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New Formulas and Numbers Arising from Analyzing Combinatorial Numbers and Polynomials

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Abstract

In this paper, we derive various identities involving the negative higher-order combinatorial numbers and polynomials and other kinds of special numbers and polynomials such as the Stirling numbers, the Lah numbers, the negative higher-order Changhee numbers and polynomials, and the positive higher-order Bernoulli numbers and polynomials. Furthermore, by using the integral formulas of not only the negative higher-order combinatorial numbers and polynomials but also their generating functions, we obtain some identities and combinatorial sums. We give some infinite series, involving the negative higher-order combinatorial numbers, with their values in terms of the falling factorials, the Catalan numbers, the Daehee numbers (linear combination of the Stirling numbers and the Bernoulli numbers) and the Changhee numbers (linear combination of the Stirling numbers and the Euler numbers). As application of these infinite series, we also set two new sequences of special numbers with their generating functions, and investigate their properties. We pose an open question related to one of these number sequences. By using an infinite series arising from the integral of the generating functions for the negative higher-order combinatorial numbers and polynomials, we also introduce a new family of polynomials associated with the Bernstein basis functions. In addition, we derive symmetry property, integral formulas and derivative formula for these newly introduced polynomials. Moreover, by implementing an explicit formula of these newly introduced polynomials in Mathematica with the aid of the Wolfram programming language, we present some plots of these newly introduced polynomial functions for some of their randomly selected special cases. We also give some further results including series representations, combinatorial sums, integral formulas and relations for some of combinatorial numbers and poynomials. Finally, we present some observations and comments on our results.

Keywords: Generating functions, Special numbers and polynomials, Bernoulli numbers and polynomials, Stirling numbers, Bell numbers, Lah numbers, Catalan numbers, Daehee and Changhee numbers, Bernstein basis functions

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1. Introduction

Throughout this paper, we use the following standard notations:

$$\mathbb{N} := \{1, 2, 3, \dots\} \quad \text{and} \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Here, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the set of real numbers and the set of complex numbers. For $z \in \mathbb{C}$, we assume that $\ln z$ denotes the principal branch of the many-valued function with the imaginary part $\mathrm{Im}(\ln z)$ constrained by

$$-\pi < \mathrm{Im}(\ln z) \leq \pi.$$

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It is also assumed that

$$0^n = \begin{cases} 1, & n = 0 \\ 0, & n \in \mathbb{N}. \end{cases}$$

In this paper, it is aimed to derive various identities and formulas involving the negative higher-order combinatorial numbers and polynomials and some kinds of special numbers and polynomials. All preliminaries required to achieve this goal are as follows:

For $m, r \in \mathbb{N}_0$, the binomial coefficient $\binom{m}{r}$ is given, as usual, by

$$\binom{m}{r} = \frac{m!}{(m-r)!r!}$$

such that it is assumed that $\binom{m}{r} = 0$ when $r < 0$ and $r > m$.

The falling factorial and the rising factorial are respectively defined by

$$(x)_n = x(x-1)(x-2)\dots(x-n+1)$$

and

$$(x)^{(n)} = x(x+1)(x+2)\dots(x+n-1)$$

where $n \in \mathbb{N}$ such that $(x)_0 = 1$ and $(x)^{(0)} = 1$ (*cf.* [2]-[52]).

The Stirling numbers of the first kind, $S_1(n,k)$, are defined as follows:

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)\, x^k \tag{1.1}$$

and

$$\frac{(\ln(1+t))^k}{k!} = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}; \quad (k \in \mathbb{N}_0) \tag{1.2}$$

so that these numbers satisfy the following recurrence relation:

$$S_1(n+1,k) = -nS_1(n,k) + S_1(n,k-1)$$

such that $S_1(0,0) = 1$, $S_1(0,k) = 0$ if $k > 0$, $S_1(n,0) = 0$ if $n > 0$, $S_1(n,k) = 0$ if $k > n$ (*cf.* [2], [4], [6], [7], [15], [30], [33], [48], [49], [51]; and the references cited therein).

The Stirling numbers of the second kind, $S_2(n,k)$, are defined by

$$\frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!}, \tag{1.3}$$

so that these numbers are computed by the following explicit formula:

$$S_2(n,k) = \frac{1}{k!}\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} j^n \tag{1.4}$$

which satisfy the following recurrence relation:

$$S_2(n+1,k) = S_2(n,k-1) + kS_2(n,k)$$

such that $S_2(0,0) = 1$, $S_2(n,k) = 0$ if $k > n$, $S_2(n,0) = 0$ if $n > 0$ (*cf.* [2], [9], [30], [33], [51]; and the references cited therein).

The Bell polynomials (i.e., exponential polynomials), $Bl_n(x)$, are defined by

$$Bl_n(x) = \sum_{v=1}^{n} S_2(n,v)\, x^v \tag{1.5}$$

whose exponential generating function is given by

$$e^{(e^t-1)x} = \sum_{n=0}^{\infty} Bl_n(x)\frac{t^n}{n!} \tag{1.6}$$

(*cf.* [7], [30]).

Substituing $x = 0$ into (1.6) yields the generating functions for the Bell numbers, i.e.:

$$Bl_n = Bl_n(0)$$

so that these numbers enumerate all partitions of a set with $n \geq 1$ elements (*cf.* [2], [7]; and see also the cited references therein).

In [13, Vol. 7, Eq-(2.29)], Gould gave the following identity:

$$\sum_{n=0}^{\infty} \frac{n^r x^n}{n!} = e^x \sum_{k=1}^{r} S_2(r,k)x^k = e^x Bl_r(x) \tag{1.7}$$

which, by the Cauchy product, is equivalent to

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^r \right) \frac{x^n}{n!} = \sum_{k=1}^{r} S_2(r,k)x^k = Bl_r(x) \tag{1.8}$$

(*cf.* [13], [24]).

The Catalan numbers, C_n, are defined by the following explicit formula:

$$C_n = \frac{1}{n+1}\binom{2n}{n}; \qquad (n \in \mathbb{N}_0) \tag{1.9}$$

whose ordinary generating function is given by

$$\frac{1-\sqrt{1-4z}}{2z} = \sum_{n=0}^{\infty} C_n z^n,$$

where $0 < |z| \leq \frac{1}{4}$ (*cf.* [2], [8, pp. 96-106], [22, pp. 109-110], [31]).

The Lah numbers, $L(n,k)$, are defined by

$$L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1} \tag{1.10}$$

which encounter the ways for the partition of a set containing n elements into k nonempty linearly ordered subsets, and for these numbers the following identity holds true:

$$(x)_n = \sum_{k=1}^{n} (-1)^{n-k} L(n,k)(x)^{(k)} \tag{1.11}$$

(*cf.* [16] and the cited references therein).

The combinatorial numbers, $y_1(n,k;\lambda)$, are defined by

$$y_1(n,k;\lambda) = \frac{1}{k!}\sum_{j=0}^{k} \binom{k}{j} \lambda^j j^n \tag{1.12}$$

whose exponential generating function is given by

$$\frac{1}{k!}\left(\lambda e^t + 1\right)^k = \sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!} \qquad (k \in \mathbb{N}_0;\ \lambda \in \mathbb{C}) \tag{1.13}$$

(*cf.* [40]).

One of the most important features of the combinatorial numbers $y_1(n,k;\lambda)$ is that these numbers are of the following equality:

$$y_1(n,k;\lambda) = \frac{1}{k!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\lambda e^t + 1\right)^k\Big|_{t=0} \tag{1.14}$$

(*cf.* [53, p. 64]).

