|\right]\}$ is non decreasing and positive for $j \ge 2$, then we have

$$\sqrt{2\left[1 + (j-1)(1-\delta)^n\right]^m \{4\lambda - \beta|d| + (\lambda+1)\left[1 + \gamma(\lambda - 3) - \gamma\beta|d|\right]\}} \sum_{j=2}^{\infty} a_j \le$$

$$\sum_{j=2}^{\infty} \sqrt{\frac{\left[1 + (j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \{\lambda j^2 + \left[\gamma(2\lambda - 2 + \beta|d|) + 1\right] j + \left[\gamma(\lambda - 1) + 1\right](\lambda - 1 + \beta|d|)\} a_j}$$

$$\le \sqrt{\frac{(\gamma\lambda + 1)(\beta|d| - \lambda)}{\Gamma(\lambda + 2)}},$$

which is equivalent to,

$$\sum_{j=2}^{\infty} a_j \le \sqrt{\frac{(\gamma\lambda + 1)(\lambda + 2)(\beta|d| - \lambda)\Gamma(\lambda)}{2\left[1 + (j-1)(1-\delta)^n\right]^m \{4\lambda - \beta|d| + (\lambda+1)\left[1 + \gamma(\lambda - 3) - \gamma\beta|d|\right]\}}}. \tag{3.1}$$

Using (3.1), we obtain

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

$$|f(z)| \le |z| + \sum_{j=2}^{\infty} a_j |z|^j \le r + \sum_{j=2}^{\infty} a_j r^j \le r + r^2 \sum_{j=2}^{\infty} a_j$$

$$\le r + \sqrt{\frac{(\gamma\lambda + 1)(\lambda + 2)(\beta|d| - \lambda)\Gamma(\lambda)}{2\left[1 + (j-1)(1-\delta)^n\right]^m \{4\lambda - \beta|d| + (\lambda+1)\left[1 + \gamma(\lambda - 3) - \gamma\beta|d|\right]\}}}\, r^2.$$

Similarly,

$$|f(z)| \ge r - \sqrt{\frac{(\gamma\lambda + 1)(\lambda + 2)(\beta|d| - \lambda)\Gamma(\lambda)}{2\left[1 + (j-1)(1-\delta)^n\right]^m \{4\lambda - \beta|d| + (\lambda+1)\left[1 + \gamma(\lambda - 3) - \gamma\beta|d|\right]\}}}\, r^2.$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{D}_{m,n}(\delta,\lambda,d,\gamma,\beta)$. Then for $|z| = r < 1$, we have*

$$-\sqrt{\frac{2(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)\Gamma(\lambda)}{[1+(j-1)(1-\delta)^n]^m\{4\lambda-\beta|d|+(\lambda+1)[1+\gamma(\lambda-3)-\gamma\beta|d|]\}}}\,r \le |f'(z)|$$

$$\le \sqrt{\frac{2(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)\Gamma(\lambda)}{[1+(j-1)(1-\delta)^n]^m\{4\lambda-\beta|d|+(\lambda+1)[1+\gamma(\lambda-3)-\gamma\beta|d|]\}}}\,r.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z + \sqrt{\frac{2(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)\Gamma(\lambda)}{[1+(j-1)(1-\delta)^n]^m\{4\lambda-\beta|d|+(\lambda+1)[1+\gamma(\lambda-3)-\gamma\beta|d|]\}}}\,z^2.$$

Proof. From (3.1)

$$f'(z) = 1 + \sum_{j=2}^{\infty} j a_j z^{j-1}$$

$$|f'(z)| \le 1 - \sum_{j=2}^{\infty} j a_j |z|^{j-1} \le 1 + \sum_{j=2}^{\infty} j a_j r^{j-1} \le 1 + \sqrt{\frac{2(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)\Gamma(\lambda)}{[1+(j-1)(1-\delta)^n]^m\{4\lambda-\beta|d|+(\lambda+1)[1+\gamma(\lambda-3)-\gamma\beta|d|]\}}}\,r.$$

Similarly,

$$|f'(z)| \ge 1 - \sqrt{\frac{2(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)\Gamma(\lambda)}{[1+(j-1)(1-\delta)^n]^m\{4\lambda-\beta|d|+(\lambda+1)[1+\gamma(\lambda-3)-\gamma\beta|d|]\}}}\,r.$$

