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Contour Integration for the Improper Rational Functions

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Abstract

In this paper, we give the intact results for the contour integral of the rational functions in series of the complete Bell polynomials. As applications, we show several interesting examples for the contour integral of the improper rational functions.

Keywords: Rational function, Improper rational functions, Contour integration, Complete Bell polynomials

2010 MSC: 30E20, 30C15

1. Introduction

The complete Bell polynomials $\mathbf{B}_n(z_1, z_2, \ldots, z_n)$ are defined by (see [1] and [3, p.173])

$$\exp\left(\sum_{k=1}^{\infty} z_k \frac{z^k}{k!}\right) = \sum_{n=0}^{\infty} \mathbf{B}_n(z_1, z_2, \ldots, z_n)\frac{z^n}{n!}, \quad \mathbf{B}_0 := 1,$$

which exact expression is

$$\mathbf{B}_n(z_1, z_2, \cdots, z_n) = \sum_{\pi(n)} \frac{n!}{m_1!m_2!\cdots m_n!}\left(\frac{z_1}{1!}\right)^{m_1}\left(\frac{z_2}{2!}\right)^{m_2}\cdots\left(\frac{z_n}{n!}\right)^{m_n},$$

where $\pi(n)$ denotes over all partitions of n into arbitrarily many non-negative parts, i.e., over all non-negative integer solutions of the single equation

$$m_1 + 2m_2 + \cdots + nm_n = n.$$

Here we also define that $\mathbf{B}_n(z_1, z_2, \cdots, z_n)$ are zeros when $n < 0$.

Let $P(z)$ and $Q(z)$ be polynomials (in the complex variable z) of degrees m and n respectively, given by

$$P(z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_m \quad \text{and} \quad Q(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n,$$

where a_i and b_j are the complex numbers for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$.

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It is assumed that $P(z)$ and $Q(z)$ have no zeros in common. If Γ is a simple closed path containing the poles of $P(z)/Q(z)$ in its interior, it is known that

$$\oint_\Gamma \frac{P(z)}{Q(z)}\mathrm{d}z = \begin{cases} \dfrac{2\pi i a_0}{b_0}, & n-m=1, \\ 0, & n-m \geqslant 2. \end{cases}$$

By applying the decomposition of $P(z)/Q(z)$ into partial fractions and combining residue's definition, E. Just and N. Schaumberger [2] gave a simple proof of the above assertions. In the end, they said that " It may be noted, further, that if $m \geqslant n$ then $P(z)/Q(z) = F(z) + R(z)/Q(z)$, where $F(z)$ is a polynomial and $R(z)$ is a polynomial of degree $n-1$. Since $F(z)$ has no poles $\oint_\Gamma P(z)/Q(z)\mathrm{d}z = \oint_\Gamma R(z)/Q(z)\mathrm{d}z$, and the latter integral has already been considered."

We easily see that the above results are true when the $P(z)/Q(z)$ is only a proper rational function, in other words, when the rational function is an improper rational fraction, they did not give an explicit computation formula on $\oint_\Gamma \frac{P(z)}{Q(z)}\mathrm{d}z$. In fact, it is usually difficult that a *general* improper rational function is decomposed into a polynomial plus a proper rational fraction. Therefore, what are the polynomials $F(z)$ and $R(z)$? We do not know their details. Consequently, they do not really resolve this question to compute the contour integrals to the general improper rational functions. However, when the $P(z)/Q(z)$ is an improper rational function, what are the deep-dyed results for the contour integral $\oint_\Gamma P(z)/Q(z)\mathrm{d}z$?

In the present paper, we will completely answer this question and give the explicit formulas for the contour integral of the general rational functions with the complete Bell polynomials, by using Cauchy's residue theorem without the use of the partial fractions decomposition. As applications, we also show several interesting examples for the contour integral of the improper rational functions.

2. The contour integration for the rational functions

We below give the explicit formulas of the contour integral of the general rational functions.

Theorem 2.1. *For m and n are any non-negative integers. Suppose that complex numbers $c_1, c_2, \ldots, c_r$ are zeros of order $t_1, t_2, \ldots, t_r$ of the polynomial $P(z)$ of degree m, and $d_1, d_2, \ldots, d_s$ are zeros of order $l_1, l_2, \ldots, l_s$ of the polynomial $Q(z)$ of degree n, and suppose that $P(z)$ and $Q(z)$ do not have common zeros. Then*