The beta function, $B(\alpha,\beta)$, is defined by

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t, \qquad (\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0), \tag{1.15}$$

(*cf.* [29], [49, p. 9, Eq.-(60)], [51]), and the relation between the beta function $B(\alpha,\beta)$ and the Euler gamma function $\Gamma(\alpha)$ is given by

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.16}$$

which, replacing α and β by natural numbers n and m respectively, yields

$$B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} = \frac{(n-1)!\,(m-1)!}{(n+m-1)!}, \tag{1.17}$$

(*cf.* [29], [49], [51]).

Let $t \in \mathbb{C}$, $x \in [0,1]$ and $k \in \mathbb{N}_0$. Then, the generating function for the Bernstein basis functions, $B_k^n(x)$, is given as follows:

$$\frac{(xt)^k e^{(1-x)t}}{k!} = \sum_{n=0}^{\infty} B_k^n(x)\frac{t^n}{n!}, \tag{1.18}$$

where

$$B_k^n(x) = \binom{n}{k}x^k(1-x)^{n-k}, \qquad (k = 0,1,\ldots,n;\ \ n \in \mathbb{N}_0) \tag{1.19}$$

which have relationships with a large number of concepts including the Catalan numbers, the binomial distribution, the Poisson distribution and etc.; for details, see [1, 27, 34, 35, 36, 44] and also the cited references therein.

Note that the integral formula for the Bernstein basis functions is given as follows:

$$\int_0^1 B_k^n(x)\mathrm{d}x = \binom{n}{k}B(k+1, n-k+1) \tag{1.20}$$

$$= \frac{1}{n+1}, \tag{1.21}$$

(*cf.* [36]; see also [14, Eq.-(5.28), p.254], [27]).

The positive higher-order Bernoulli numbers $B_k^{(n)}$ and the positive higher-order Bernoulli polynomials $B_k^{(n)}(x)$ are respectively defined by the following exponential generating functions:

$$\left(\frac{t}{e^t-1}\right)^n = \sum_{k=0}^{\infty} B_k^{(n)}\frac{t^k}{k!} \tag{1.22}$$

and

$$\left(\frac{t}{e^t-1}\right)^n e^{tx} = \sum_{k=0}^{\infty} B_k^{(n)}(x)\frac{t^k}{k!}, \tag{1.23}$$

where $n \in \mathbb{N}_0$. By (1.22) and (1.23), we have

$$B_k^{(n)}(x) = \sum_{v=0}^{k}\binom{k}{v}x^v B_{k-v}^{(n)}, \tag{1.24}$$

in which $n, k \in \mathbb{N}_0$ (*cf.* [5], [7], [10], [11], [12], [20], [28], [32], [50], [51]).

Using (1.22) yields the well-known recurrence relation for the numbers $B_k^{(n)}$ as follows:

$$B_k^{(n)} = \left(1 - \frac{k}{n-1}\right) B_k^{(n-1)} - k B_{k-1}^{(n-1)}, \tag{1.25}$$

where $n, k \in \mathbb{N} \setminus \{1\}$. Substituting $n = k+1$ into (1.25), one has the following well-known interesting identity including factorial:

$$B_k^{(k+1)} = (-1)^k k!, \tag{1.26}$$

(*cf.* [5]).

The Daehee numbers, D_n, are defined by the following generating function:

$$\frac{\ln(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} \tag{1.27}$$

so that the explicit formula for these numbers is given by

$$D_n = \frac{(-1)^n n!}{n+1} = \sum_{k=0}^{n} S_1(n,k) B_k \tag{1.28}$$

where B_k denotes the Bernoulli numbers (*cf.* [19]; and see also [42]).

The Changhee numbers, Ch_n, are defined by the following generating function:

$$\frac{2}{t+2} = \sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!} \tag{1.29}$$

so that the explicit formula for these numbers is given by

$$Ch_n = \frac{(-1)^n n!}{2^n} = \sum_{k=0}^{n} S_1(n,k) E_k \tag{1.30}$$

where E_k denotes the Euler numbers (*cf.* [21]; and see also [17], [42]).

Exponential generating functions for the negative higher-order Changhee polynomials $Ch_n^{(-k)}(x)$ are given as follows:

$$\frac{(2+t)^k}{2^k} (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(-k)}(x) \frac{t^n}{n!}, \tag{1.31}$$

(*cf.* [18]).

A computation formula for the negative higher-order Changhee polynomials $Ch_n^{(-k)}(x)$ is given by the following combinatorial sum:

$$Ch_n^{(-k)}(x) = 2^{-k} \sum_{j=0}^{k} \sum_{l=0}^{n} \binom{k}{j} \binom{n}{l} (j)_{n-l} (x)_l, \tag{1.32}$$

(*cf.* [18, p. 7, Eq.-(25)]).

Setting $x = 0$ into (1.31) yields the generating functions for the negative higher-order Changhee numbers $Ch_n^{(-k)} = Ch_n^{(-k)}(0)$ whose computation formula is given by the following combinatorial sum:

$$Ch_n^{(-k)} = 2^{-k} \sum_{j=0}^{k} \binom{k}{j} (j)_n \tag{1.33}$$

(*cf.* [18]).

1.1. A certain family of the negative higher-order combinatorial numbers and polynomials

Recall that Kucukoglu et al. [26] introduced the negative higher-order combinatorial numbers $Y_n^{(-k)}(\lambda)$ and polynomials $Q_n(x;\lambda,k)$ by the following generating functions, respectively:

$$\mathcal{G}(t,k;\lambda) = 2^{-k}(\lambda(1+\lambda t)-1)^k = \sum_{n=0}^{\infty} Y_n^{(-k)}(\lambda)\frac{t^n}{n!} \tag{1.34}$$

and

$$\mathcal{G}(t,x,k;\lambda) = \mathcal{G}(t,k;\lambda)(1+\lambda t)^x = \sum_{n=0}^{\infty} Q_n(x;\lambda,k)\frac{t^n}{n!}, \tag{1.35}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$).

By (1.34) and (1.35), we have

$$Q_n(x;\lambda,k) = \sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}Y_j^{(-k)}(\lambda)(x)_{n-j}, \tag{1.36}$$

and

$$Y_n^{(-k)}(\lambda) = \begin{cases} 2^{-k}(k)_n\lambda^{2n}(\lambda-1)^{k-n} & \text{if} \quad n \le k \\ 0 & \text{if} \quad n > k \end{cases} \tag{1.37}$$

where $k, n \in \mathbb{N}_0$ (*cf.* [26]).

By (1.36) and (1.37), the values of the numbers $Y_n^{(-k)}(\lambda)$ and the polynomials $Q_n(x;\lambda,k)$ are respectively given as follows:

$$\begin{aligned} Y_0^{(-k)}(\lambda) &= 2^{-k}(\lambda-1)^k, \\ Y_1^{(-k)}(\lambda) &= 2^{-k}(k)_1\lambda^2(\lambda-1)^{k-1}, \\ Y_2^{(-k)}(\lambda) &= 2^{-k}(k)_2\lambda^4(\lambda-1)^{k-2}, \\ &\vdots \\ Y_k^{(-k)}(\lambda) &= 2^{-k}k!\lambda^{2k}, \\ Y_j^{(-k)}(\lambda) &= 0 \quad for \quad j > k, \end{aligned}$$

and

$$\begin{aligned} Q_0(x;\lambda,k) &= 2^{-k}(\lambda-1)^k, \\ Q_1(x;\lambda,k) &= 2^{-k}(\lambda-1)^k\lambda x + 2^{-k}k\lambda^2(\lambda-1)^{k-1}, \\ Q_2(x;\lambda,k) &= 2^{-k}(\lambda-1)^k\lambda^2x^2 + \left(-2^{-k}(\lambda-1)^k\lambda^2 + 2^{-k+1}k\lambda^3(\lambda-1)^{k-1}\right)x \\ &\quad +2^{-k}k(k-1)\lambda^4(\lambda-1)^{k-1}, \end{aligned}$$

and so on (*cf.* [26]).