This completes the proof of Theorem 3.2. □

4. Closure Theorems

Theorem 4.1. *Let the functions f_k, $k = 1, 2, ..., l$, defined by*

$$f_k(z) = z + \sum_{j=2}^{\infty} a_{j,k} z^j, \quad a_{j,k} \ge 0, \tag{4.1}$$

be in the class $\mathcal{D}_{m,n}(\delta,\lambda,d,\gamma,\beta)$. Then the function $h(z)$ defined by

$$h(z) = \sum_{k=1}^{l} \mu_k f_k(z), \quad \mu_k \ge 0,$$

is also in the class $\mathcal{D}_{m,n}(\delta,\lambda,d,\gamma,\beta)$, where

$$\sum_{k=1}^{l} \mu_k = 1.$$

Proof. We can write

$$h(z) = \sum_{k=1}^{l} \mu_k z + \sum_{k=1}^{l}\sum_{j=2}^{\infty} \mu_k a_{j,k} z^j = z + \sum_{j=2}^{\infty}\sum_{k=1}^{l} \mu_k a_{j,k} z^j.$$

Furthermore, since the functions $f_k(z)$, $k = 1, 2, ..., l$, are in the class $\mathcal{D}_{m,n}(\delta,\lambda,d,\gamma,\beta)$, then from Corollary 2.2 we have

$$\sum_{j=2}^{\infty} \sqrt{\frac{[1+(j-1)(1-\delta)^n]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}\{\lambda j^2 + [\gamma(2\lambda-2+\beta|d|)+1]j + [\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}a_j}$$

$$\leq \sqrt{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}$$

Thus it is enough to prove that

$$\sum_{j=2}^{\infty}\sqrt{\frac{\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}}\sqrt{\{\lambda j^2+[\gamma(2\lambda-2+\beta|d|)+1]j+[\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}}$$

$$\left(\sum_{k=1}^{m}\mu_k a_{j,k}\right)=\sum_{k=1}^{m}\mu_k\sum_{j=2}^{\infty}\sqrt{\frac{\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}}\sqrt{\{\lambda j^2+[\gamma(2\lambda-2+\beta|d|)+1]j+[\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}}a_{j,k}$$

$$\leq \sum_{k=1}^{m}\mu_k\sqrt{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}=\sqrt{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}.$$

Hence the proof is complete. □

Corollary 4.2. *Let the functions f_k, $k=1,2$, defined by (4.1) be in the class $\mathcal{D}_{m,n}(\delta,\lambda,d,\gamma,\beta)$. Then the function $h(z)$ defined by*

$$h(z)=(1-\zeta)f_1(z)+\zeta f_2(z),\quad 0\leq\zeta\leq 1,$$

is also in the class $\mathcal{D}_{m,n}(\delta,\lambda,d,\gamma,\beta)$.

Theorem 4.3. *Let*

$$f_1(z)=z,$$

and

$$f_j(z)=z+\sqrt{\frac{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}{\frac{\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}\{\lambda j^2+[\gamma(2\lambda-2+\beta|d|)+1]j+[\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}}}z^j,\quad j\geq 2.$$

Then the function $f(z)$ is in the class $\mathcal{D}_{m,n}(\delta,\lambda,d,\gamma,\beta)$ if and only if it can be expressed in the form:

$$f(z)=\mu_1 f_1(z)+\sum_{j=2}^{\infty}\mu_j f_j(z),$$

where $\mu_1\geq 0$, $\mu_j\geq 0$, $j\geq 2$ and $\mu_1+\sum_{j=2}^{\infty}\mu_j=1$.

Proof. Assume that $f(z)$ can be expressed in the form

$$f(z)=\mu_1 f_1(z)+\sum_{j=2}^{\infty}\mu_j f_j(z)=z+\sum_{j=2}^{\infty}\sqrt{\frac{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}{\frac{\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}\{\lambda j^2+[\gamma(2\lambda-2+\beta|d|)+1]j+[\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}}}\mu_j z^j.$$

Thus

$$\sum_{j=2}^{\infty}\sqrt{\frac{\frac{\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}\{\lambda j^2+[\gamma(2\lambda-2+\beta|d|)+1]j+[\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}}{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}}\cdot$$

$$\sqrt{\frac{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}{\frac{\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}\{\lambda j^2+[\gamma(2\lambda-2+\beta|d|)+1]j+[\gamma(\lambda-1)+1](\lambda-1+\beta|d|)\}}}\mu_j=\sum_{j=2}^{\infty}\mu_j=1-\mu_1\leq 1.$$

Hence $f(z) \in \mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$.

Conversely, assume that $f(z) \in \mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$.

