Fekete-Szegö Inequalities for Certain Subclasses of Analytic Functions Related with Leaf-Like Domain

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Abstract
The purpose of this paper is to consider coefficient estimates in a class of functions $M_{\alpha,\lambda}(q)$ consisting of analytic functions $f$ normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ subordinating with leaf like domain, to derive certain coefficient estimates $a_2, a_3$ and Fekete-Szegö inequality for $f \in M_{\alpha,\lambda}(q)$. A similar results have been done for the function $f^{-1}$. Further application of our results to certain functions defined by convolution products with a normalized functions analytic is given, and in particular we obtain Fekete-Szegö inequalities for certain subclasses of functions defined through Poisson distribution series.

Keywords: Analytic functions, Starlike functions, Convex functions, Subordination, Fekete-Szegö inequality, Poisson distribution series, Hadamard product

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1. Introduction

Let $A$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\Delta := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and $S$ be the subclass of $A$ consisting of univalent functions. A function $f \in S$ is said to be starlike in $\Delta$ if and only if

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > 0, \quad (z \in \Delta)$$

(1.2)

and on the other hand, a function $f \in S$ is said to be convex in $\Delta$ if and only if

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad (z \in \Delta).$$

(1.3)
The class of functions satisfying the analytic criteria given by (1.2) and (1.3) are denoted by \( \mathcal{S}^* \) and \( \mathcal{C} \) respectively.

Let \( f \) and \( g \) be functions analytic in \( \Delta \). Then we say that the function \( f \) is subordinate to \( g \) if there exists a Schwarz function \( w(z) \), analytic in \( \Delta \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \((z \in \Delta)\), such that \( f(z) = g(w(z)) \) \((z \in \Delta)\). We denote this subordination by

\[
f < g \quad \text{or} \quad f(z) < g(z) \quad (z \in \Delta).
\]

In particular, if the function \( g \) is univalent in \( \Delta \), the above subordination is equivalent to

\[
f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).
\]

**Definition 1.1.** [14] Let \( \mathcal{S}^*(q) \) denote the class of analytic functions \( f \) in the unit disc \( \Delta \) normalized by \( f(0) = f'(0) - 1 = 0 \) and satisfying the condition that

\[
zf'(z) < \sqrt{1 + z^2} + z =: q(z), \quad z \in \Delta,
\]

where the branch of the square root is chosen to be \( q(0) = 1 \).

It may be noted from (1.4) of Definition 1 that the set \( q(\Delta) \) lies in the right half-plane and it is not a starlike domain with respect to the origin, see Fig. 1 (below).

![Figure 1. The boundary of the set \( q(\Delta) \)](image)

Recently, Raina and Sokol [14] have studied and obtained some coefficient inequalities for the class \( \mathcal{S}^*(q) \) and these results are further improved by Sokol and Thomas [16], for the class \( \mathcal{C}(q) \) in view of the Alexander result between the class \( f \in \mathcal{C}(q) \leftrightarrow zf'(z) \in \mathcal{S}^*(q) \), further the Fekete-Szeg"{o} inequality for functions in \( \mathcal{S}^*(q) \) were also obtained. For a brief history of Fekete-Szeg"{o} problem for the class of starlike, convex and various other subclasses of analytic functions, we refer the interested reader to [5, 20, 19, 22, 18, 21, 17].

Let \( \alpha \geq 0, \lambda \geq 0 \) and \( 0 \leq \rho < 1 \) and \( f \in \mathcal{S} \). We say that \( f \in M(\alpha, \lambda, \rho) \) if it satisfies the condition

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^\alpha + \lambda \left[ 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right) \right] \right\} > \rho.
\]

The class \( M(\alpha, \lambda, \rho) \) was introduced very recently by Guo and Liu [2].

Motivated essentially by the aforementioned works (see [1, 9, 14, 15]), in this paper we define the following class \( \mathcal{M}_{a, \lambda}(q) \) due to Guo and Liu [2], of functions which unifies the class \( \mathcal{S}^*(q), \mathcal{C}(q) \) and \( \mathcal{M}_a(q) \). First, we shall find
estimations of first few coefficients of functions $f$ of the form (1.1) belonging to $\mathcal{M}_0(q)$ and we prove the Fekete-Szegö inequality for a more general class of analytic functions which we define below in Definition 1.2 and also for $f^{-1} \in \mathcal{M}_0(q)$. Also we give applications of our results to certain functions defined through Poisson distribution.

