Montes Taurus J. Pure Appl. Math. 3 (3), 227–237, 2021

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Water Engineering Modeling Controlled by Generalized Tsallis Entropy

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Abstract

Water engineering is a real live, study that combines engineering and non-engineering factors that are realized for operating water schemes. These facets and the connected problems applying various procedures. We formulate a new type of the chi-square distributions, which is given in terms of the local fractional integral (fractal integral operator). This concept is a special part of fractional calculus. Then the fractal chi-square will employ to generalize Tsallis entropy. These types of entropy have been seen in numerous applications in almost all the sciences, including the social sciences and humanities studies. We scheme a unique form of the fractal Tsallis entropy using fractal chi-square test. A test method is talented of studying water engineering modeling.

Keywords: Chi-square, Fractional calculus, Fractal, Laplace transform, Wave modeling, Fractional differential equation, Fractional operator

2010 MSC: 34A12

1. Introduction

In the environment, most of the arrangements have nonlinear communications, and this nonlinearity effort the system in a confused state, an occurrence that is reasonably difficult to train. In dynamics, such organizations have often indicated to as dynamical systems. Water waves are particularly the conjoint dynamical system in nature. Currently, many investigators are concentrating on water waves. This can be styled as the revolution of wave energy into wind kinetic energy at a serious amplitude rate of the wave [1]. Since the cluster velocity is lower at narrow waters, the rate of flouting waves is reasonably potential at the coasts [2]. For this reason, these properties have affected on the solution the wave equation, which can help ocean engineers. Entropy is a most active concept to treat with the more physical problems. That is can be utilized to indicate the evaluation of many events taking place in the world. Entropy carries a deterministic viewpoint to our sense of physical proceedings. From the physical point of view, in reports of forecasting an event pattern, entropy is reasonably positive in replicating the leading sets of the physical occurrence, though it desertions some characteristics of the occurrence. These method efforts are very well in systems where communications are linear or nonlinear [3].

Water engineering is a mixture of engineering and non-engineering sides that are requested for scheduling, scheming and handling water systems. Engineering features involve arrangement, progress scheme, process and organization,

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Received:10 November 2020, Accepted:10 March 2021, Published:25 April 2021

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while non-engineering features consist of conservation effect valuation, socioeconomic investigation, policy assembly and control on civilization. These characteristics and the related problems have been concerned with in the works utilizing diverse methods that are created by different ideas and rules. An essential query that still rests is: Can we improve a combining theory for lecturing these? One of these solutions is by using the second law of thermodynamics, which helps statement these in a combined method. This theory can be indicated to the information theory (Entropy concept and its positions). The Tsallis entropy has been utilized for a varied range of difficulties in water engineering. This paper provides an overview of Tsallis entropy theory in water engineering. There are three types of problems can be studied and presented for researches. The first problem needs entropy maximization; the second problem is demanding coupling Tsallis entropy theory with another theory; and the third problem is including physical associations. The entropy theory has been employed to seek an extensive range of difficulties in water engineering, containing rainfall-runoff, infiltration, soil moisture, system scheme, velocity distributions, residue application, hydraulic system, reliability, and survival analysis (see [4]).

Survival analysis (SA) is a significant matter in actuarial science. SA has a rich path of studied and there are various varieties of methods. The conjoint methodology to evaluate the survival distribution is the parametric one, with which a hypothetical survival distribution is quantified and the constraints convoluted are resolute by certain approaches. One of the important branches of the SA is the entropy of parametric recognition. Consequently, one of the most popular parametric entropy is the Tsallis entropy. The Tsallis method entropy studies the structures as non- extensive, which are categorized by the enterprise index [5]

$$\Lambda_\alpha(\varrho) = \frac{1}{\alpha - 1}\left(1 - \sum_{\jmath}^{n} \varrho_{\jmath}^{\alpha}\right), \quad \alpha \neq 1,$$

where α is a real parameter which is famed as the entropy-index. Moreover, for definite, specific positions, devices to calculate this index are well recognized. There occurs not a common methodology to assess the index for a specified arrangement (biological [6, 7], physical [8], economical [9], etc.). Recently, different structures are introduced assimilated to Tsallis entropy. For example, it formulates in fractional differential polynomials [10, 11], complex Tsallis entropy [12], convolution with the Riesz fractional derivative [13] and others studies [14].

The chi-square distribution (C-SD) has been one of the greatest normally utilized distributions in science

$$\chi_\upsilon^2 = \sum_{\imath=0}^{n} \left(\frac{x_\imath - \mu_\imath}{\sigma_\imath}\right)^2,$$

where n indicates the number of monitors, $x_\imath$ represents the observed variable, $\mu_\imath$ admits the expected value, $\sigma_\imath$ forms the standard deviation, and $\upsilon \leq n$. The definition is derived by many theories, which can be seen in [15].