Setting

$$\mu_j = \sqrt{\frac{\frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{\lambda j^2 + \left[\gamma\left(2\lambda - 2 + \beta\left|d\right|\right) + 1\right] j + \left[\gamma\left(\lambda - 1\right) + 1\right]\left(\lambda - 1 + \beta\left|d\right|\right)\right\}}{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}} a_j,$$

since

$$\mu_1 = 1 - \sum_{j=2}^{\infty} \mu_j.$$

Thus

$$f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z).$$

Hence the proof is complete. □

Corollary 4.4. *The extreme points of the class $\mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$ are the functions*

$$f_1(z) = z,$$

and

$$f_j(z) = z + \sqrt{\frac{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}{\frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{\lambda j^2 + \left[\gamma\left(2\lambda - 2 + \beta\left|d\right|\right) + 1\right] j + \left[\gamma\left(\lambda - 1\right) + 1\right]\left(\lambda - 1 + \beta\left|d\right|\right)\right\}}} z^j, \ \ j \geq 2.$$

5. Inclusion and Neighborhood Results

We define the k- neighborhood of a function $f(z) \in \mathcal{A}$ by

$$N_k(f) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j \ \text{ and } \ \sum_{j=2}^{\infty} j|a_j - b_j| \leq k\}. \tag{5.1}$$

In particular, for $e(z) = z$

$$N_k(e) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j \ \text{ and } \sum_{j=2}^{\infty} j|b_j| \leq k\}. \tag{5.2}$$

Furthermore, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{D}^{\xi}_{m,n}(\delta, \lambda, d, \gamma, \beta)$ if there exists a function $h(z) \in \mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$ such that

$$\left|\frac{f(z)}{h(z)} - 1\right| < 1 - \xi, \ \ z \in U, \ \ 0 \leq \xi < 1. \tag{5.3}$$

Theorem 5.1. *If*

$$k = \sqrt{\frac{2\left(\gamma\lambda + 1\right)\left(\lambda + 2\right)\left(\beta\left|d\right| - \lambda\right)\Gamma\left(\lambda\right)}{\left[1 + \left(j - 1\right)\left(1 - \delta\right)^n\right]^m \left\{4\lambda - \beta\left|d\right| + \left(\lambda + 1\right)\left[1 + \gamma\left(\lambda - 3\right) - \gamma\beta\left|d\right|\right]\right\}}},$$

then

$$\mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta) \subset N_k(e).$$

Proof. Let $f \in \mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$. Then in view of assertion of Corollary 2.2 and since

$$\frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{\lambda j^2 + \left[\gamma(2\lambda-2+\beta|d|)+1\right] j + \left[\gamma(\lambda-1)+1\right](\lambda-1+\beta|d|)\right\} \geq$$
$$\frac{2\left[1+(1-\delta)^n\right]^m}{\Gamma(\lambda+3)} \{4\lambda-\beta|d|+(\lambda+1)[1+\gamma(\lambda-3)-\gamma\beta|d|]\} \text{ for } j \geq 2, \text{we get}$$

$$\sqrt{\frac{2\left[1+(1-\delta)^n\right]^m}{\Gamma(\lambda+3)} \left\{\beta|d|\left[1+\gamma(\lambda+1)\right]+\gamma(\lambda+1)^2+5(\lambda-\gamma)+1\right\}} \sum_{j=2}^{\infty} a_j \leq$$

$$\sum_{j=2}^{\infty} \sqrt{\frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \{\lambda j^2 + \left[\gamma(2\lambda-2+\beta|d|)+1\right] j + \left[\gamma(\lambda-1)+1\right](\lambda-1+\beta|d|)\} a_j}$$

$$\leq \sqrt{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}},$$

which implies

$$\sum_{j=2}^{\infty} a_j \leq \sqrt{\frac{(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)\Gamma(\lambda)}{2\left[1+(j-1)(1-\delta)^n\right]^m \{4\lambda-\beta|d|+(\lambda+1)[1+\gamma(\lambda-3)-\gamma\beta|d|]\}}}. \tag{5.4}$$

Applying assertion of Corollary 2.2 in conjunction with (5.4), we obtain

$$\sum_{j=2}^{\infty} j a_j \leq \sqrt{\frac{2(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)\Gamma(\lambda)}{\left[1+(j-1)(1-\delta)^n\right]^m \{4\lambda-\beta|d|+(\lambda+1)[1+\gamma(\lambda-3)-\gamma\beta|d|]\}}} = k,$$

by virtue of (5.1), we have $f \in N_k(e)$.
This completes the proof of the Theorem 5.1. □