Now, we define the following class $\mathcal{M}_0(q)$:

**Definition 1.2.** For $\alpha \geq 0$, $\lambda \geq 0$ a function $f \in \mathcal{M}_0(q)$ if

$$
\left\{ \frac{zf'(z)}{f(z)} - \frac{zq}{f(z)} + \alpha \left[ 1 + \frac{zf''(z)}{f'(z)} - \frac{zq}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right) \right] \right\} < z + \sqrt{1 + z^2} = q(z); \quad z = re^{\theta} \in \Delta.
$$

(1.5)

Note that $\mathcal{M}_{0,0}(q) \equiv \mathcal{P}(q)$ (see [14]); $\mathcal{M}_{0,1}(q) \equiv \mathcal{M}_0(q)$ (see [15]) and $\mathcal{M}_{0,1}(q) \equiv \mathcal{E}(q)$ given in [16].

2. A Coefficient Estimate

To prove our main result, we need the following:

**Lemma 2.1.** [7] If $p_1(z) = 1 + c_1z + cz^2 + \cdots$ is a function with positive real part in $\Delta$, then

$$
|c_2 - ve^{\eta}|^2 \leq \begin{cases} -4v + 2, & \text{if } v \leq 0, \\ 2, & \text{if } 0 \leq v \leq 1, \\ 4v - 2, & \text{if } v \geq 1. 
\end{cases}
$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $\frac{1 + z}{1 - z}$ or one of its rotations. If $0 < v < 1$, then equality holds if and only if $p_2(z) = \frac{1 + z^2}{1 - z^2}$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$
p_3(z) = \left( \frac{1}{2} + \frac{1}{2} \eta \right) \frac{1 + z}{1 - z} + \left( \frac{1}{2} - \frac{1}{2} \eta \right) \frac{1 - z}{1 + z}; \quad (0 \leq \eta \leq 1)
$$

or one of its rotations. If $v = 1$, the equality holds if and only if $p_1$ is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Although the above upper bound is sharp, when $0 < v < 1$, it can be improved as follows:

$$
|c_2 - ve^{\eta}| + v|c_1|^2 \leq 2 \quad (0 < v \leq 1/2)
$$

and

$$
|c_2 - ve^{\eta}| + (1 - v)|c_1|^2 \leq 2 \quad (1/2 < v \leq 1).
$$

We also need the following:

**Lemma 2.2.** [3] If $p_1(z) = 1 + c_1z + cz^2 + \cdots$ is a function with positive real part in $\Delta$, then

$$
|c_n| \leq 2 \quad \text{for all } n \geq 1 \quad \text{and} \quad |c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}.
$$

The class of all such functions with positive real part is denoted by $\mathcal{P}$.

**Lemma 2.3.** [6] If $p_1(z) = 1 + c_1z + cz^2 + \cdots$ is a function with positive real part in $\Delta$, and vis a complex number, then

$$
|c_2 - ve^{\eta}|^2 \leq 2 \max(1, |2v - 1|).
$$

The result is sharp for the functions

$$
p(z) = \frac{1 + z^2}{1 - z}; \quad p(z) = \frac{1 + z}{1 - z}.
$$

307
Lemma 2.4. [4] Let \( P(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots \) be in \( \mathcal{P} \) then for any complex number \( \mu \),

\[
\left| c_2 - \mu \frac{c_1^2}{2} \right| \leq \max\{2, 2|\mu| - 1\} = \begin{cases} 2, & 0 \leq \mu \leq 2; \\ 2|\mu| - 1, & \text{elsewhere.} \end{cases}
\]

The result is sharp for the functions defined by \( P(z) = \frac{z^2}{1-z} \) or \( P(z) = \frac{1-z^2}{z} \).

Theorem 2.5. Let \( \alpha \geq 0 \) and \( \lambda \geq 0 \). If \( f(z) \) given by (1.1) belongs to \( \mathcal{M}_{\alpha, \lambda}(q) \), then

\[
|a_2| \leq \frac{1}{(1+\alpha)(1+\lambda)},
\]

\[
|a_3| \leq \frac{1}{(\alpha+2)(1+2\lambda)} \max\{1,\left|\left(\frac{\alpha^2 + \alpha - 2(\alpha + 3)\lambda - 2}{2(1+\alpha)(1+\lambda)^2} - \frac{1}{2}\right)\right|\}.
\]

These results are sharp.

Proof. If \( f \) \in \mathcal{M}_{\alpha, \lambda}(q) \), then there is a Schwarz function \( w(z) \), analytic in \( \Delta \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( \Delta \) such that

\[
z f'(z) \left( \frac{f(z)}{z} \right)^\alpha + \lambda \left[ 1 + z f''(z)f(z) - z f'(z) + \alpha \left( \frac{z f'(z)}{f(z)} - 1 \right) \right] = q(w(z)) = w(z) + \sqrt{1+|w(z)|^2}.
\]

(2.1)

Define the function \( P(z) \) by

\[
P(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \cdots.
\]

It is easy to see that

\[
w(z) = \frac{P(z) - 1}{P(z) + 1} = \frac{1}{2} \left[ c_1z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \left( c_3 - c_1c_2 + \frac{c_1^3}{4} \right) z^3 + \cdots \right].
\]