It is an excellent example of the gamma distribution, which has been a fundamental distribution in critical physics, for occurrence, as kinetic energy distribution of atoms in an ideal gas (Maxwell-Boltzmann) or the kinetic energy distribution of atoms created from interested nuclei in nuclear reactions. An earliest situation for the growth of the C-SD is expressed in [16, 17]. There are numerous of roots of the integral creation, such as multiple integrals over normal variables and substitutions, integration in the complex domain to calculate multiple integrals and a classic transformation to derive C-SD uses Jacobian determinant (see [18]). Commonly, the summarized derivations use the part that C-SD is a singular example of the gamma distribution. Continuing to the integral and recursive possessions of the gamma distribution, as well as its moment generating function, shortened derivations of C-SD are demarcated in the texts [19].

In this work, we utilize the idea of fractal integral operator without singularity to transport a new interested C-SD. The fractal concept offers outline the elementary ideologies and terms of fractal geometry. Fractal procedures did not improve general attention until the 1960s. Fractals typically cover an adequate, recursive, self-similar structure and have an anticipated appearance. Their enormousness is dependent on the quantity at which they are measured, and they do not path the rubrics of traditional geometry (see [20]). Consequently, we employ the fractal C-SD to introduce a fractal Tsallis entropy. As application, we test the method to design water engineering modeling. Different studies will be presented such as the optimization and oscillation.

2. Preliminaries

I n this section, we illustrate some basic definitions to create our method.

Definition 2.1. For the interval $\Pi := [a, b]$, we represent to the fractal space by $C_\beta(\Pi)$; that is for all $\epsilon > 0$ there occurs a fixed constant $\delta > 0$ such that $g \in C_\beta(\Pi)$

$$|g(t) - g(t_0)| \leq \epsilon^\beta, \quad \beta \in (0, 1]$$

when $|t - t_0| \leq \delta$.

Definition 2.2. For a function $g \in C_\beta(\Pi)$, which is formulated on a fractal set (a fractal set is a set that the fractal dimension strictly increases the topological dimension) involving the fractal value β, the fractal integral operator (local fractional integral) is introduced as follows:

$$I^\beta g(t) = \frac{1}{\Gamma(1+\beta)} \int_a^b g(t)(dt)^\beta, \quad \beta \in (0, 1],$$

where

$$(dt)^\beta = \frac{t^{1-\beta}}{\Gamma(2-\beta)} dt^\beta.$$

Moreover, since g is fractal continuous functions then by the mean value theorem for local fractional integrals

$$I^\beta g(t) = \frac{1}{\Gamma(1+\beta)} \int_a^b g(t)(dt)^\beta = g(\zeta)\frac{(b-a)^\beta}{\Gamma(1+\beta)}, \quad \zeta \in (a, b).$$

Example 2.3. The fractal exponential function can be described by the series

$$\Xi_\beta(t^\beta) = \sum_{n=0}^{\infty} \frac{t^{n\beta}}{\Gamma(1+n\beta)}, \quad \beta \in (0, 1].$$

Moreover, we have

$$\Xi_\beta(-t^\beta) = \sum_{n=0}^{\infty} \frac{(-1)^n \, t^{n\beta}}{\Gamma(1+n\beta)}, \quad \beta \in (0, 1].$$

Definition 2.4. For a function $g \in C_\beta(\Pi)$, the fractal Laplace transform can be organized by the integral

$$\Upsilon(g(\varsigma)) = \left(\frac{1}{\Gamma(1+\beta)}\right) \int_0^\infty g(t)\, \Xi_\beta(-\varsigma^\beta t^\beta)(dt)^\beta, \quad \beta \in (0, 1]$$

for a fixed number ς.

Definition 2.5. For a function $g \in C_\beta(\Pi)$, the fractal integral on a simple closed contour C can be structured by

$$(g(\zeta_0))^{(n\beta)} = \left(\frac{1}{(2\pi i)^\beta}\right) \cdot \left(\frac{1}{\Gamma(1+\beta)}\right) \oint_C \frac{g(\zeta)}{(\zeta - \zeta_0)^{(n+1)\beta}} (d\zeta)^\beta$$

for $\beta \in (0, 1]$.

Definition 2.6. For a function $g \in C_\beta(\Pi)$, the inverse fractal Laplace transform

$$L^{-1}(g(t)) = \frac{1}{(2\pi)^\beta} \int_0^\infty \Upsilon(g(\varsigma))\, \Xi_\beta(\varsigma^\beta t^\beta)(d\varsigma)^\beta, \quad \beta \in (0, 1]$$

for a fixed number t.