Remark 1.1. It follows from (1.19) and (1.37) that there exist a relationship between the numbers $Y_n^{(-k)}(\lambda)$ and the Bernstein basis functions as follows:

$$Y_n^{(-k)}(\lambda) = \frac{(-1)^{k-n}n!}{2^k}\lambda^n B_n^k(\lambda) \tag{1.38}$$

in the case when $n, k \in \mathbb{N}_0$ and $\lambda \in [0,1]$ (*cf.* [26]). The interested reader may refer to [26] for further identities containing the numbers $Y_n^{(-k)}(\lambda)$, the Poisson–Charlier polynomials, the Bell polynomials (i.e., exponential polynomials) and other kinds of combinatorial numbers and polynomials. Observe also that the combination of (1.26), (1.30) and (1.38) yields the following identity:

$$Y_n^{(-k)}(\lambda) = \frac{\lambda^n Ch_k B_n^{(n+1)} B_n^k(\lambda)}{k!} \tag{1.39}$$

which combines the numbers $Y_n^{(-k)}(\lambda)$, the positive higher-order Bernoulli numbers, the Bernstein basis functions and the Changhee numbers in a single formula.

Some formulas for the integral representations of the combinatorial numbers $Y_n^{(-k)}(\lambda)$ are given by

$$\int_0^1 Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = (-1)^{k-n}2^{-k}n!\binom{k}{n}\frac{\Gamma(2n+1)\Gamma(k-n+1)}{\Gamma(k+n+2)}, \tag{1.40}$$

and

$$\int_0^1 Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = (-1)^{k-n}2^{-k}n!\binom{k}{n}\sum_{j=0}^{k-n}(-1)^j\binom{k-n}{j}\frac{1}{2n+j+1}. \tag{1.41}$$

where $n \leq k$ (*cf.* [23]).

Another integral representation of the combinatorial numbers $Y_n^{(-k)}(\lambda)$, in terms of the Gauss hypergeometric function, is given by

$$\int_0^z Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = \frac{(-1)^{k-n}2^{-k}(k)_n z^{2n+1}}{2n+1}\,{}_2F_1(n-k, 2n+1; 2n+2; z),$$

where $n \leq k$ and ${}_2F_1$ denotes the Gauss hypergeometric function (*cf.* [23]).

Integral representations of the combinatorial polynomials $Q_n(x;\lambda,k)$ is given by

$$\int_0^1 Q_n(x;\lambda,k)\,\mathrm{d}\lambda = \frac{2^{-k}k!}{(k+n+1)!}\sum_{j=0}^{n}(-1)^{k-j}\binom{n}{j}(x)_{n-j}(n+j)!, \tag{1.42}$$

(*cf.* [23]).

In the following, we give a brief summary outlining what results were achieved in each section:

In Section 2, various identities, involving the negative higher-order combinatorial numbers and polynomials and other kinds of special numbers and polynomials such as the Stirling numbers, the Lah numbers, the negative higher-order Changhee numbers and polynomials, and the positive higher-order Bernoulli numbers and polynomials, are derived. In Section 3, by using the integral formulas of not only the negative higher-order combinatorial numbers and polynomials but also their generating functions, some identities and combinatorial sums are obtained. Also, some infinite series involving the negative higher-order combinatorial numbers are given, and the values of these infinite series are presented in terms of the falling factorials, the Catalan numbers, the Daehee numbers and the Changhee numbers.In Section 4, in order to give applications of these infinite series, two new sequences of special numbers are defined by their generating functions, and some properties of these number sequences are investigated. Some computation formulas and explicit formulas for these new number sequences. Moreover, we pose an open question related to one of these number sequences. In Section 5, by using an infinite series arising from the integration of the generating functions for the negative higher-order combinatorial numbers and polynomials, a new family of polynomials, associated with the Bernstein basis functions, are introduced. Additionally, some properties, such as symmetry property, integral formulas and derivative formulas, of these polynomials are investigated. Over and above, by implementing an explicit formula of these polynomials in Mathematica with the aid of the Wolfram programming language, we present some figures containing two dimensional plots of these newly introduced polynomial functions for some of their randomly selected special cases. In Section 6, we give some further results including series representations, combinatorial sums, integral formulas and relations for some of combinatorial numbers and poynomials. In Section 7, we conclude the present paper by providing some observations and comments on our results.

2. Some identities involving the negative higher-order combinatorial numbers and polynomials

In this section, we derive some identities that involve the negative higher-order combinatorial numbers $Y_n^{(-k)}(\lambda)$ and polynomials $Q_n(x;\lambda,k)$.

Theorem 2.1. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$Y_n^{(-k)}(\lambda) = 2^{-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (j)_n \lambda^{j+n}. \tag{2.1}$$

Proof. By using the binomial theorem in (1.34), we have

$$\begin{aligned} \sum_{n=0}^{\infty} Y_n^{(-k)}(\lambda) \frac{t^n}{n!} &= 2^{-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \lambda^j (1+\lambda t)^j \\ &= 2^{-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \lambda^j \sum_{n=0}^{\infty} (j)_n \lambda^n \frac{t^n}{n!}. \end{aligned}$$

Comparing coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, we get the desired result. □

In the case of when $\lambda = -1$, (2.1) reduces to

$$Y_n^{(-k)}(-1) = \frac{(-1)^{k+n}}{2^k} \sum_{j=0}^{k} \binom{k}{j} (j)_n. \tag{2.2}$$

Combining the above equation with (1.33) yields a relation between the numbers $Y_n^{(-k)}(\lambda)$ and the negative higher-order Changhee numbers given by the following corollary:

Corollary 2.2. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$Y_n^{(-k)}(-1) = (-1)^{k+n} Ch_n^{(-k)}. \tag{2.3}$$

Combining (1.11) with (2.1) yields the following corollary:

Corollary 2.3. *Let $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Then we have*

$$Y_n^{(-k)}(\lambda) = 2^{-k} \sum_{j=0}^{k} \sum_{r=1}^{n} (-1)^{k+n-j-r} \binom{k}{j} L(n,r) (j)^{(r)} \lambda^{j+n}. \tag{2.4}$$

Combining (1.1) with (2.1) yields the following corollary:

Corollary 2.4. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$Y_n^{(-k)}(\lambda) = 2^{-k} \sum_{j=0}^{k} \sum_{r=0}^{n} (-1)^{k-j} \binom{k}{j} S_1(n,r) j^r \lambda^{j+n}. \tag{2.5}$$

Remark 2.5. Combining (2.5) with (1.12) yields the following result:

$$Y_n^{(-k)}(\lambda) = (-1)^k \frac{k!\lambda^n}{2^k} \sum_{r=0}^{n} S_1(n,r) y_1(r,k;-\lambda) \tag{2.6}$$

whose different proof was given by Kucukoglu et al. [26, Theorem 3, p.7].