(2.2)

Since \( w(z) \) is a Schwarz function, we see that \( \Re(p_1(z)) > 0 \) and \( p_1(0) = 1 \). Let us define the function \( p(z) \) by

\[
p(z) := \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^\alpha + \lambda \left[ 1 + z f''(z)f(z) - z f'(z) + \alpha \left( \frac{z f'(z)}{f(z)} - 1 \right) \right] = 1 + b_1z + b_2z^2 + \cdots.
\]

(2.3)

In view of the equations (2.1), (2.2), (2.3), we have

\[
p(z) = q \left( \frac{P(z) - 1}{P(z) + 1} \right) = \sqrt{1 + \left( \frac{P(z) - 1}{P(z) + 1} \right)^2} + \frac{P(z) - 1}{P(z) + 1} = 1 + \frac{c_1}{2}z + \left( \frac{c_2}{2} - \frac{c_1^2}{8} \right)z^2 + \left( \frac{c_3}{3} - \frac{c_1c_2}{4} \right)z^3 + \cdots.
\]

(2.4)

Now by (2.3) and (2.4),

\[
b_1 = \frac{c_1}{2} \quad \text{and} \quad b_2 = \frac{c_2}{2} - \frac{c_1^2}{8}.
\]
For given $f(z)$ of the form (1.1), a computation shows that
\[ \frac{zf''(z)}{f(z)} = 1 + a_2z + (2a_3 - a_2^2)z^2 + (3a_4 + a_2^3 - 3a_3a_2)z^3 + \cdots. \]
Similarly we have
\[ 1 + \frac{zf''(z)}{f(z)} = 1 + 2a_2z + (6a_3 - 4a_2^2)z^2 + \cdots. \]
An easy computation shows that
\[ \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right) + \lambda \left[ 1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} \right) - 1 \right] = 1 + (1 + \alpha)(1 + \lambda)a_2z + (\alpha + 2)(1 + 2\lambda)a_3z^2 + \left( \frac{\alpha^2 + \alpha}{2} - (\alpha + 3)\lambda - 1 \right)a_2^2z^2 + \cdots. \]
In view of the equation (2.3), we see that
\[ b_1 = (1 + \alpha)(1 + \lambda)a_2, \]
\[ b_2 = (\alpha + 2)(1 + 2\lambda)a_3 + \left( \frac{\alpha^2 + \alpha}{2} - (\alpha + 3)\lambda - 1 \right)a_2^2 \]
or equivalently, we have
\[ a_2 = \frac{c_1}{2(1 + \alpha)(1 + \lambda)}, \]
\[ a_3 = \frac{1}{(\alpha + 2)(1 + 2\lambda)} \left( c_2 - \frac{1}{2} \right) - \frac{1}{8} \left( 1 + \frac{\alpha^2 + \alpha - 2(\alpha + 3)\lambda - 2}{((1 + \alpha)(1 + \lambda))^2} \right)c_1^2 \]
Therefore, we have
\[ a_3 \leq \frac{1}{(\alpha + 2)(1 + 2\lambda)} \left( c_2 - \frac{1}{4} \right) \left( 1 + \frac{\alpha^2 + \alpha - 2(\alpha + 3)\lambda - 2}{((1 + \alpha)(1 + \lambda))^2} \right)c_1^2. \]
Using the estimate $|c_2 - vc_1^2| \leq 2\max(1, |2v - 1|)$ given in Lemma 2.3, we have
\[ |a_3| \leq \frac{1}{(\alpha + 2)(1 + 2\lambda)} \max \left\{ 1, \frac{1}{2} \times 1 \right\} \left( 1 + \frac{\alpha^2 + \alpha - 2(\alpha + 3)\lambda - 2}{((1 + \alpha)(1 + \lambda))^2} \right) - 1 \]
To show that the bounds are sharp, we define the functions $K_{\phi_n}(z) \phi_n = q(z^{n-1})(n = 2, 3, \ldots)$ with $K_{\phi_n}(0) = 0 = [K_{\phi_n}]'(0) - 1$, by
\[ \frac{z(K_{\phi_n})'(z)}{K_{\phi_n}(z)} \left( \frac{K_{\phi_n}(z)}{z} \right)^{\alpha} \left[ 1 + \frac{z(K_{\phi_n})'(z)}{K_{\phi_n}(z)} - \frac{z(K_{\phi_n})'(z)}{K_{\phi_n}(z)} + \alpha \left( \frac{z(K_{\phi_n})'(z)}{K_{\phi_n}(z)} - 1 \right) \right] = q(z^{n-1}). \]
Clearly the functions $K_n \in \mathcal{K}_{a,\lambda}(q)$, we write $K_q := K_{\phi_q}$. That is, when $n = 2$ we get $q(z) = z + \sqrt{1 + z^2} = z + z^2/2 - z^4/8 + z^6/16 + \cdots$
Here we have \( b_1 = 1 \) and \( b_2 = 1/2 \). By using (2.5) we get
\[
|a_2| = \frac{1}{(1 + \alpha)(1 + \lambda)}
\]
and again by using (2.6) we have
\[
\frac{1}{2} = (\alpha + 2)(1 + 2\lambda)a_3 + \left(\frac{\alpha^2 + \alpha}{2} - (\alpha + 3)\lambda - 1\right)a_2^2.
\]
Substituting for \( a_2 = \frac{1}{(1 + \alpha)(1 + \lambda)} \) simple calculation and taking absolute value gives
\[
|a_3| = \frac{1}{(\alpha + 2)(1 + 2\lambda)} \left| \frac{\alpha^2 + \alpha - 2(\alpha + 3)\lambda - 2}{2((1 + \alpha)(1 + \lambda))^2} - \frac{1}{2} \right|.
\]
\[
\square
\]