Combining Definitions 2.5 and 2.6 to get a hybrid integral formula

$$\Upsilon^{-1}(g(\zeta_0)) = \left(\frac{1}{(2\pi i)^\beta}\right)\cdot\left(\frac{1}{\Gamma(1+\beta)}\right)\oint_C \frac{\Upsilon(g(\zeta))\,\Xi_\beta(\zeta^\beta t^\beta)}{(\zeta-\zeta_0)^{(n+1)\beta}}(d\zeta)^\beta \tag{2.1}$$

Finally, we have the fractal derivative

Definition 2.7. For a function $g \in C_\beta(\Pi)$, the fractal derivative is defined by the formula

$$D^\beta g(t) = \lim_{t\to t_0}\frac{d^\beta[g(t)-g(t_0)]}{[(t-t_0)]^\beta},$$

where

$$d^\beta[g(t)-g(t_0)] \approx \Gamma(1+\beta)[g(t)-g(t_0)].$$

Hence,

$$D^\beta g(t) = \lim_{t\to t_0}\frac{\Gamma(1+\beta)[g(t)-g(t_0)]}{[(t-t_0)]^\beta}.$$

In general,

$$D^\beta\, g(t) \approx \frac{\Gamma(1+\beta)[g(t)-g(t_0)]}{[(t-t_0)]^\beta}.$$

3. Fractal indicators

We have two facts to present as fractal indicators, C-Ds and Tsallis entropy.

3.1. Fractal C-SD

The probability density function is formulated by the construction

$$\Delta(\chi^2_\upsilon|\upsilon) = \frac{(\chi^2_\upsilon)^{\upsilon/2-1}\; e^{-(\chi^2_\upsilon)/2}}{2^{\upsilon/2}\;\Gamma(\upsilon/2)}, \tag{3.1}$$

where Γ represents the gamma function, where $\mathfrak{E}(\chi^2_\upsilon) = \upsilon$ and $\mathfrak{Var}[\chi^2_\upsilon] = 2\upsilon$. Now, the fractal interval of the normal single variable x_1 can be recognized by the structure $[x_1, x_1 + (dx_1)^\beta]$ then the probability

$$\left(\mathfrak{N}^\beta\, x_1\right)(dx_1)^\beta = \left(\frac{1}{\sqrt{2\pi}\sigma_1}e^{-\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2/2}\right)^\alpha\,(dx_1)^\beta.$$

By letting

$$\chi_1^2 = \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2,$$

we have

$$\begin{aligned}\left(\mathfrak{N}^\beta\,(\chi_1^2|1)\right)(d\chi_1^2)^\beta &= \frac{2}{\sqrt{2\pi}\sigma_1}e^{-\chi_1^2/2}\left(\left|\frac{dx_1}{d\chi_1^2}\right|\right)^\alpha(d\chi_1^2)^\beta\\ &= \frac{1}{(2^{1/2}\Gamma(1/2))^\beta}\,e^{-(\chi_1^2/2)\beta}\left(\chi_1^2\right)^{(-1/2)\beta}(d\chi_1^2)^\beta\\ &= \frac{1}{(2^{1/2}\Gamma(1/2))^\beta}\,e^{-(\chi_1^2/2)\beta}\left(\chi_1^2\right)^{(1/2-1)\beta}(d\chi_1^2)^\beta\\ &\approx \gamma\left(\chi_1^2|(1/2)\beta,2\right)(d\chi_1^2)^\beta,\end{aligned} \tag{3.2}$$

where γ indicates the gamma distribution and $\Gamma(1/2) = \sqrt{\pi}$.

We proceed to introduce the fractal formula of C-SD by using the fractal Laplace transform in Definition 2.4 and its inverse in Definition 2.5. The fractal Laplace transform of (3.2) is given by the integral formula

$$\begin{aligned}\Upsilon(\Re^{\beta}(\chi_1^2|1)) &= \left(\frac{1}{\Gamma(1+\beta)}\right)\int_0^{\infty}\left(\frac{e^{-(\chi_1^2/2)\beta}\left(\chi_1^2\right)^{(1/2-1)\beta}}{(2^{1/2}\Gamma(1/2))^{\beta}}\right) e(-\varsigma^{\alpha}(\chi_1^2)^{\beta})(d\chi_1^2)^{\beta} \\ &:= \left(\frac{1}{\Gamma(1+\beta)}\right)\int_0^{\infty}\Theta(\chi_1^2)\; e(-\varsigma^{\beta}(\chi_1^2)^{\beta})(d\chi_1^2)^{\beta} \\ &\approx \frac{1}{\Gamma(1+\beta)}\left(\frac{1/2}{1/2+\varsigma}\right)^{\beta/2}.\end{aligned} \tag{3.3}$$