$$\begin{aligned}\oint_\Gamma \frac{P(z)}{Q(z)}\mathrm{d}z &= \frac{a_0}{b_0}\frac{2\pi i}{(m-n+1)!}\mathbf{B}_{m-n+1}(z_1, z_2, \ldots, z_{m-n+1}) \\ &= \begin{cases} 0, & m-n+1<0, \\ \dfrac{2\pi i a_0}{b_0}, & m-n+1=0, \\ \dfrac{a_0}{b_0}\dfrac{2\pi i}{(m-n+1)!}\mathbf{B}_{m-n+1}(z_1, z_2, \ldots, z_{m-n+1}), & m-n+1>0, \end{cases}\end{aligned}$$

where Γ is a simple closed contour which surrounds all the poles of $P(z)/Q(z)$,

$$z_k = (k-1)!\left(\sum_{j=1}^s l_j d_j^k - \sum_{j=1}^r t_j c_j^k\right), \quad k = 1, 2, \ldots, m-n+1.$$

Proof. We reformulate the polynomials $P(z)$ and $Q(z)$ respectively

$$P(z) = a_0(z-c_1)^{t_1}(z-c_2)^{t_2}\cdots(z-c_r)^{t_r} = a_0\prod_{j=1}^r (z-c_j)^{t_j}, \quad \sum_{j=1}^r t_j = m,$$

$$Q(z) = b_0(z-d_1)^{l_1}(z-d_2)^{l_2}\cdots(z-d_s)^{l_s} = b_0\prod_{j=1}^s (z-d_j)^{l_j}, \quad \sum_{j=1}^s l_j = n.$$

In the extended complex plane, by using Cauchy's residue theorem, we obtain

$$\oint_{\Gamma} \frac{P(z)}{Q(z)} \mathrm{d}z = 2\pi i \sum_{j=1}^{s} \operatorname*{Res}_{z=d_j} \frac{P(z)}{Q(z)} = -2\pi i \operatorname*{Res}_{z=\infty} \frac{P(z)}{Q(z)}.$$

To calculate the residue of the rational function $P(z)/Q(z)$ in $z = \infty$ yields

$$\operatorname*{Res}_{z=\infty} \frac{P(z)}{Q(z)} = -\operatorname*{Res}_{t=0} \frac{1}{t^2} \frac{P\left(\frac{1}{t}\right)}{Q\left(\frac{1}{t}\right)} = -\frac{a_0}{b_0} \operatorname*{Res}_{t=0} \frac{1}{t^{m-n+2}} \frac{\prod_{j=1}^{r}(1-c_j t)^{t_j}}{\prod_{j=1}^{s}(1-d_j t)^{l_j}}.$$

When $m - n + 1 < 0$, then $t = 0$ is not the pole.

$$\operatorname*{Res}_{z=\infty} \frac{P(z)}{Q(z)} = -\frac{a_0}{b_0} \operatorname*{Res}_{t=0} \frac{1}{t^{m-n+2}} \frac{\prod_{j=1}^{r}(1-c_j t)^{t_j}}{\prod_{j=1}^{s}(1-d_j t)^{l_j}} \equiv 0.$$

When $m - n + 1 = 0$, then $t = 0$ is a single pole of order 1.

$$\operatorname*{Res}_{z=\infty} \frac{P(z)}{Q(z)} = -\frac{a_0}{b_0} \operatorname*{Res}_{t=0} \frac{1}{t} \frac{\prod_{j=1}^{r}(1-c_j t)^{t_j}}{\prod_{j=1}^{s}(1-d_j t)^{l_j}} = -\frac{a_0}{b_0} \lim_{t\to 0} \frac{\prod_{j=1}^{r}(1-c_j t)^{t_j}}{\prod_{j=1}^{s}(1-d_j t)^{l_j}} = -\frac{a_0}{b_0}.$$

When $m - n + 1 > 0$, then $t = 0$ is a single pole of order $m - n + 2$. By utilizing the power series expansion of the logarithmic function:

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}, \quad (|z| < 1)$$

and combining the definition of complete Bell polynomials, we obtain

$$\begin{aligned}
\operatorname*{Res}_{z=\infty} \frac{P(z)}{Q(z)} &= -\frac{a_0}{b_0} \operatorname*{Res}_{t=0} \frac{1}{t^{m-n+2}} \frac{\prod_{j=1}^{r}(1-c_j t)^{t_j}}{\prod_{j=1}^{s}(1-d_j t)^{l_j}} \\
&= -\frac{a_0}{b_0} [t^{m-n+1}] \exp\left[\sum_{j=1}^{r} \left[t_j \log(1-c_j t)\right] - \sum_{j=1}^{s} \left[l_j \log(1-d_j t)\right]\right] \\
&= -\frac{a_0}{b_0} [t^{m-n+1}] \exp\left\{\sum_{k=1}^{\infty} \left[(k-1)!\left(\sum_{j=1}^{s} l_j d_j^k - \sum_{j=1}^{r} t_j c_j^k\right)\right] \frac{t^k}{k!}\right\} \\
&= -\frac{a_0}{b_0} \frac{1}{(m-n+1)!} \mathbf{B}_{m-n+1}(z_1, z_2, \ldots, z_{m-n+1}).
\end{aligned}$$

This completes the proof of Theorem 2.1. □

3. Applications

We give some interesting examples for the contour integrals of the improper rational functions by utilizing Theorem 2.1.