Remark 2.6. Let \( \alpha = 0 \) and \( \lambda \geq 0 \). If \( f(z) \) given by (1.1) belongs to \( M_{0,1}(q) \), then
\[
|a_2| \leq \frac{1}{1 + \lambda},
\]
\[
|a_3| \leq \frac{1}{2(1 + 2\lambda)} \max\{1, \left| \frac{\alpha^2 + \alpha - 2(\alpha + 3)\lambda - 2}{2((1 + \alpha)(1 + \lambda))^2} - \frac{1}{2} \right| \} = \frac{\alpha^2 + 8\lambda + 3}{4(1 + 2\lambda)(1 + \lambda)^2}.
\]

Remark 2.7. Let \( \alpha = 0 \) and \( \lambda = 0 \). If \( f(z) \) given by (1.1) belongs to \( M_{0,0}(q) \), then
\[
|a_2| \leq 1, \quad \text{and} \quad |a_3| \leq \frac{1}{2} \max\{1, \frac{3}{2}\} = \frac{3}{4}.
\]

Remark 2.8. Let \( \alpha = 0 \) and \( \lambda = 1 \). If \( f(z) \) given by (1.1) belongs to \( M_{0,0}(q) \), then
\[
|a_2| \leq \frac{1}{2}, \quad \text{and} \quad |a_3| \leq \frac{1}{6} \max\{1, \frac{3}{2}\} = \frac{1}{4}.
\]

Theorem 2.9. Let \( 0 \leq \mu \leq 1, \alpha \geq 0 \) and \( \lambda \geq 0 \). If \( f(z) \) given by (1.1) belongs to \( M_{\alpha,\lambda}(q) \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2\xi} \left(1 - \frac{\gamma}{2\tau^2}\right), & \text{if } \mu \leq \sigma_1, \\ \frac{1}{\xi}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{1}{2\xi} \left(1 + \frac{\gamma}{2\tau^2}\right), & \text{if } \mu \geq \sigma_2, \end{cases}
\]
where, for convenience,
\[
\sigma_1 = \frac{-1 + 2(\alpha + 3)\lambda - \rho}{2\xi}; \sigma_2 = \frac{3\tau^2 + 2(\alpha + 3)\lambda - \rho}{2\xi}; \sigma_3 = \frac{\tau^2 + 2(\alpha + 3)\lambda - \rho}{2\xi},
\]
\[
\gamma := \rho - 2(\alpha + 3)\lambda + 2\mu\xi,
\]
\[
\rho := \alpha^2 + \alpha - 2,
\]
\[
\xi := (\alpha + 2)(1 + 2\lambda),
\]
and
\[
\tau := (1 + \alpha)(1 + \lambda).
\]
Further, if \( \sigma_1 \leq \mu \leq \sigma_3 \), then
\[
|a_3 - \mu a_2^2| + \frac{r^2}{2\xi} \left( 1 + \frac{\gamma}{r^2} \right) |a_2|^2 \leq \frac{1}{\xi}.
\]
If \( \sigma_3 \leq \mu \leq \sigma_2 \), then
\[
|a_3 - \mu a_2^2| + \frac{r^2}{2\xi} \left( 3 - \frac{\gamma}{r^2} \right) |a_2|^2 \leq \frac{1}{\xi}.
\]
These results are sharp.