By operating the n-th convolution, we obtain the general iterative expression

$$\begin{aligned}&\Upsilon(\Re^{\beta}(\chi_1^2|1)) * \ldots * \Upsilon(\Re^{\beta}(\chi_n^2|n)) \\ &= \frac{1}{\Gamma(1+\beta)}\left(\frac{1/2}{1/2+\varsigma}\right)^{\beta/2}\times\ldots\times\frac{1}{\Gamma(1+\beta)}\left(\frac{1/2}{1/2+\varsigma}\right)^{\alpha/2} \\ &= \left(\frac{1}{\Gamma(1+\beta)}\right)^{n}\left(\frac{1/2}{1/2+\varsigma}\right)^{n\beta/2}.\end{aligned} \tag{3.4}$$

Clearly, when $\beta = 1$ we have the normalized style. Now, suppose that $\vartheta := \chi_n^2$ then the fractal inverse Laplace transform outcomes in the probability density function of ϑ is

$$\begin{aligned}\Delta_{\beta}(\vartheta|n) &= \left(\frac{1}{(2\pi i)^{\beta}}\right)\cdot\left(\frac{1}{\Gamma(1+\beta)}\right)^{n}\oint_C \Upsilon(\vartheta(\zeta))\,\Xi_{\beta}(\zeta^{\beta}t^{\beta})(d\zeta)^{\beta} \\ &= \left(\frac{1}{\Gamma(1+\beta)}\right)^{n}\left(\frac{1}{(2\pi i)^{\beta}}\right)\oint_C\left(\frac{1/2}{1/2+\varsigma}\right)^{n\beta/2} e^{(\varsigma^{\alpha}\vartheta^{\beta})}(d\varsigma)^{\beta} \\ &= (\frac{1}{2})^{n\beta/2}\left(\frac{1}{\Gamma(1+\beta)}\right)^{n}\left(\frac{1}{(2\pi i)^{\beta}}\right)\oint_C\frac{e^{(\varsigma^{\beta}\vartheta^{\beta})}}{(1/2+\varsigma)^{n\beta/2}}(d\varsigma)^{\beta}.\end{aligned} \tag{3.5}$$

By utilizing the n-th convolution of the fractal integral (see [20]), we obtain

$$\Delta_{\beta}(\vartheta|n) \approx \frac{\vartheta^{(n/2-1)\beta}e^{-n\beta/2}}{2^{n\beta/2}\Gamma(n\beta/2)}. \tag{3.6}$$

Note that, when $\beta \to 1$, we obtain the normalized probability density function. Thus, Eq.(2.7) represents the fractal C-SD by utilizing the fractal Laplace transform. The essential benefit of the fractal Laplace transform is that it does not demand prior material about the gamma distribution and it disregards any contractions integral by scheming one initiation stage for the outstanding total of degrees of freedom. The recommended structure technique by the fractal Laplace transform is advanced because it employed integration in the complex domain. Next, we shall use it to formulate the Tsallis entropy.

3.2. Fractal Tsallis entropy

We proceed to introduce Tsallis entropy in terms of $\Delta_\beta(\vartheta|n)$ as follows: for the discrete formula we obtain

$$\begin{aligned}\Lambda_\alpha(\Delta_\beta) &= \frac{1}{\alpha-1}\left(1-\sum_{\jmath}^{n}\varrho_{\jmath}^{\alpha}\right)\\ &= \frac{1}{\alpha-1}\left(1-\sum_{\jmath}^{n}[\Delta_\beta(\vartheta|\jmath)]^{\alpha}\right)\\ &= \frac{1}{\alpha-1}\left(1-\sum_{\jmath}^{n}\Big[\frac{\vartheta^{(\jmath/2-1)\beta}e^{-\jmath\beta/2}}{2^{\jmath\beta/2}\Gamma(\jmath\beta/2)}\Big]^{\alpha}\right),\end{aligned} \tag{3.7}$$

where α is a real number different from 1 and $\beta \in (0,1]$. We have varieties to select α, as a special case, when $\alpha = \beta \in (0,1)$ we get the fractal Tsallis entropy

$$\Lambda_\beta(\Delta_\beta) = \frac{1}{\beta-1}\left(1-\sum_{\jmath}^{n}\Big[\frac{\vartheta^{(\jmath/2-1)\beta}e^{-\jmath\beta/2}}{2^{\jmath\beta/2}\Gamma(\jmath\beta/2)}\Big]^{\beta}\right). \tag{3.8}$$

For the continuous case, we have

$$\Lambda_\alpha(\Delta_\beta) = \frac{1}{\alpha-1}\left(1-\int\Big[\frac{\vartheta^{(-1/2)\beta}e^{-\beta/2}}{2^{\beta/2}\Gamma(\beta/2)}\Big]^{\alpha}d\vartheta\right), \tag{3.9}$$

and for $\alpha = \beta \in (0,1)$, we have

$$\Lambda_\beta(\Delta_\beta) = \frac{1}{\beta-1}\left(1-\int\Big[\frac{\vartheta^{(-1/2)\beta}e^{-\beta/2}}{2^{1\beta/2}\Gamma(\beta/2)}\Big]^{\beta}d\vartheta\right). \tag{3.10}$$

Clearly, the optimization of the fractal Tsallis entropy in Eq.s (3.7)-(3.10) can be determined by the optimization of $\Delta_\beta, \beta \in (0,1]$ and $\alpha \neq 1$. There are different methods to have the optimization of Tsallis entropy such as using integration method, Lagrange Multipliers, Laplace principle, velocity distribution and Kullback-Leibler principle. All these methods indicated a list of steps to determine the maximum entropy [3]. Our method is simple and clear to hit the maximization. We have use Mathematica 11.2 to evaluate our data.