By combining (2.6) with (1.26), we arrive at the following corollary:

Corollary 2.6. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$Y_n^{(-k)}(\lambda) = \frac{\lambda^n B_k^{(k+1)}}{2^k} \sum_{r=0}^{n} S_1(n,r) y_1(r,k;-\lambda). \tag{2.7}$$

Also, combining (2.6) with (1.30) yields the following corollary:

Corollary 2.7. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$Y_n^{(-k)}(\lambda) = \lambda^n Ch_k \sum_{r=0}^{n} S_1(n,r)\, y_1(r,k;-\lambda). \tag{2.8}$$

Summing the equation (2.6) over all $0 \leq k \leq v$ and then using (1.12), we obtain

$$\sum_{k=0}^{v} Y_n^{(-k)}(\lambda) = \sum_{r=0}^{n} S_1(n,r) \sum_{k=0}^{v} (-1)^k \frac{\lambda^n}{2^k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \lambda^j j^r. \tag{2.9}$$

By using (2.9), we get

$$\sum_{k=0}^{v} Y_n^{(-k)}(\lambda) = \sum_{r=0}^{n} S_1(n,r) \sum_{k=0}^{v} (-1)^k \frac{\lambda^n k!}{2^k} \sum_{j=0}^{k} (-1)^j \frac{\lambda^j}{j!(k-j)!} j^r.$$

Combining (1.30) and (1.26) with the above equation, we get

$$\sum_{k=0}^{v} Y_n^{(-k)}(\lambda) = \sum_{k=0}^{v} \sum_{r=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} \frac{Ch_k S_1(n,r)}{B_j^{(j+1)} B_{k-j}^{(k-j+1)}} \lambda^{j+n} j^r.$$

which yields the following theorem:

Theorem 2.8. *Let $n, k, v \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{v} \left(Y_n^{(-k)}(\lambda) - \sum_{r=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} \frac{\lambda^{j+n} Ch_k S_1(n,r)}{B_j^{(j+1)} B_{k-j}^{(k-j+1)}} j^r \right) = 0.$$

By substituting $\lambda = 1$ into (2.5), we get

$$\sum_{k=0}^{\infty} Y_n^{(-k)}(1) \frac{2^k}{k!} = \sum_{r=0}^{n} S_1(n,r) \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^r \right) \frac{1}{k!}. \tag{2.10}$$

Combining the above equation with (1.8), we get

$$\sum_{k=0}^{\infty} Y_n^{(-k)}(1) \frac{2^k}{k!} = \sum_{r=0}^{n} \sum_{m=1}^{r} S_1(n,r)\, S_2(r,m). \tag{2.11}$$

which, by using the well-known orthogonality relation of the Stirling numbers (see, for details, [45], [3]), reduces to an identity given by the following theorem:

Theorem 2.9. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{\infty} Y_n^{(-k)}(1) \frac{2^k}{k!} = 1. \tag{2.12}$$

By (1.34) and (1.35), we obtain

$$\mathcal{G}(t,k;\lambda) = e^{-x \ln(1+\lambda t)} \sum_{n=0}^{\infty} Q_n(x;\lambda,k) \frac{t^n}{n!}.$$

By using Taylor series expansion of $e^{-x \ln(1+\lambda t)}$ in the above equation, we have

$$\sum_{n=0}^{\infty} Y_n^{(-k)}(\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n x^n \frac{(\ln(1+\lambda t))^n}{n!} \sum_{n=0}^{\infty} Q_n(x;\lambda,k) \frac{t^n}{n!}.$$

By using (1.2) in the above equation, we have

$$\sum_{n=0}^{\infty} Y_n^{(-k)}(\lambda)\frac{t^n}{n!} = \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^n x^n \lambda^m S_1(m,n)\frac{t^m}{m!}\sum_{n=0}^{\infty} Q_n(x;\lambda,k)\frac{t^n}{n!},$$

which, by using the Cauchy product, yields

$$\sum_{m=0}^{\infty} Y_m^{(-k)}(\lambda)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\sum_{n=0}^{j}(-1)^n\binom{m}{j}Q_{m-j}(x;\lambda,k)\lambda^j x^n S_1(j,n)\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both-sides of the above equation yields the following theorem:

Theorem 2.10. *Let $m,k \in \mathbb{N}_0$. Then we have*

$$Y_m^{(-k)}(\lambda) = \sum_{j=0}^{m}\sum_{n=0}^{j}(-1)^n\binom{m}{j}S_1(j,n)Q_{m-j}(x;\lambda,k)\lambda^j x^n.$$

3. Identities and combinatorial sums arising from the integrals of the generating functions for the negative higher-order combinatorial numbers

In this section, we derive some identities and combinatorial sums with the aid of the integrals of not only the negative higher-order combinatorial numbers $Y_n^{(-k)}(\lambda)$, but also their generating functions.

Integrating both-sides of the equation (1.34), with respect to the parameter t, from 0 to z, we get the following integral formula:

$$\int_0^z \mathcal{G}(t,k;\lambda)\,\mathrm{d}t = \int_0^z 2^{-k}(\lambda(1+\lambda t)-1)^k\,\mathrm{d}t \tag{3.1}$$

$$= \frac{2^{-k}\left((\lambda(1+\lambda z)-1)^{k+1}-(\lambda-1)^{k+1}\right)}{\lambda^2(k+1)} \tag{3.2}$$

$$= \frac{2}{\lambda^2(k+1)}\left(\mathcal{G}(z,k+1;\lambda)-\mathcal{G}(0,k+1;\lambda)\right). \tag{3.3}$$

On the other hand, by integrating both-sides of the equation (1.34), with respect to the parameter t, from 0 to z, we also get

$$\int_0^z \mathcal{G}(t,k;\lambda)\,\mathrm{d}t = \sum_{n=0}^{\infty}\frac{Y_n^{(-k)}(\lambda)}{n!}\int_0^z t^n\mathrm{d}t \tag{3.4}$$

$$= \sum_{n=0}^{\infty}\frac{Y_n^{(-k)}(\lambda)}{(n+1)!}z^{n+1}. \tag{3.5}$$

Combining the above two results, we get the following theorem:

Theorem 3.1. *Let $k \in \mathbb{N}_0$ and $\lambda \neq 0$. Then we have*

$$\frac{(\lambda(1+\lambda z)-1)^{k+1}-(\lambda-1)^{k+1}}{\lambda^2(k+1)2^k} = \sum_{n=0}^{\infty}\frac{Y_n^{(-k)}(\lambda)}{(n+1)!}z^{n+1}. \tag{3.6}$$

Next, we give some applications of (3.6) as follows:

If we substitute $\lambda = z = \frac{1}{2}$ into (3.6), then we get the following corollary:

Corollary 3.2. *Let $k \in \mathbb{N}_0$. Then we have*

$$\sum_{n=0}^{\infty} \frac{Y_n^{(-k)}\left(\frac{1}{2}\right)}{2^n (n+1)!} = \frac{(-1)^k}{k+1}\left(\frac{4^{k+1}-3^{k+1}}{2^{4k}}\right). \tag{3.7}$$

Combining (3.7) with (1.28) yields the following corollary:

Corollary 3.3. *Let $k \in \mathbb{N}_0$. Then we have*

$$\sum_{n=0}^{\infty} \frac{Y_n^{(-k)}\left(\frac{1}{2}\right)}{2^n (n+1)!} = \left(\frac{4^{k+1}-3^{k+1}}{2^{4k}}\right)\frac{D_k}{k!}. \tag{3.8}$$

Combining (3.7) with (1.9) also yields the following corollary:

Corollary 3.4. *Let $k \in \mathbb{N}_0$. Then we have*

$$\sum_{n=0}^{\infty} \frac{Y_n^{(-k)}\left(\frac{1}{2}\right)}{2^n (n+1)!} = \frac{(-1)^k\left(4^{k+1}-3^{k+1}\right)}{2^{4k}\binom{2k}{k}} C_k. \tag{3.9}$$

As an application, if we combine (3.7) with (1.37), we get the following convergent series:

Corollary 3.5. *Let $k \in \mathbb{N}_0$. Then we have*

$$\sum_{n=0}^{\infty} (-1)^n \binom{k}{n} \frac{1}{4^n (n+1)} = \frac{1}{k+1}\left(\frac{4^{k+1}-3^{k+1}}{4^k}\right). \tag{3.10}$$

Note that the equation (3.10) is reduced to a novel finite sum formula since $\binom{k}{n} = 0$ when $k < n$.