**Proof.** Now by making use of (2.7) and (2.8), we get
\[
a_3 - \mu a_2^2 = \frac{1}{(\alpha + 2)(1 + 2\lambda)} \left( c_2 - c_1 \right) \left( \frac{\alpha^2 + \alpha - 2 - 2(\alpha + 3)\lambda + 2\mu(\alpha + 2)(1 + 2\lambda)}{8((1 + \alpha)(1 + \lambda))^2} \right) \left( c_1 \right)
\]
\[
= \frac{1}{2(\alpha + 2)(1 + 2\lambda)} \left( c_2 - c_1 \right) \left( \frac{\alpha^2 + \alpha - 2 - 2(\alpha + 3)\lambda + 2\mu(\alpha + 2)(1 + 2\lambda)}{2((1 + \alpha)(1 + \lambda))^2} \right)
\]
where
\[
v := \frac{1}{2} \left( \frac{\alpha^2 + \alpha - 2 - 2(\alpha + 3)\lambda + 2\mu(\alpha + 2)(1 + 2\lambda)}{2((1 + \alpha)(1 + \lambda))^2} \right).
\]
The assertion of Theorem 2.9 now follows by an application of Lemma 2.1.

To show that the bounds are sharp, we define the functions \( F_\eta \) and \( G_\eta \) (0 \leq \eta \leq 1), respectively, with \( F_\eta(0) = 0 = F_\eta(0) - 1 \) and \( G_\eta(0) = 0 = G_\eta(0) - 1 \) by
\[
\frac{z(F_\eta)'(z)}{F_\eta(z)} - \frac{z(F_\eta)'(z)}{F_\eta(z)} + \alpha \left( \frac{z(F_\eta)'(z)}{F_\eta(z)} - 1 \right) = \Upsilon \left( \frac{z + \eta}{1 + \eta z} \right),
\]
and
\[
\frac{z(G_\eta)'(z)}{G_\eta(z)} - \frac{z(G_\eta)'(z)}{G_\eta(z)} + \alpha \left( \frac{z(G_\eta)'(z)}{G_\eta(z)} - 1 \right) = \Upsilon \left( -\frac{z + \eta}{1 + \eta z} \right),
\]
respectively.

Clearly the functions \( K_\gamma \), \( F_\eta, G_\eta \in \mathcal{M}_{a,\lambda}(q) \). In addition, we write \( K_\gamma := K_\gamma(z) = z + \sqrt{1 + z^2} \).

If \( \mu < \sigma_1 \) or \( \mu > \sigma_2 \), then the equality holds if and only if \( f \) is \( K_\gamma \) or one of its rotations. When \( \sigma_1 < \mu < \sigma_2 \), then the equality holds if and only if \( f \) is \( K_\gamma \) or \( q(z^2) = z^2 + \sqrt{1 + z^2} \), that is \( q(z^2) = z^2 + \sqrt{1 + z^2} = z^2 + z^2/2 - z^4/8 + \cdots \) or one of its rotations. Here we note that \( b_1 = 0; b_2 = \) using these values in (2.5) and (2.6) one can get sharp results. If \( \mu = \sigma_1 \) then the equality holds if and only if \( f \) is \( F_\eta \) or one of its rotations. If \( \mu = \sigma_2 \) then the equality holds if and only if \( f \) is \( G_\eta \) or one of its rotations.

By making use of Lemma 2.3, we immediately obtain the following:

**Theorem 2.10.** Let \( 0 \leq \alpha \leq 1 \), and \( 0 \leq \lambda \leq 1 \). If \( f \in \mathcal{M}_{a,\lambda}(q) \), then for complex \( \mu \), we have
\[
|a_3 - \mu a_2^2| = \frac{2}{(\alpha + 2)(1 + 2\lambda)} \max \left\{ 1, \frac{1}{2} \right\} \left[ 1 + \frac{\alpha^2 + \alpha - 2 - 2(\alpha + 3)\lambda + 2\mu(\alpha + 2)(1 + 2\lambda)}{(1 + \alpha)(1 + \lambda)^2} \right].
\]
The result is sharp.

**Remark 2.11.** 1. For the choice \( \alpha = 0 \), and \( \lambda = 1 \), Theorem 2.9, coincides with the result obtained for the class \( \mathcal{C}(q) \) [16].

2. For the choices \( \alpha = 0 \), and \( \lambda = 0 \), Theorem 2.9 reduces to the result for the class \( \mathcal{S}(q) \) [14].