Theorem 5.2. *If* $h \in \mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$ *and*

$$\xi = 1 + \frac{k}{2}\sqrt{\frac{(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)\Gamma(\lambda)}{2\left[1+(j-1)(1-\delta)^n\right]^m \{4\lambda-\beta|d|+(\lambda+1)[1+\gamma(\lambda-3)-\gamma\beta|d|]\}}}, \tag{5.5}$$

then

$$N_k(h) \subset \mathcal{D}^{\xi}_{m,n}(\delta, \lambda, d, \gamma, \beta).$$

Proof. Suppose that $f \in N_k(h)$, we then find from (5.1) that

$$\sum_{j=2}^{\infty} j|a_j - b_j| \leq k,$$

which readily implies the following coefficient inequality

$$\sum_{j=2}^{\infty} |a_j - b_j| \leq \frac{k}{2}. \tag{5.6}$$

Next, since $h \in \mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$ in the view of (5.4), we have

$$\sum_{j=2}^{\infty} b_j \leq \sqrt{\frac{(\gamma\lambda + 1)(\lambda + 2)(\beta|d| - \lambda)\Gamma(\lambda)}{2\left[1 + (j-1)(1-\delta)^n\right]^m \{4\lambda - \beta|d| + (\lambda + 1)[1 + \gamma(\lambda - 3) - \gamma\beta|d|]\}}}. \tag{5.7}$$

Using 5.6) and (5.7), we get

$$\left|\frac{f(z)}{h(z)} - 1\right| \leq \frac{\sum_{j=2}^{\infty} |a_j - b_j|}{1 - \sum_{j=2}^{\infty} b_j} \leq \frac{k}{2\left(1 - \sqrt{\frac{(\gamma\lambda+1)(\lambda+2)(\beta|d|-\lambda)\Gamma(\lambda)}{2\left[1+(j-1)(1-\delta)^n\right]^m\left\{4\lambda-\beta|d|+(\lambda+1)\left[1+\gamma(\lambda-3)-\gamma\beta|d|\right]\right\}}}\right)} = 1 - \xi,$$

provided that ξ is given by (5.5), thus by condition (5.3), $f \in \mathcal{D}^{\xi}_{m,n}(\delta, \lambda, d, \gamma, \beta)$, where ξ is given by (5.5). □

6. Radii of Starlikeness, Convexity and Close-to-Convexity

Theorem 6.1. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$. Then f is univalent starlike of order k, $0 \leq k < 1$, in $|z| < r_1$, where*

$$r_1 = \inf_j \left\{\frac{(1-k)^2 \frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{\lambda j^2 + [\gamma(2\lambda - 2 + \beta|d|) + 1] j + [\gamma(\lambda - 1) + 1](\lambda - 1 + \beta|d|)\right\}}{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)} (j-k)^2}\right\}^{\frac{1}{2(j-1)}}.$$

The result is sharp for the function $f(z)$ given by

$$f_j(z) = z + \sqrt{\frac{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}{\frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \{\lambda j^2 + [\gamma(2\lambda - 2 + \beta|d|) + 1] j + [\gamma(\lambda - 1) + 1](\lambda - 1 + \beta|d|)\}}} z^j, \ \ j \geq 2.$$

Proof. It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - k, \quad |z| < r_1.$$

Since

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{\sum_{j=2}^{\infty}(j-1)a_j z^{j-1}}{1 + \sum_{j=2}^{\infty} a_j z^{k-1}}|\right| \leq \frac{\sum_{j=2}^{\infty}(j-1)a_j|z|^{j-1}}{1 - \sum_{j=2}^{\infty} a_j|z|^{j-1}}.$$

To prove the theorem, we must show that

$$\frac{\sum_{j=2}^{\infty}(j-1)a_j|z|^{j-1}}{1 - \sum_{j=2}^{\infty} a_j|z|^{j-1}} \leq 1 - k.$$

It is equivalent to

$$\sum_{j=2}^{\infty}(j-k)a_j|z|^{j-1} \leq 1 - k,$$

using Theorem 2.1, we obtain

$$|z| \leq \left\{\frac{(1-k)^2 \frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{\lambda j^2 + [\gamma(2\lambda - 2 + \beta|d|) + 1] j + [\gamma(\lambda - 1) + 1](\lambda - 1 + \beta|d|)\right\}}{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)} (j-k)^2}\right\}^{\frac{1}{2(j-1)}}.$$