By substituting $\lambda = -1$ into (3.6), and then combining the final equation with (2.3), we also get an infinite series involving generating functions for the negative higher-order Changhee numbers or the numbers $Y_n^{(-k)}(-1)$ by the following corollary:

Corollary 3.6. *Let $k \in \mathbb{N}_0$. Then we have*

$$\sum_{n=0}^{\infty} (-1)^n \frac{Ch_n^{(-k)}}{(n+1)!} z^{n+1} = \frac{2}{k+1}\left(1-\left(1-\frac{z}{2}\right)^{k+1}\right). \tag{3.11}$$

Remark 3.7. Differentiating (3.11) with respect to z yields

$$\sum_{n=0}^{\infty} (-1)^n Ch_n^{(-k)} \frac{z^n}{n!} = \left(1-\frac{z}{2}\right)^k \tag{3.12}$$

which is the special case of (1.31) when $x = 0$ and $t = -z$.

4. New sequences of special numbers and their generating functions

In this section, we define two new sequences of special numbers with their generating functions, and we investigate some of their properties. We give some computation formulas and explicit formulas for these new number sequences. Moreover, we pose an open question related to one of these number sequences.

4.1. Generating functions for a new sequence of special numbers $\gamma_n(k)$

Using (3.11), we define a new sequence of special numbers by the following definition:

Definition 4.1. A new sequence of special numbers $\gamma_n(k)$ is defined by means of the following generating function:

$$f_1(z,k) := \frac{2}{(k+1)z}\left(1-\left(1-\frac{z}{2}\right)^{k+1}\right) = \sum_{n=0}^{\infty}\gamma_n(k)\frac{z^n}{n!}, \tag{4.1}$$

where $k \in \mathbb{N}_0$, $z \in \mathbb{C}$ with $z \neq 0$ and $|z| < 2$.

Combining (4.1) with (3.11) and (1.33), we get a computation formula for the numbers $\gamma_n(k)$ by the following theorem:

Theorem 4.2. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$\gamma_n(k) = \frac{(-1)^n}{2^k(k+1)}\sum_{j=0}^{k}\binom{k}{j}(j)_n\,. \tag{4.2}$$

Combining (4.1) with (3.11) and (1.28), we get an explicit formula for the numbers $\gamma_n(k)$ by the following theorem:

Theorem 4.3. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$\gamma_n(k) = \frac{D_n Ch_n^{(-k)}}{n!}. \tag{4.3}$$

If we combine (4.3) with (2.3), we get another explicit formula for the numbers $\gamma_n(k)$ as in the following theorem:

Theorem 4.4. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$\gamma_n(k) = \frac{(-1)^{k+n} D_n Y_n^{(-k)}(-1)}{n!}. \tag{4.4}$$

By (4.4), first few values of the numbers $\gamma_n(k)$ are given as follows:

$$\gamma_0(k) = 1, \quad \gamma_1(k) = -\frac{k}{4}, \quad \gamma_2(k) = \frac{(k)_2}{12},$$
$$\gamma_3(k) = -\frac{(k)_3}{32}, \quad \gamma_4(k) = \frac{(k)_4}{80}, \quad \gamma_5(k) = -\frac{(k)_5}{192},$$

and so on.

By using (1.1), the numbers $\gamma_n(k)$ can also be represented in term of the Stirling numbers of the first kind as follows:

$$\gamma_n(k) = \frac{(-1)^n}{A}\sum_{j=0}^{n} S_1(n,j)\,k^j. \tag{4.5}$$

As a result of the above representation, we arrive at the following open question:

Open Question: *How can we compute the value of A in the equation* (4.5)*?*

4.2. Generating functions for a new sequence of special numbers $\beta_n(k)$

Using (3.12), we define another new sequence of special numbers by the following definition:

Definition 4.5. A new sequence of special numbers $\beta_n(k)$ is defined by means of the following generating function:

$$f_2(z,k) := \left(1-\frac{z}{2}\right)^k = \sum_{n=0}^{\infty}\beta_n(k)\frac{z^n}{n!}, \tag{4.6}$$

where $k \in \mathbb{N}_0$, $z \in \mathbb{C}$ with $|z| < 2$.

Combining (4.6) with (3.12), we get an explicit formula for the numbers $\beta_n(k)$ by the following theorem:

Theorem 4.6. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$\beta_n(k) = (-1)^n \, Ch_n^{(-k)}. \tag{4.7}$$

Combining (4.7) with (2.3), we get another explicit formula for the numbers $\beta_n(k)$ as in the following theorem:

Theorem 4.7. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$\beta_n(k) = (-1)^k \, Y_n^{(-k)}(-1). \tag{4.8}$$

By (4.8), first few values of the numbers $\beta_n(k)$ are given as follows:

$$\beta_0(k) = 1, \quad \beta_1(k) = -\frac{k}{2}, \quad \beta_2(k) = \frac{(k)_2}{4},$$
$$\beta_3(k) = -\frac{(k)_3}{8}, \quad \beta_4(k) = \frac{(k)_4}{16}, \quad \beta_5(k) = -\frac{(k)_5}{32},$$

and so on.

More generally, by using (4.6), we get

$$\sum_{n=0}^{\infty} \beta_n(k) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \binom{k}{n} \left(-\frac{z}{2}\right)^n.$$

Comparing the coefficients of $\frac{z^n}{n!}$ on both-sides of the above equation yields the following theorem which involves a computation formula for the numbers $\beta_n(k)$:

Theorem 4.8. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$\beta_n(k) = \frac{(-1)^n n!}{2^n} \binom{k}{n}. \tag{4.9}$$

By combining (4.9) with (1.30), we arrive at the following corollary:

Corollary 4.9. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$\beta_n(k) = \binom{k}{n} Ch_n. \tag{4.10}$$

Substituting $k = 2n$ into (4.9) and combining the final equation with (1.9), we also arrive at the following theorem:

Theorem 4.10. *Let $n \in \mathbb{N}_0$. Then we have*

$$\beta_n(2n) = (-1)^n \frac{(n+1)! C_n}{2^n}.$$

5. A new family of polynomials associated with the Bernstein basis functions

In this section, we introduce a new family of polynomials by their ordinary generating functions. We also investigate some properties of these polynomials, and give some of their graphical presentations for some randomly selected special cases.

Substituting (1.37) into (3.6) yields the following corollary:

Corollary 5.1. *Let $k \in \mathbb{N}_0$ and $\lambda \neq 0$. Then we have*

$$\frac{(\lambda(1+\lambda z) - 1)^{k+1} - (\lambda - 1)^{k+1}}{\lambda^2 (k+1)} = \sum_{n=0}^{\infty} \frac{(k)_n \, \lambda^{2n} (\lambda - 1)^{k-n}}{(n+1)!} z^{n+1}. \tag{5.1}$$

Remark 5.2. Alternate form of (5.1) is given by

$$\frac{(\lambda(1+\lambda z)-1)^{k+1}-(\lambda-1)^{k+1}}{\lambda^2(\lambda-1)^k(k+1)}=\sum_{n=0}^{\infty}\frac{(k)_n}{(n+1)!}\left(\frac{\lambda^2}{\lambda-1}\right)^n z^{n+1}.$$

By the aid of (5.1), we define a new family of polynomials as in the following definition:

Definition 5.3. Let $k \in \mathbb{N}_0$ and $\lambda, z \in \mathbb{R} \setminus \{0\}$ (or $\mathbb{C} \setminus \{0\}$). A new family of polynomials, denoted by $\alpha_n(\lambda; k)$, are defined by the following ordinary generating functions:

$$\begin{aligned}\mathcal{F}_\alpha(z,\lambda;k) &:= \frac{(\lambda(1+\lambda z)-1)^{k+1}-(\lambda-1)^{k+1}}{z\lambda^2(k+1)} \\ &= \sum_{n=0}^{\infty}\alpha_n(\lambda;k)z^n. \qquad (5.2)\end{aligned}$$

Notice here that there is one generating function for each value of k.