Example 3.1. For $n \geqslant 1, m \geqslant 0$. Then

$$\oint_{\Gamma} \frac{z^{n+m}}{z^n - 1} \mathrm{d}z = 0,$$

where Γ surrounds all the zeros of $z^n - 1$.

Example 3.2. For $n \geqslant 1, m \geqslant 0$. Then

$$\oint_{\Gamma} \frac{z^{n+m}}{(z-1)^n} \mathrm{d}z = 2\pi i \binom{n+m}{n-1},$$

where Γ surrounds a single zero 1 of order n of $(z-1)^n$.

Example 3.3. For $n, s \geqslant 1, m, r \geqslant 0$ and $(r-s)n + rm \geqslant 0$. Let $(z)_n = z(z+1)\cdots(z+n-1)$. Then

$$\oint_{\Gamma} \frac{(z-n-m)_{n+m}^r}{(z)_n^s} \mathrm{d}z = \frac{2\pi i}{((r-s)n+rm+1)!} \mathbf{B}_{(r-s)n+rm+1}(z_1, z_2, \ldots, z_{(r-s)n+rm+1}),$$

where Γ surrounds all the zeros of order s of $(z)_n^s$,

$$z_k = -(k-1)! \left[\left((-1)^{k-1} s + r \right) \sum_{j=1}^{sn-1} j^k + r \sum_{j=0}^{(r-s)n+rm} (sn+j)^k \right], \; k = 1, 2, \ldots, (r-s)n + rm + 1.$$

The first few complete Bell polynomials $\mathbf{B}_n(z_1, z_2, \ldots, z_n)$ may be displayed as follows (see [3, p.175]):

$$\begin{aligned}
&\mathbf{B}_0 = 1, \\
&\mathbf{B}_1(z_1) = z_1, \\
&\mathbf{B}_2(z_1, z_2) = z_1^2 + z_2, \\
&\mathbf{B}_3(z_1, z_2, z_3) = z_1^3 + 3z_1 z_2 + z_3, \\
&\mathbf{B}_4(z_1, z_2, z_3, z_4) = z_1^4 + 6z_1^2 z_2 + 4z_1 z_3 + 3z_2^2 + z_4.
\end{aligned}$$

Applying the above complete Bell polynomials, further special cases of Example 3.3 can be shown as follows:

Cases 1. Taking $r = s = 1$, Γ surrounds all the zeros of $(z)_n$.

1. When $m = 0$.

$$\oint_{\Gamma} \frac{(z-n)_n}{(z)_n} \mathrm{d}z = -2\pi i n^2.$$

2. When $m = 1$.

$$\oint_{\Gamma} \frac{(z-n-1)_{n+1}}{(z)_n} \mathrm{d}z = \pi i n^2 (n+1)^2.$$

3. When $m = 2$.

$$\oint_{\Gamma} \frac{(z-n-2)_{n+2}}{(z)_n} \mathrm{d}z = -\frac{\pi i}{3} n^2 (n+1)^2 (n+2)^2.$$

Cases 2. Taking $r = s = 2$, Γ surrounds all the zeros of $(z)_n^2$.

1. When $m = 0$.

$$\oint_{\Gamma} \left[\frac{(z-n)_n}{(z)_n} \right]^2 \mathrm{d}z = -16\pi i n^2.$$

2. When $m = 1$.

$$\oint_{\Gamma}\left[\frac{(z-n-1)_{n+1}}{(z)_n}\right]^2 \mathrm{d}z = -\frac{\pi i}{3}\Big(512\,n^6 + 1536\,n^5 + 2144\,n^4 + 1728\,n^3 + 920\,n^2 + 312\,n + 72\Big).$$

3. When $m = 2$.

$$\begin{aligned}\oint_{\Gamma}\left[\frac{(z-n-2)_{n+2}}{(z)_n}\right]^2 \mathrm{d}z = -\frac{\pi i}{60}\Big(&32768\,n^{10} + 327680\,n^9 + 1536000\,n^8 + 4423680\,n^7 + 8707584\,n^6\\ &+ 12334080\,n^5 + 12901120\,n^4 + 10009600\,n^3 + 5665248\,n^2 + 2182080\,n\\ &+ 477600\Big).\end{aligned}$$

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