3. For the choices \( \alpha = 0 \), Theorem 2.9, coincides with the result obtained for the class \( \mathcal{M}(q) \) [15].
3. Coefficient Inequalities for the Function $f^{-1}$

**Theorem 3.1.** If $f \in \mathcal{M}_{\alpha, \lambda}(q)$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ is the inverse function of $f$ with $|w| < r_0$ where $r_0$ is greater than the radius of the Koebe domain of the class $f \in \mathcal{M}_{\alpha, \lambda}(q)$, then for any complex number $\mu$, we have

$$|d_1 - \mu d_2^2| \leq \frac{1}{\xi} \max \left\{ 1, \frac{\tau^2 + \rho - 2(\alpha + 3)\lambda + (4 - 2\mu)\xi}{2\tau^2} - 1 \right\}. \quad (3.1)$$

**Proof.** As $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n \quad (3.2)$

is the inverse function of $f$, it can be seen that

$$f^{-1}(f(z)) = f(f^{-1}(z)) = z. \quad (3.3)$$

From equations (1.1) and (3.3), it can be reduced to

$$f^{-1}(z + \sum_{n=2}^{\infty} a_n z^n) = z. \quad (3.4)$$

From (3.3) and (3.4), one can obtain

$$z + (a_2 + d_2)z^2 + (a_3 + 2a_2d_2 + d_3)z^3 + \ldots = z. \quad (3.5)$$

By comparing the coefficients of $z$ and $z^2$ from relation (3.5), it can be seen that

$$d_2 = -a_2 \quad (3.6)$$

$$d_3 = 2a_2^2 - a_3. \quad (3.7)$$

From relations (2.7),(2.8),(3.6) and (3.7)

$$d_2 = -\frac{c_1}{2(1 + \alpha)(1 + \lambda)}; \quad (3.8)$$

$$d_3 = -\frac{1}{2(\alpha + 2)(1 + 2\lambda)} \left( (1 + \alpha)(1 + \lambda) \frac{\tau^2 + \rho + 2(\alpha + 3)\lambda - 2(\alpha + 3)\lambda + (4 - 2\mu)\xi}{4((1 + \alpha)(1 + \lambda))^2} c_1^2 - c_2 \right); \quad (3.9)$$

For any complex number $\mu$, consider

$$d_3 - \mu d_2^2 = -\frac{1}{2\xi} \left( c_2 - \frac{\tau^2 + \rho + 2(\alpha + 3)\lambda + (4 - 2\mu)\xi}{4\tau^2} c_1^2 \right). \quad (3.10)$$

Taking modulus on both sides and by applying Lemma 2.3 on the right hand side of (3.10), one can obtain the result as in (3.1). Hence this completes the proof. \(\square\)

**Remark 3.2.** Suitably specializing the parameters in Theorem 3.1 one can easily state above result for the function classes

$$\mathcal{M}_{0, \lambda}(q) \equiv \mathcal{M}(q); \quad \mathcal{M}_{0, 0}(q) \equiv \mathcal{S}^*(q) \quad \text{and} \quad \mathcal{M}_{0, 1}(q) \equiv \mathcal{C}(q).$$

312
4. Application to Functions Defined by Poisson Distribution

For the application of the results given in the previous section, we define the class $\mathcal{M}^q_{\alpha,\lambda}(q)$, which requires the following definition. A variable $X$ is said to be Poisson distributed if it takes the values $0, 1, 2, 3, \ldots$ with probabilities $e^{-m}, m^2e^{-m}, m^3e^{-m}, \ldots$ respectively, where $m$ is called the parameter. Thus

$$P(X = r) = \frac{m^r e^{-m}}{r!}, \quad r = 0, 1, 2, 3, \ldots$$

In [12], Porwal introduced a power series whose coefficients are probabilities of Poisson distribution

$$K(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, \quad z \in \mathcal{U},$$

where $m > 0$. By ratio test the radius of convergence of above series is infinity. Using the Hadamard product, Porwal [12] (see also, [1, 8, 10, 13] introduced a new linear operator $I^m(z) : \mathcal{A} \to \mathcal{A}$ defined by

$$I^m f = K(m, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n,$$

$$= z + \sum_{n=2}^{\infty} \psi_n a_n z^n, \quad z \in \Delta,$$

where $\psi_n = \frac{m^{n-1}}{(n-1)!} e^{-m}$, and $*$ denote the convolution or Hadamard product of two series. We define the class $\mathcal{M}^q_{\alpha,\lambda}(q)$ in the following way:

$$\mathcal{M}^q_{\alpha,\lambda}(q) := \{ f \in \mathcal{A} \text{ and } I^m f \in \mathcal{M}_{\alpha,\lambda}(q) \}$$

where $\mathcal{M}_{\alpha,\lambda}(q)$ is given by Definition 1.2.

Let $\varphi(z) = z + \sum_{n=2}^{\infty} \varphi_n z^n$, $(\varphi_n > 0)$. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{M}^q_{\alpha,\lambda}(q)$ then $F(z) = (f * \varphi)(z) = z + \sum_{n=2}^{\infty} \varphi_n a_n z^n \in \mathcal{M}_{\alpha,\lambda}(q)$.

In this section we obtain the coefficient estimate for functions in the class $\mathcal{M}^q_{\alpha,\lambda}(q)$, from the corresponding estimate for functions in the class $\mathcal{M}_{\alpha,\lambda}(q)$. Applying Theorem 2.9 for the function

$$F(z) = (f * \varphi)(z) = z + \varphi_2 a_2 z^2 + \varphi_3 a_3 z^3 + \cdots$$

we get the following Theorems 4.1 and 4.2 after an obvious change of the parameter $\mu$.