Using (5.2) and (5.1) yields an explicit formula for the polynomials $\alpha_n(\lambda; k)$ as following theorem:

Theorem 5.4. *Let $k, n \in \mathbb{N}_0$. Then we have*

$$\alpha_n(\lambda;k)=\frac{(k)_n\lambda^{2n}(\lambda-1)^{k-n}}{(n+1)!}. \qquad (5.3)$$

By (5.3), the polynomials $\alpha_n(\lambda; k)$ are computed as follows:

$$\begin{aligned}\alpha_0(\lambda;k) &= (\lambda-1)^k, \\ \alpha_1(\lambda;k) &= \frac{k\lambda^2}{2}(\lambda-1)^{k-1}, \\ \alpha_2(\lambda;k) &= \frac{(k)_2\lambda^4}{6}(\lambda-1)^{k-2}, \\ &\vdots \\ \alpha_{k-1}(\lambda;k) &= \lambda^{2k-2}(\lambda-1), \\ \alpha_k(\lambda;k) &= \frac{\lambda^{2k}}{k+1},\end{aligned}$$

so that $\alpha_n(\lambda; k) = 0$ if $n > k$.

Combining (5.3) with (1.19), we arrive at the relation between the Bernstein basis functions and the polynomials $\alpha_n(\lambda; k)$ given by the following theorem:

Theorem 5.5. *Let $k \in \mathbb{N}_0$ and $n = 0, 1, \ldots, k$. Then we have*

$$\alpha_n(\lambda;k)=\frac{(-1)^{k-n}\lambda^n}{n+1}B_n^k(\lambda). \qquad (5.4)$$

Some properties of the polynomials $\alpha_n(\lambda; k)$, such as symmetry property, integral formulas and derivative formula, are given as follows:

By combining (5.4) with the following identity (*cf.* [1, 27, 34, 35, 36, 44]):

$$B_{k-n}^k(1-\lambda)=B_n^k(\lambda),$$

we arrive at a symmetry property for the polynomials $\alpha_n(\lambda; k)$ given by the following theorem:

Theorem 5.6. *Let $k \in \mathbb{N}_0$ and $n = 0, 1, \ldots, k$. Then, the polynomials $\alpha_n(\lambda; k)$ satisfy the following symmetry property:*

$$\alpha_{k-n}(1-\lambda;k)=\frac{(n+1)(\lambda-1)^k}{(k-n+1)\lambda^n(1-\lambda)^n}\alpha_n(\lambda;k). \qquad (5.5)$$

Integrating (5.4), with respect to the parameter λ, from 0 to 1, we get

$$\int_0^1 \alpha_n(\lambda;k)\,\mathrm{d}\lambda = \frac{(-1)^{k-n}}{n+1}\int_0^1 \lambda^n B_n^k(\lambda)\mathrm{d}\lambda \tag{5.6}$$

which, by using (1.19), yields

$$\int_0^1 \alpha_n(\lambda;k)\,\mathrm{d}\lambda = \frac{(-1)^{k-n}}{n+1}\binom{k}{n}\int_0^1 \lambda^{2n}(1-\lambda)^{k-n}\,\mathrm{d}\lambda. \tag{5.7}$$

By using (1.15) in the right-hand side of the above equation, we arrive at an integral formula for the polynomials $\alpha_n(\lambda;k)$ given by the following theorem:

Theorem 5.7. *Let $k \in \mathbb{N}_0$ and $n = 0, 1, \ldots, k$. Then we have*

$$\int_0^1 \alpha_n(\lambda;k)\,\mathrm{d}\lambda = \frac{(-1)^{k-n}}{n+1}\binom{k}{n}B(2n+1, k-n+1). \tag{5.8}$$

By combining (1.16) and (1.17) with (5.8), we get another integral formula for the polynomials $\alpha_n(\lambda;k)$ given by the following corollary:

Corollary 5.8. *Let $k \in \mathbb{N}_0$ and $n = 0, 1, \ldots, k$. Then we have*

$$\begin{aligned}\int_0^1 \alpha_n(\lambda;k)\,\mathrm{d}\lambda &= \frac{(-1)^{k-n}}{n+1}\binom{k}{n}\frac{\Gamma(2n+1)\Gamma(k-n+1)}{\Gamma(n+k+2)} &(5.9)\\ &= \frac{(-1)^{k-n}}{n+1}\binom{k}{n}\frac{(2n)!\,(k-n)!}{(n+k+1)!} &(5.10)\\ &= \frac{(-1)^{k-n}\,k!n!}{(n+k+1)!}C_n. &(5.11)\end{aligned}$$

Differentiating (5.4), with respect to the parameter λ, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\{\alpha_n(\lambda;k)\} &= \frac{(-1)^{k-n}}{n+1}\frac{\mathrm{d}}{\mathrm{d}\lambda}\{\lambda^n B_n^k(\lambda)\}\\ &= \frac{(-1)^{k-n}}{n+1}\left(n\lambda^{n-1}B_n^k(\lambda) + \lambda^n\frac{\mathrm{d}}{\mathrm{d}\lambda}\{B_n^k(\lambda)\}\right).\end{aligned}$$

Combining the above equation with the following identity (*cf.* [14], [27], [34, Corollary 3.9, p.7]):

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\{B_n^k(\lambda)\} = k\left(B_{n-1}^{k-1}(\lambda) - B_n^{k-1}(\lambda)\right),$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\{\alpha_n(\lambda;k)\} = \frac{(-1)^{k-n}}{n+1}\left(n\lambda^{n-1}B_n^k(\lambda) + k\lambda^n\left(B_{n-1}^{k-1}(\lambda) - B_n^{k-1}(\lambda)\right)\right).$$

which, by using (5.4), yields a derivative formula for the polynomials $\alpha_n(\lambda;k)$ given by the following theorem:

Theorem 5.9. *Let $k \in \mathbb{N}$ and $n = 1, \ldots, k$. Then we have*

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\{\alpha_n(\lambda;k)\} = n\alpha_n(\lambda;k) + k\lambda\left(\frac{n\lambda}{n+1}\alpha_{n-1}(\lambda;k-1) + \alpha_n(\lambda;k-1)\right).$$

5.1. Some graphical presentations of the functions arising from the polynomials $\alpha_n(\lambda;k)$

Observe that $\alpha_n(\lambda;k)$ are polynomial functions of variable λ. Therefore, in order to investigate the behaviour of these polynomial functions, we implement the formula, given by (5.3), with the aid of the Wolfram programming language in Mathematica [54], and as result of this implementation we here give some two dimensional plots of the polynomial functions arising from the polynomials $\alpha_n(\lambda;k)$.

Figure 1 includes some plots of the polynomial functions $\alpha_n(\lambda;k)$ for randomly selected special cases when $n \in \{0,1,2,3\}$ with $k = 3$ and $\lambda \in \left[\frac{1}{2}, 1\right]$.

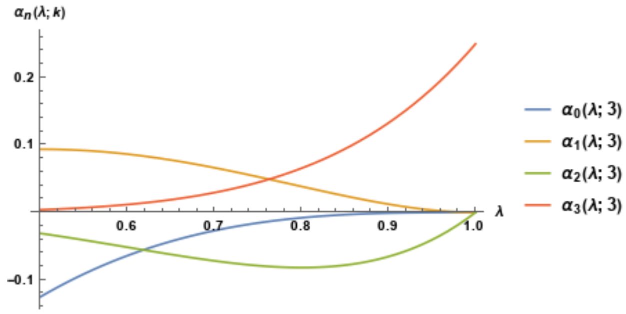

Figure 1. Plots of the polynomial functions $\alpha_n(\lambda;k)$ for randomly selected special cases when $n \in \{0,1,2,3\}$ with $k = 3$ and $\lambda \in \left[\frac{1}{2}, 1\right]$.