3.3. Application in ocean Studies

There are many directions to present the ocean studies, and the most important one is to develop and investigate the shallow water equations. This class of wave equations indicates various corners to focus on. We deal with the 1-dimensional Saint-Venant equations of diffusive wave. This equation simply takes the formula

$$\frac{d\Psi(\chi)}{d\chi} + \Delta(\varsigma) = 0, \tag{3.11}$$

where Ψ is the height deviation of the horizontal pressure surface at position χ and $\Delta(\varsigma)$ represents the difference of bed slop. The diffusive wave is usable when the inertial acceleration is considerably lesser than all other arrangements of acceleration. There are many recent investigations on this wave equation, for 1-dimensional equation with entropy and 3-dimensional equation with symmetric diffusive ([1, 2]). It has been shown in the approximated solution of Eq.(3.11) is given by the integral formula

$$\Psi_j \simeq \frac{\Delta(\varsigma)}{\Delta(\chi)}\int_{\chi_{j-1/2}}^{\chi_{j+1/2}}\Psi(\chi,\varsigma^n)d\chi$$

consequently, by using the numerical entropy H (is a cell-average approximation to the entropy)

$$H_j \simeq \frac{\Delta(\varsigma)}{\Delta(\chi)}\int_{\chi_{j-1/2}}^{\chi_{j+1/2}}H(\chi,\varsigma^n)d\chi.$$

By letting $\lambda := |\frac{\Delta(\varsigma)}{\Delta(\chi)}|$, we have

$$H_j \simeq \lambda \int_{\chi_{j-1/2}}^{\chi_{j+1/2}} H(\chi, \varsigma^n) d\chi.$$

In this effort, we generalize the 1-dimensional Saint-Venant equations of diffusive wave in terms of fractal (see Definition 2.7) as follows:

$$D^{\beta}\Psi(\chi) + \Delta(\varsigma) = 0. \tag{3.12}$$

Following the process in [2], we have that the approximated entropy solution of Eq.(3.12) in terms of the fractal Tsallis entropy

$$\Big(\Lambda_{\beta}(\Delta_{\beta})\Big)_j = \frac{\lambda}{\beta - 1}\left(1 - \int_{\chi_{j-1/2}}^{\chi_{j+1/2}} \Big[\frac{\chi^{(-1/2)\beta} e^{-\beta/2}}{2^{\beta/2}\Gamma(\beta/2)}\Big]^{\beta} d\chi\right). \tag{3.13}$$

For a special case, when $\lambda \to \beta$ and $\beta \in [0, 1)$, we have

$$\Big(\Lambda_{\beta}(\Delta_{\beta})\Big)_j = \frac{\beta}{1 - \beta}\left(\int_{\chi_{j-1/2}}^{\chi_{j+1/2}} \Big[\frac{\chi^{(-1/2)\beta} e^{-\beta/2}}{2^{\beta/2}\Gamma(\beta/2)}\Big]^{\beta} d\chi - 1\right). \tag{3.14}$$

Example 3.1. Consider the fractal 1-dimensional Saint-Venant equations of diffusive wave

$$D^{\beta}\Psi(\chi) + \Delta(\varsigma) = 0, \quad \beta = 0.1, 0.5, 0.9. \tag{3.15}$$

The the entropy solution is given by (see Fig.1-respectively)

$$\begin{aligned}
(\Lambda_{0.1}(\Delta_{0.1}))_j &= (1/9)\left(\int_{\chi_{j-1/2}}^{\chi_{j+1/2}} \Big[\frac{\chi^{(-0.1/2)} e^{-0.1/2}}{2^{0.1/2}\Gamma(0.1/2)}\Big]^{0.1} d\chi - 1\right).\\
(\Lambda_{0.5}(\Delta_{0.5}))_j &= \left(\int_{\chi_{j-1/2}}^{\chi_{j+1/2}} \Big[\frac{\chi^{(-0.5/2)} e^{-0.5/2}}{2^{0.5/2}\Gamma(0.5/2)}\Big]^{0.5} d\chi - 1\right).\\
(\Lambda_{0.9}(\Delta_{0.9}))_j &= 9\left(\int_{\chi_{j-1/2}}^{\chi_{j+1/2}} \Big[\frac{\chi^{(-0.9/2)} e^{-0.9/2}}{2^{0.9/2}\Gamma(0.9/2)}\Big]^{0.9} d\chi - 1\right).
\end{aligned} \tag{3.16}$$