**Theorem 4.1.** Let $0 \leq \alpha \leq 1$, and $0 \leq \lambda \leq 1$. If $f \in \mathcal{M}^q_{\alpha,\lambda}(q)$, then for complex $\mu$, we have

$$|a_3 - \mu a_2^2| = \frac{2}{(\alpha + 2)(1 + 2\lambda)\varphi_3} \max \left(1, \frac{\alpha^2 + \alpha - 2 - 2(\alpha + 3)\lambda}{(1 + \alpha)(1 + \lambda)} + \frac{2\mu(\alpha + 2)(1 + 2\lambda)\varphi_3}{((1 + \alpha)(1 + \lambda)\varphi_3)^2} \right).$$

Our main result is the following:

**Proof.** For $f(z) \in M^q_{\alpha,\lambda}(q)$, let $F(z) = (f * \varphi)(z)$ given by (4.1) we can state the condition as

$$p(z) = \frac{z F'(z)}{F(z)} \left( \frac{F(z)}{z} \right)^{\alpha} + 1 + \frac{z F''(z)}{F(z)} \left( \frac{F(z)}{z} \right)^{\alpha} \left( - \frac{F'(z)}{F(z)} - 1 \right) \frac{F'(z)}{F(z)} = 1 + b_1 z + b_2 z^2 + \cdots$$

Proceeding as in Theorem 2.5 we get

$$p(z) = 1 + (1 + \alpha)(1 + \lambda)\varphi_2 a_2 z + (\alpha + 2)(1 + 2\lambda)\varphi_3 a_3 z^2 + \left( \frac{\alpha^2 + \alpha}{2} - (\alpha + 3)\lambda - 1 \right) \varphi_3^2 a_2^2 z^2 + \cdots.$$
From 2.5-2.8 and from this equation (4.3), we obtain

\[ a_2 = \frac{c_1}{2(1 + \alpha)(1 + \lambda)\varphi_2} \quad (4.4) \]

\[ a_3 = \frac{1}{(\alpha + 2)(1 + 2\lambda)\varphi_3} \left( \frac{c_2}{2} - \frac{1}{8} \left( \frac{1 + \frac{\alpha^2 + \alpha - 2(\alpha + 3)\lambda - 2}{(1 + \alpha)(1 + \lambda)\varphi_3^2} \right) c_1^2 \right). \quad (4.5) \]

\[ a_3 - \mu a_2^2 = \frac{1}{(\alpha + 2)(1 + 2\lambda)\varphi_3} \left( \frac{c_2}{2} - \frac{1}{8} \left( \frac{1 + \frac{\alpha^2 + \alpha - 2(\alpha + 3)\lambda - 2}{(1 + \alpha)(1 + \lambda)\varphi_3^2} \right) c_1^2 \right) - \frac{c_1^2}{4(1 + \alpha)^2(1 + \lambda)^2\varphi_2^2}. \]

Therefore our result now follows by an application of Lemma 2.3. The result is sharp for the function defined by

\[ \frac{zF'(z)}{F(z)} \left( \frac{F(z)}{z} \right)^{\alpha} + \lambda \left[ 1 + \frac{zF''(z)}{F'(z)} \cdot \frac{\varphi'(z)}{\varphi(z)} + \alpha \left( \frac{\varphi'(z)}{\varphi(z)} - 1 \right) \right] = q(z) \]

and

\[ \frac{zF'(z)}{F(z)} \left( \frac{F(z)}{z} \right)^{\alpha} + \lambda \left[ 1 + \frac{zF''(z)}{F'(z)} \cdot \frac{\varphi'(z)}{\varphi(z)} + \alpha \left( \frac{\varphi'(z)}{\varphi(z)} - 1 \right) \right] = q(z) \]

Theorem 4.2. Let \( 0 \leq \mu \leq 1, \ \alpha \geq 0, \ \lambda \geq 0 \) and \( \varphi_n > 0 \). If \( f(z) \) given by (1.1) belongs to \( \mathcal{H}^\mu_{\alpha,\lambda}(q) \), then

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2\xi\varphi_3} \left( 1 - \frac{\gamma_2}{2\tau^2} \right), & \text{if} \quad \mu \leq \sigma_1, \\ \frac{1}{\xi\varphi_3}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{1}{2\xi\varphi_3} \left( -1 + \frac{\gamma_2}{2\tau^2} \right), & \text{if} \quad \mu \geq \sigma_2, \end{cases} \]

where, for convenience,

\[ \sigma_1 := \frac{\phi_2}{\phi_3} \left( \frac{2(\alpha + 3)\lambda - \rho - \tau^2}{2\xi} \right), \quad \sigma_2 := \frac{\phi_2}{\phi_3} \left( \frac{3\tau^2 + 2(\alpha + 3)\lambda - \rho}{2\xi} \right), \]

\[ \sigma_3 := \frac{\phi_2}{\phi_3} \left( \frac{\tau^2 + 2(\alpha + 3)\lambda - \rho}{2\xi} \right), \quad \gamma_2 := \rho - 2(\alpha + 3)\lambda + 2\mu \frac{\varphi_2}{\phi_2} \xi, \]

and \( \rho, \xi, \tau \) are as defined in (2.10), (2.11) and (2.12).