In addition to the Figure 1, by setting the range of the variable λ to be $\lambda \in [0,1]$, we also give Figure 2 as follows:

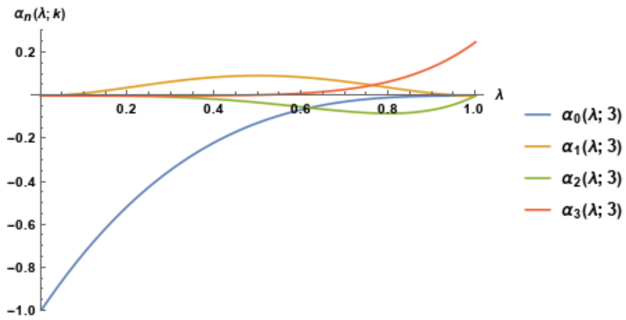

Figure 2. Plots of the polynomial functions $\alpha_n(\lambda;k)$ for randomly selected special cases when $n \in \{0,1,2,3\}$ with $k = 3$ and $\lambda \in [0,1]$.

In addition to the Figure 1 and Figure 2, by widening the range of the variable λ further, we also give Figure 3 which allows us to examine the behavior of functions in a wider perspective compared to the Figure 1 and Figure 2.

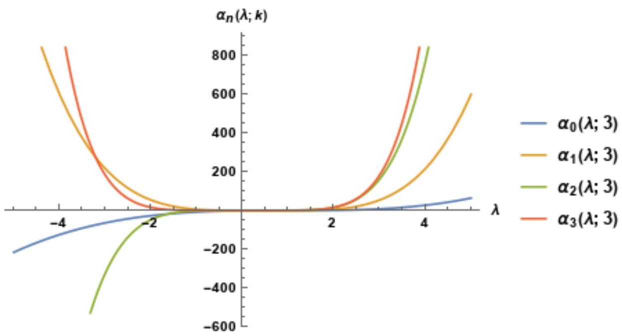

Figure 3. Plots of the polynomial functions $\alpha_n(\lambda;k)$ for randomly selected special cases when $n \in \{0,1,2,3\}$ with $k = 3$ and $\lambda \in [-5,5]$.

Figure 4 includes some plots of the polynomial functions $\alpha_n(\lambda; k)$ for randomly selected special cases when $n \in \{0, 1, 2, 3, 4\}$ with $k = 4$ and $\lambda \in \left[\frac{1}{2}, 1\right]$.

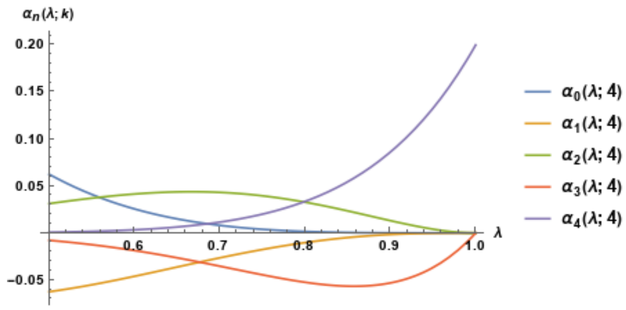

Figure 4. Plots of the polynomial functions $\alpha_n(\lambda; k)$ for randomly selected special cases when $n \in \{0, 1, 2, 3, 4\}$ with $k = 4$ and $\lambda \in \left[\frac{1}{2}, 1\right]$.

In addition to the Figure 4, by widening the range of the variable λ further, we also give Figure 5 which allows us to examine the behavior of functions in a wider perspective compared to the Figure 4.

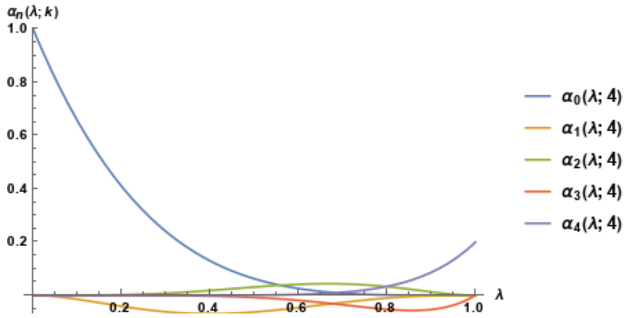

Figure 5. Plots of the polynomial functions $\alpha_n(\lambda; k)$ for randomly selected special cases when $n \in \{0, 1, 2, 3, 4\}$ with $k = 4$ and $\lambda \in [0, 1]$.

Figure 6 includes some plots of the polynomial functions $\alpha_n(\lambda; k)$ for randomly selected special cases when $k \in \{5, 6, 7, 8\}$ with $n = 4$ and $\lambda \in \left[\frac{1}{2}, 1\right]$.

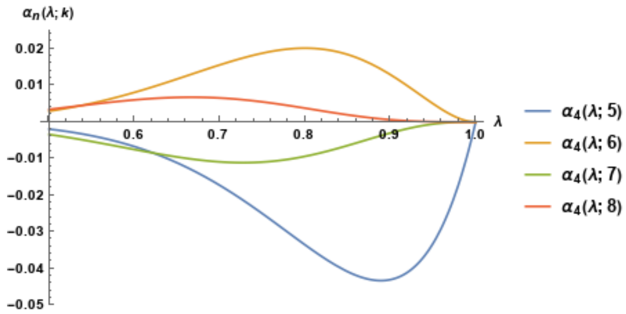

Figure 6. Plots of the polynomial functions $\alpha_n(\lambda; k)$ for randomly selected special cases when $k \in \{5, 6, 7, 8\}$ with $n = 4$ and $\lambda \in \left[\frac{1}{2}, 1\right]$.

Figure 7 includes some plots of the polynomial functions $\alpha_n(\lambda; k)$ for randomly selected special cases when $k \in \{6, 7, 8, 9, 10\}$ with $n = 5$ and $\lambda \in [0, 1]$.

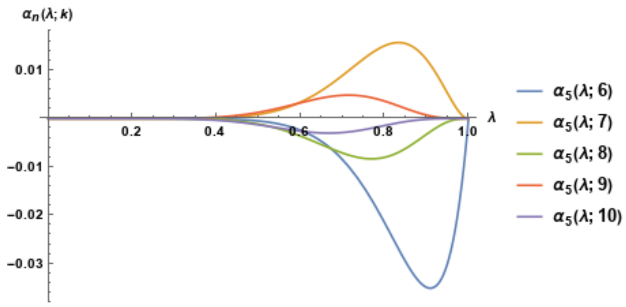

Figure 7. Plots of the polynomial functions $\alpha_n(\lambda; k)$ for randomly selected special cases when $k \in \{6, 7, 8, 9, 10\}$ with $n = 5$ and $\lambda \in [0, 1]$.

In addition to the Figure 7, by narrowing the range of the variable λ slightly, we also give Figure 8 which allows us to examine the function's behavior more closely compared to the Figure 7.

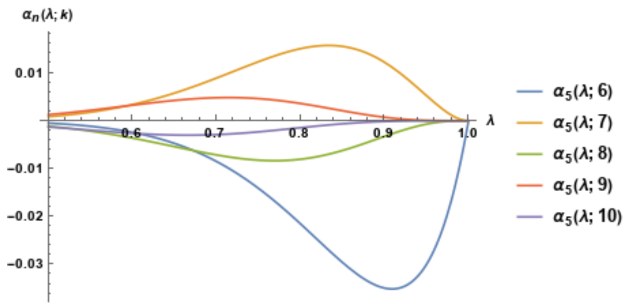

Figure 8. Plots of the polynomial functions $\alpha_n(\lambda; k)$ for randomly selected special cases when $k \in \{6, 7, 8, 9, 10\}$ with $n = 5$ and $\lambda \in \left[\frac{1}{2}, 1\right]$.