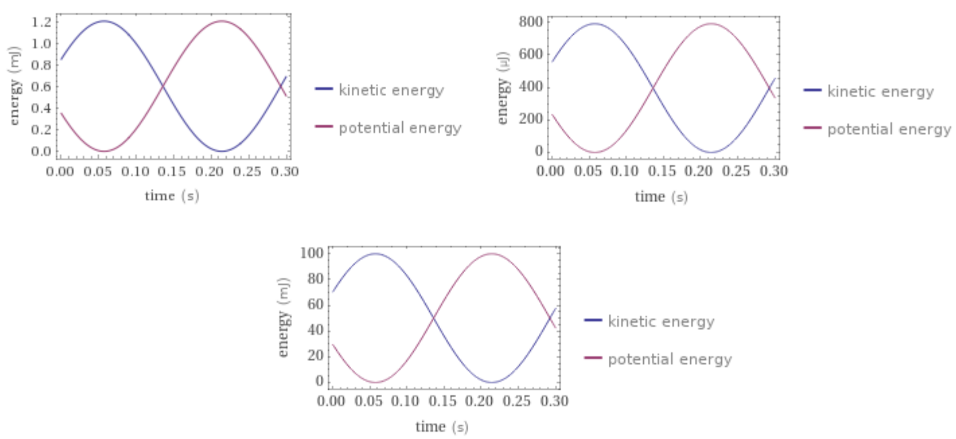

Figure 1. The entropy solution of the fractal the fractal 1-dimensional Saint-Venant equations (FSVE) for $\beta = 0.1, 0.5, 0.9$ respectively

3.4. Algorithm

Our algorithm is based on the following steps

- Input all data for calculation: β, array wave, FSVE etc.;
- Find the fractal Chi-square for data set;
- Substitute the fractal Chi-square in the Tsallis integral formula;
- Approximated the solution of FSVE by fractal entropy.

4. Numerical examples

Here, we test our fractal method and make a comparison with the normal case. We shall suggest a simulation data for a quality of water in the interval [0,1]. We have the following cases $C = \{0.76, 0.82, 0.91\}$.

- Case I: Consider the observation value $0.76 \in [0,1]$ and the degree of freedom is 1 from the data set. We receive the probability density function $Er = \frac{\sqrt{x}}{\sqrt{2}}$ with area under the integral $\chi^2 = 0.6166$ (see Fig.2 in the left, first row). The maximum value is achieved the functional $\frac{e^{-x/2}}{\sqrt{2\pi x}}$ with maximal point 0.878 (see Fig.2 in the meddle graph, first row). This yields the maximum value of the complement in the interval [0,1] $\Lambda_{0.5}(\Delta_{0.5}|1) = 1 - 0.12 = 0.874$ (see Fig.2 in the meddle graph, first row). The second row indicates the degree of freedom =2 to get the value $\chi^2 = 0.316$ with the functional $1 - e^{x/2}$ with maximum value 0.657. While by using the entropy, the maximum value $\Lambda_{0.5}(\Delta_{0.5}|2) = 1 - 0.378 = 0.621$ and $\Lambda_{0.1}(\Delta_{\beta}|2) = 1 - 0.045 = 0.954$. Therefore, the maximum value that optimizes the entropy is in the interval [0.1,0.5).

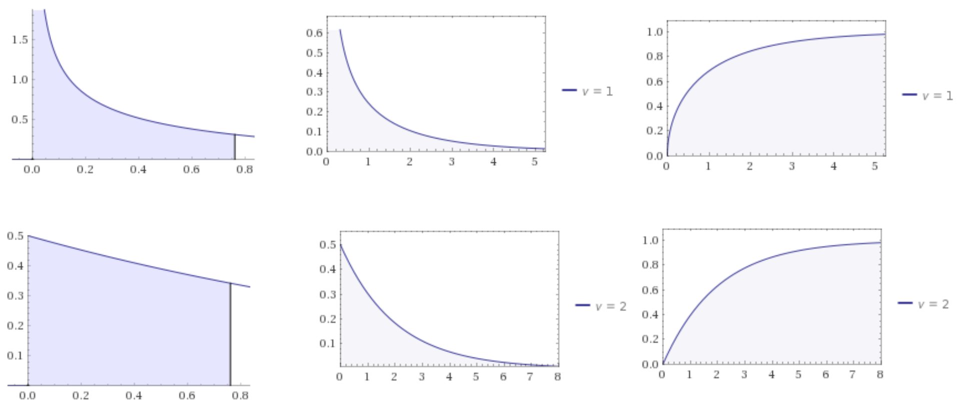

Figure 2. Case I: 0.76 for $\beta = 0.5$ to optimize the formula $\Lambda_{0.5}(\Delta_{0.5}|1)$ and $\Lambda_{0.5}(\Delta_{0.5}|2)$ respectively.