Proof. Using (4.4), (4.5) and proceeding as in Theorem 2.9 and Theorem 4.1 we get desired result. \( \square \)

Now, we obtain the coefficient estimate for \( f \in \mathcal{H}^\mu_{\alpha,\lambda}(q) \), from the corresponding estimate for \( f \in \mathcal{H}_{\alpha,\lambda}(q) \). Applying Theorem 2.9 for the function

\[ I^n f = z + \psi_2 a_2 z^2 + \psi_3 a_3 z^3 + \cdots \]

we get the following Theorems 4.3 and 4.4 after an obvious change of the parameter \( \mu \) as in above theorems.

314
Since, \( I^m f = z + \sum_{n=2}^{\infty} \psi_n a_n e^n \), where \( \psi_n = \frac{m^{n-1}}{(n-1)!} e^{-m} \), we have
\[
\psi_2 = me^{-m}
\] (4.6)
and
\[
\psi_3 = \frac{m^2}{2} e^{-m}.
\] (4.7)

For \( \psi_2 \) and \( \psi_3 \) given by (4.6) and (4.7), Theorems 4.1 and 4.2 yields to the following results by taking \( \varphi_2 = me^{-m} \) and \( \varphi_3 = \frac{m^2}{2} e^{-m} \).

**Theorem 4.3.** Let \( 0 \leq \alpha \leq 1, \) and \( 0 \leq \lambda \leq 1. \) If \( f \in \mathcal{M}_m^{\alpha, \lambda}(q) \), then for complex \( \mu \), we have
\[
|a_3 - \mu a_2^2| = \frac{4}{(\alpha + 2)(1 + 2\lambda) m^2 e^{-m}} \max \left\{ \frac{1}{2} \left| \frac{1}{2} + \frac{\alpha^2 + \alpha - 2 - 2(\alpha + 3)\lambda}{((1 + \alpha)(1 + \lambda))^2} \pm \frac{\mu(\alpha + 2)(1 + 2\lambda)}{(1 + \alpha)(1 + \lambda)^2 e^{-m}} \right| \right\}.
\]

**Theorem 4.4.** Let \( 0 \leq \mu \leq 1, \) \( \alpha \geq 0, \) \( \lambda \geq 0 \) and \( \psi_n > 0. \) If \( f(z) \) given by (1.1) belongs to \( \mathcal{M}_m^{\alpha, \lambda}(q) \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} \\
\frac{1}{\xi m e^{-m}} \left( 1 - \frac{\gamma_2}{\tau^2} \right), & \text{if } \mu \leq \sigma_1, \\
\frac{2}{\xi m^2 e^{-m}}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\
\frac{1}{\xi m^2 e^{-m}} \left( -1 + \frac{\gamma_2}{\tau^2} \right), & \text{if } \mu \geq \sigma_2,
\end{cases}
\]
where, for convenience,
\[
\sigma_1 := e^{-m} \left( \frac{2(\alpha + 3)\lambda - \rho - \tau^2}{2\xi} \right), \quad \sigma_2 := e^{-m} \left( \frac{3\tau^2 + (2(\alpha + 3)\lambda - \rho)}{2\xi} \right),
\]
\[
\sigma_3 := e^{-m} \left( \frac{\tau^2 - (\rho - 2(\alpha + 3)\lambda)}{2\xi} \right), \quad \gamma_2 := \rho - 2(\alpha + 3)\lambda + \frac{\mu}{e^{-m} \xi},
\]
and \( \rho, \xi, \tau \) are as defined in (2.10), (2.11) and (2.12).

**Remark 4.5.** Suitably specializing the parameters in Theorems 4.3 and 4.4 one can easily state the results for the function classes associated with Poisson distribution as listed below:

1. \( \mathcal{M}_{0,1}^m(q) \equiv \mathcal{M}_{l}^m(q) \)
2. \( \mathcal{M}_{0,0}^m(q) \equiv \mathcal{J}^m_m(q) \)
3. \( \mathcal{M}_{0,1}^m(q) \equiv \mathcal{J}^m_0(q) \)

which are new and not been studied so far.

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References