6. Further series representations, integral formulas and relations for combinatorial numbers and poynomials

In this section, we give further series representations, integral formulas and relations for combinatorial numbers and poynomials.

Setting $z = 2$ into (3.11) and combining the final equation with (1.30) yields the following theorem:

Theorem 6.1. *Let $k \in \mathbb{N}_0$. Then we have*

$$\sum_{n=0}^{\infty} \frac{Ch_n^{(-k)}}{(n+1)\,Ch_n} = \frac{1}{k+1}. \tag{6.1}$$

Combining (6.1) with (1.20) and (1.21), we also have the following corollary:

Corollary 6.2. *Let $k \in \mathbb{N}_0$. Then we have*

$$\sum_{n=0}^{\infty} \frac{Ch_n^{(-k)}}{(n+1)\,Ch_n} = \binom{k}{n} B\,(n+1, k-n+1)\,, \tag{6.2}$$

and

$$\sum_{n=0}^{\infty} \frac{Ch_n^{(-k)}}{(n+1)\,Ch_n} = \int_0^1 B_n^k(x)\mathrm{d}x. \tag{6.3}$$

Theorem 6.3. *Let $k, n \in \mathbb{N}_0$. Then we have*

$$\int_0^1 Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = 2^{-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(j)_n}{n+j+1}. \tag{6.4}$$

Proof. Integrating both sides of the equation (2.1) with respect to the parameter λ, from 0 to 1, we get

$$\int_0^1 Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = 2^{-k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(j)_n\int_0^1 \lambda^{j+n}\mathrm{d}\lambda$$

which yields the desired result. □

Combining (6.4) with (1.40) and (1.17), we arrive at the following corollary including combinatorial sums:

Corollary 6.4. *Let $k, n \in \mathbb{N}_0$ with $k \geq n$. Then we have*

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{(j)_n}{n+j+1} = (-1)^{k-n}\frac{\Gamma(2n+1)\Gamma(k+1)}{\Gamma(k+n+2)}.$$

and

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{(j)_n}{n+j+1} = (-1)^{k-n}(k)_n\frac{(2n)!\,(k-n)!}{(k+n+1)!}.$$

Combining the above corollary with (1.1), we also arrive at the following corollary:

Corollary 6.5. *Let $k, n \in \mathbb{N}_0$ with $k \geq n$. Then we have*

$$\sum_{j=0}^{k}\sum_{k=0}^{n}(-1)^{k-j}\binom{k}{j}\frac{S_1(n,k)\,j^k}{n+j+1} = (-1)^{k-n}(k)_n\frac{(2n)!\,(k-n)!}{(k+n+1)!}.$$

By combining (6.4) with (1.41), we also get the following corollary:

Corollary 6.6. *Let $k, n \in \mathbb{N}_0$ with $k \geq n$. Then we have*

$$\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\frac{(j)_n}{n+j+1} = (k)_n\sum_{j=0}^{k-n}(-1)^{n+j}\binom{k-n}{j}\frac{1}{2n+j+1}.$$

Theorem 6.7. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$\int_0^1 Q_n(x;\lambda,k)\,\mathrm{d}\lambda = 2^{-k}\sum_{j=0}^{n}\sum_{r=0}^{k}(-1)^{k-r}\binom{n}{j}\binom{k}{r}\frac{(x)_{n-j}(r)_j}{n+r+1}. \tag{6.5}$$

Proof. Substituting (2.1) into (1.36) yields

$$Q_n(x;\lambda,k) = 2^{-k}\sum_{j=0}^{n}\sum_{r=0}^{k}(-1)^{k-r}\binom{n}{j}\binom{k}{r}\lambda^{n+r}(x)_{n-j}(r)_j\,. \tag{6.6}$$

Integrating both sides of the above equation with respect to the parameter λ, from 0 to 1, we get

$$\int_0^1 Q_n(x;\lambda,k)\,\mathrm{d}\lambda = 2^{-k}\sum_{j=0}^{n}\sum_{r=0}^{k}(-1)^{k-r}\binom{n}{j}\binom{k}{r}(x)_{n-j}(r)_j\int_0^1 \lambda^{n+r}\mathrm{d}\lambda$$

which yields the desired result. □

Combining (6.5) with (1.42), we arrive at the following corollary including combinatorial sums:

Corollary 6.8. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n}\sum_{r=0}^{k}(-1)^{k-r}\binom{n}{j}\binom{k}{r}\frac{(x)_{n-j}(r)_j}{n+r+1}=\frac{k!}{(k+n+1)!}\sum_{j=0}^{n}(-1)^{k-j}\binom{n}{j}(x)_{n-j}(n+j)!.$$

In the case of when $\lambda = -1$, (6.6) yields

$$Q_n(x;-1,k)=(-1)^{n+k}2^{-k}\sum_{j=0}^{n}\sum_{r=0}^{k}\binom{n}{j}\binom{k}{r}(x)_{n-j}(r)_j. \tag{6.7}$$

Combining the above equation with (1.32) yields a relation between the following result the polynomials $Q_n(x;\lambda,k)$ and the negative higher-order Changhee polynomials given by the following corollary:

Corollary 6.9. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$Q_n(x;-1,k)=(-1)^{n+k}Ch_n^{(-k)}(x). \tag{6.8}$$

7. Conclusion

Some observations and comments on our results are given as follows:

In this paper, we have presented an investigation on a certain family of negative higher-order combinatorial numbers and polynomials, and we have derived some identities, involving not only the negative higher-order combinatorial numbers and polynomials, but also some kinds of special numbers and polynomials such as the Stirling numbers, the Lah numbers, the negative higher-order Changhee numbers and polynomials, and the positive higher-order Bernoulli numbers and polynomials. As a result of the application of the Riemann integral to the generating functions for the negative higher-order combinatorial numbers, we have also defined two new sequences of special numbers, denoted respectively by $\gamma_n(k)$ and $\beta_n(k)$, with the aid of their generating functions. Moreover, we have given some computation formulas for $\gamma_n(k)$ and $\beta_n(k)$, and we have represented the numbers $\gamma_n(k)$ and $\beta_n(k)$ in terms of some other kinds of special numbers. As a result of one of the aforementioned representations, we have rised the following open question:

How can we calculate the value of A in the equation (4.5):

$$\gamma_n(k)=\frac{(-1)^n}{A}\sum_{j=0}^{n}S_1(n,j)k^j,$$

which is obtained when the numbers $\gamma_n(k)$ are represented in terms of the Stirling numbers of the first kind?

Besides, we have also introduced new family of polynomials, denoted by $\alpha_n(\lambda;k)$, associated with the Bernstein basis functions. We have given some properties of these polynomials such as symmetry property (see Theorem 5.6), integral formulas (see Theorem 5.7 and Corollary 5.8) and derivative formula (see Theorem 5.9). Moreover, in order to investigate the behaviour of the polynomial functions $\alpha_n(\lambda;k)$, the formula, given by (5.3), have been implemented, and as result of this implementation, some graphical presentations of the polynomial functions $\alpha_n(\lambda;k)$ have been given (see Figure 1-Figure 8). At the end of this paper, we have obtained some further results including series representations, combinatorial sums, integral formulas and relations for some of combinatorial numbers and poynomials.

Finally, in this paper, a considerably large number of results have been obtained. It should be stated here that the results of this paper have potential for attracting attention of researchers working on pure and applied mathematics. For future studies, it is planned to investigate higher-order derivative formulas for the negative higher-order combinatorial numbers and polynomials, and to give further applications of these numbers and polynomials.

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