- Case II: Consider the observation value $0.82 \in [0,1]$ and the degree of freedom is 1 from the data set. We arrive at the probability density function $Er = \frac{\sqrt{x}}{\sqrt{2}}$ with area under the integral $\chi^2 = 0.634$ (see Fig.3 in the left, first row). The maximum value is concluded under the functional $\frac{e^{-x/2}}{\sqrt{2\pi x}}$ with maximal point 0.981 (see Fig.3 in the meddle graph, first row). This implies the maximum value of the complement in the interval [0,1] $\Lambda_{0.5}(\Delta_{0.5}|1) = 1 - 0.019 = 0.980$ (see Fig.3 in the meddle graph, first row). The second row indicates the degree of freedom =2 to obtain the value $\chi^2 = 0.336$ with the functional $1 - e^{x/2}$ with maximum value

$0.759 < 0.82$. While by employing the entropy, the maximum value $\Lambda_{0.5}(\Delta_{0.5}|2) = 1 - 0.257 = 0.742$ and $\Lambda_{0.1}(\Delta_\beta|2) = 1 - 0.030 = 0.969$. Therefore, the maximum value that optimizes the entropy when the fractal is decreasing.

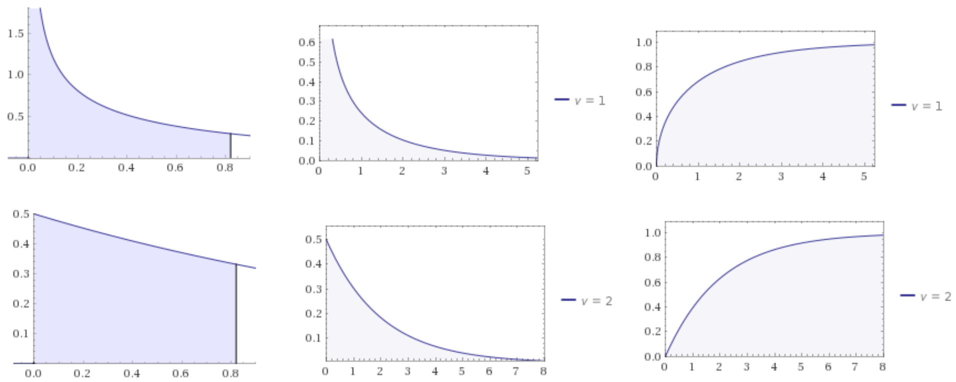

Figure 3. Case II: 0.82 for $\beta = 0.5$ to optimize the formula $\Lambda_{0.5}(\Delta_{0.5}|1)$ and $\Lambda_{0.5}(\Delta_{0.5}|2)$ respectively.

- Case III: Consider the observation value $0.91 \in [0, 1]$ and the degree of freedom is 1 from the data set. We conclude that the probability density function $Er = \frac{\sqrt{x}}{\sqrt{2}}$ with area under the integral $\chi^2 = 0.659$ (see Fig.4 in the left, first row). The maximum value is satisfied under the functional $\frac{e^{-x/2}}{\sqrt{2\pi x}}$ with maximal point 0.843 (see Fig.4 in the meddle graph, first row). This implies the maximum value of the complement in the interval [0,1] $\Lambda_{0.5}(\Delta_{0.5}|1) = 1 - 0.163 = 0.836 < 0.91$ (see Fig.4 in the meddle graph, first row). Moreover, $\Lambda_{0.1}(\Delta_{0.5}|1) = 1 - 0.018 = 0.981 > 0.91$. Therefore, the best result when $\beta < 0.5$. The second row indicates the degree of freedom =2 to obtain the value $\chi^2 = 0.365$ with the functional $1 - e^{x/2}$ with maximum value $0.613 < 0.91$. While by utilizing the entropy, the maximum value $\Lambda_{0.5}(\Delta_{0.5}|2) = 1 - 0.434 = 0.565 < 0.91$ and $\Lambda_{0.1}(\Delta_\beta|2) = 1 - 0.053 = 0.946 > 0.91$. Therefore, the maximum value that optimizes the entropy when the fractal is decreasing.