Hence the proof is complete. □

Theorem 6.2. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$. Then f is univalent convex of order k, $0 \le k \le 1$, in $|z| < r_2$, where*

$$r_2 = \inf_j \left\{ \frac{(1-k)^2 \frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{\lambda j^2 + \left[\gamma\left(2\lambda - 2 + \beta|d|\right) + 1\right] j + \left[\gamma\left(\lambda - 1\right) + 1\right]\left(\lambda - 1 + \beta|d|\right)\right\}}{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)} j^2 (j-k)^2} \right\}^{\frac{1}{2(j-1)}}.$$

The result is sharp for the function $f(z)$ given by

$$f_j(z) = z + \sqrt{\frac{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}{\frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{\lambda j^2 + \left[\gamma\left(2\lambda - 2 + \beta|d|\right) + 1\right] j + \left[\gamma\left(\lambda - 1\right) + 1\right]\left(\lambda - 1 + \beta|d|\right)\right\}}}\, z^j, \quad j \ge 2. \qquad (6.1)$$

Proof. It suffices to show that

$$\left|\frac{zf''(z)}{f'(z))}\right| \le 1-k, \quad |z| < r_2.$$

Since

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{j=2}^{\infty} j(j-1)a_j z^{j-1}}{1+\sum_{j=2}^{\infty} j a_j z^{j-1}}\right| \le \frac{\sum_{j=2}^{\infty} j(j-1)a_j |z|^{j-1}}{1-\sum_{j=2}^{\infty} j a_j |z|^{j-1}}.$$

To prove the theorem, we must show that

$$\frac{\sum_{j=2}^{\infty} j(j-1)a_j |z|^{j-1}}{1-\sum_{j=2}^{\infty} j a_j |z|^{j-1}} \le 1-k,$$

$$\sum_{j=2}^{\infty} j(j-k)a_j |z|^{j-1} \le 1-k,$$

using Theorem 2.1, we obtain

$$|z|^{j-1} \le \frac{(1-k)}{j(j-k)} \sqrt{\frac{\frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{\lambda j^2 + \left[\gamma\left(2\lambda - 2 + \beta|d|\right) + 1\right] j + \left[\gamma\left(\lambda - 1\right) + 1\right]\left(\lambda - 1 + \beta|d|\right)\right\}}{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}},$$

or

$$|z| \le \left\{ \frac{(1-k)^2 \frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{\lambda j^2 + \left[\gamma\left(2\lambda - 2 + \beta|d|\right) + 1\right] j + \left[\gamma\left(\lambda - 1\right) + 1\right]\left(\lambda - 1 + \beta|d|\right)\right\}}{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)} j^2 (j-k)^2} \right\}^{\frac{1}{2(j-1)}}.$$

Hence the proof is complete. □

Theorem 6.3. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{D}_{m,n}(\delta, \lambda, d, \gamma, \beta)$. Then f is univalent close-to-convex of order k, $0 \le k < 1$, in $|z| < r_3$, where*

$$r_3 = \inf_j \left\{ \frac{(1-k)^2 \frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{\lambda j^2 + \left[\gamma\left(2\lambda - 2 + \beta|d|\right) + 1\right] j + \left[\gamma\left(\lambda - 1\right) + 1\right]\left(\lambda - 1 + \beta|d|\right)\right\}}{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)} j^2} \right\}^{\frac{1}{2(j-1)}}.$$

The result is sharp for the function $f(z)$ given by (6.1).

Proof. It suffices to show that

$$|f'(z)-1| \leq 1-k, \quad |z| < r_3.$$

Then

$$|f'(z)-1| = \left|\sum_{j=2}^{\infty} j a_j z^{j-1}\right| \leq \sum_{j=2}^{\infty} j a_j |z|^{j-1}.$$

Thus $|f'(z)-1| \leq 1-k$ if $\sum_{j=2}^{\infty} \frac{j a_j}{1-\delta} |z|^{j-1} \leq 1$. Using Theorem 2.1, the above inequality holds true if

$$|z|^{j-1} \leq \frac{(1-k)}{j} \sqrt{\frac{\frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{\lambda j^2 + \left[\gamma(2\lambda-2+\beta|d|)+1\right] j + \left[\gamma(\lambda-1)+1\right](\lambda-1+\beta|d|)\right\}}{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)}}}$$

or

$$|z| \leq \left\{\frac{(1-k)^2 \frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} \left\{\lambda j^2 + \left[\gamma(2\lambda-2+\beta|d|)+1\right] j + \left[\gamma(\lambda-1)+1\right](\lambda-1+\beta|d|)\right\}}{\frac{(\gamma\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)} j^2}\right\}^{\frac{1}{2(j-1)}}.$$

Hence the proof is complete. □

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