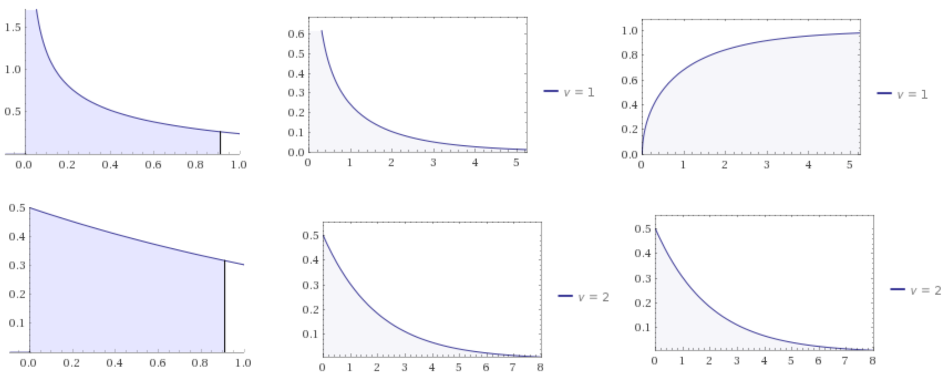

Figure 4. Case III: 0.91 for $\beta = 0.5$ to optimize the formula $\Lambda_{0.5}(\Delta_{0.5}|1)$ and $\Lambda_{0.5}(\Delta_{0.5}|2)$ respectively.

5. Conclusion

The above method is a hybrid between the local fractional C-SD and the Tsallis entropy. Fig.5 shows the relation between β and ϑ in the fractal Tsallis entropy $\Lambda_\beta(\Delta_\beta)$. The application showed the validity of the suggested method by using a sample set for the measurement of the quality of water. Fig.6 indicates the graph of $y = \Lambda_\beta(\Delta_\beta)$ with respect to $x = \vartheta$ for different values of $\beta = 0.1, 0.5$ and 0.9.

For future works, one may generalize the 1-dimensional Saint-Venant equations of diffusive wave by using another fractional calculus or conformable fractional calculus. Moreover, the recent 1-dimensional fractal Saint-Venant equations of diffusive wave can be extended in 2-dimensional fractal or 3-dimensional fractal.

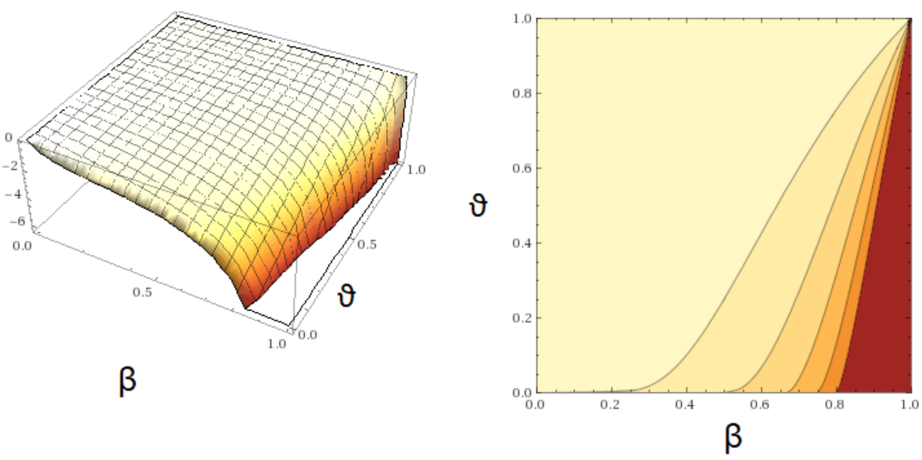

Figure 5. The relation between β and ϑ using $\Lambda_\beta(\Delta_\beta)$

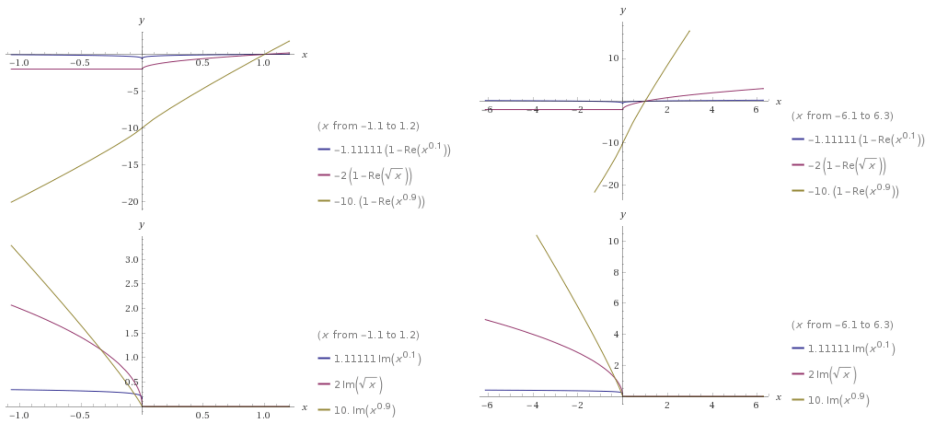

Figure 6. The graph of $y = \Lambda_\beta(\Delta_\beta)$ with respect to $x = \vartheta$ for different values of $\beta = 0.1, 0.5$ and 0.9.

Acknowledgment

This paper is dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th Birthday. The author would like to thank the editor and reviewers for their kind advice to improve the paper.

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