Montes Taurus J. Pure Appl. Math. 3 (3), 305–316, 2021

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Fekete-Szegö Inequalities for Certain Subclasses of Analytic Functions Related with Leaf-Like Domain

Gangadharan Murugusundaramoorthy [a]

[a] *School of Advanced Science, Vellore Institute of Technology, Deemed to be University, Vellore - 632014, India*

Abstract

The purpose of this paper is to consider coefficient estimates in a class of functions $\mathscr{M}_{\alpha,\lambda}(q)$ consisting of analytic functions f normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$ subordinating with leaf like domain, to derive certain coefficient estimates a_2, a_3 and Fekete-Szegö inequality for $f \in \mathscr{M}_{\alpha,\lambda}(q)$. A similar results have been done for the function f^{-1}. Further application of our results to certain functions defined by convolution products with a normalized functions analytic is given, and in particular we obtain Fekete-Szegö inequalities for certain subclasses of functions defined through Poisson distribution series.

Keywords: Analytic functions, Starlike functions, Convex functions, Subordination, Fekete-Szegö inequality, Poisson distribution series, Hadamard product

2010 MSC: 30C80, 30C45

1. Introduction

Let $\mathscr{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\Delta := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of univalent functions. A function $f \in \mathscr{S}$ is said to be *starlike* in Δ if and only if

$$\Re\Big(\frac{zf'(z)}{f(z)}\Big) > 0, \qquad (z \in \Delta) \tag{1.2}$$

and on the other hand , a function $f \in \mathscr{S}$ is said to be *convex* in Δ if and only if

$$\Re\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > 0, \qquad (z \in \Delta). \tag{1.3}$$

†*Article ID:* MTJPAM-D-20-00042

Email address: gmsmoorthy@yahoo.com (Gangadharan Murugusundaramoorthy)

Received:27 November 2020, Accepted:29 March 2021, Published:25 April 2021

⋆*Corresponding Author:* Gangadharan Murugusundaramoorthy

The class of functions satisfying the analytic criteria given by (1.2) and (1.3) are denoted by $\mathscr{S}^*$ and $\mathscr{C}$ respectively.

Let f and g be functions analytic in Δ. Then we say that the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \Delta)$, such that$f(z) = g(\omega(z))$ $(z \in \Delta)$. We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta).$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0) \quad \textit{and} \quad f(\Delta) \subset g(\Delta).$$

Definition 1.1. [14] Let $\mathscr{S}^*(q)$ denote the class of analytic functions f in the unit disc Δ normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the condition that

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z^2} + z =: q(z), \ \ z \in \Delta, \tag{1.4}$$

where the branch of the square root is chosen to be $q(0) = 1$.

It may be noted from (1.4) of Definition 1 that the set $q(\Delta)$ lies in the right half-plane and it is not a starlike domain with respect to the origin, see Fig. 1 (below).

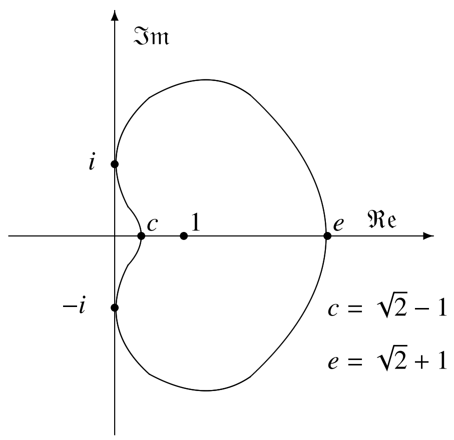

Figure 1. The boundary of the set $q(\Delta)$

Recently, Raina and Sokol [14] have studied and obtained some coefficient inequalities for the class $\mathscr{S}^\star(q)$ and these results are further improved by Sokol and Thomas [16], for the class $\mathscr{C}(q)$ in view of the Alexander result between the class $f \in \mathscr{C}(q) \Leftrightarrow zf'(z) \in \mathscr{S}^*(q)$, further the Fekete-Szegö inequality for functions in $\mathscr{S}^*(q)$ were also obtained. For a brief history of Fekete-Szegö problem for the class of starlike, convex and various other subclasses of analytic functions, we refer the interested reader to [5, 20, 19, 22, 18, 21, 17].

Let $\alpha \geqq 0$, $\lambda \geqq 0$ and $0 \leqq \rho < 1$ and $f \in \mathscr{A}$. We say that $f \in M(\alpha, \lambda, \rho)$ if it satisfies the condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\} > \rho.$$

The class $M(\alpha, \lambda, \rho)$ was introduced very recently by Guo and Liu [2].

Motivated essentially by the aforementioned works (see [1, 9, 14, 15]), in this paper we define the following class $\mathscr{M}_{\alpha,\lambda}(q)$ due to Guo and Liu [2], of functions which unifies the class $\mathscr{S}^*(q)$, $\mathscr{C}(q)$ and $\mathscr{M}_\lambda(q)$. First, we shall find estimations of first few coefficients of functions f of the form (1.1) belonging to $\mathcal{M}_{\alpha,\lambda}(q)$ and we prove the Fekete-Szegö inequality for a more general class of analytic functions which we define below in Definition 1.2 and also for $f^{-1} \in \mathcal{M}_{\alpha,\lambda}(q)$. Also we give applications of our results to certain functions defined through Poisson distribution .

Now, we define the following class $\mathcal{M}_{\alpha,\lambda}(q)$:

Definition 1.2. For $\alpha \geqq 0$, $\lambda \geqq 0$ a function $f \in \mathcal{A}$ is in the class $\mathcal{M}_{\alpha,\lambda}(q)$ if

$$\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\}$$
$$\prec z + \sqrt{1+z^2} = q(z);\ z = re^{i\theta} \in \Delta. \tag{1.5}$$

Note that $\mathcal{M}_{0,0}(q) \equiv \mathcal{S}^*(q)$(see[14]); $\mathcal{M}_{0,\lambda}(q) \equiv \mathcal{M}_{\lambda}(q)$ (see[15]) and $\mathcal{M}_{0,1}(q) \equiv \mathcal{C}(q)$ given in [16].

2. A Coefficient Estimate

To prove our main result, we need the following:

Lemma 2.1. *[7] If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in Δ, then*

$$|c_2 - vc_1^2| \leqq \begin{cases} -4v + 2, & \text{if} \quad v \leqq 0, \\ 2, & \text{if} \quad 0 \leqq v \leqq 1, \\ 4v - 2, & \text{if} \quad v \geqq 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < v < 1$, then equality holds if and only if $p_2(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p_3(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right)\frac{1-z}{1+z} \quad (0 \leqq \eta \leqq 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Although the above upper bound is sharp, when $0 < v < 1$, it can be improved as follows:

$$|c_2 - vc_1^2| + v|c_1|^2 \leqq 2 \quad (0 < v \leqq 1/2)$$

and

$$|c_2 - vc_1^2| + (1 - v)|c_1|^2 \leqq 2 \quad (1/2 < v \leqq 1).$$

We also need the following:

Lemma 2.2. *[3] If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in Δ, then*

$$|\, c_n \,| \leq 2 \ \ for\ all\ n \geq 1 \qquad \text{and} \qquad |c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}.$$

The class of all such functions with positive real part is denoted by $\mathcal{P}$.

Lemma 2.3. *[6] If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in Δ, and vis a complex number, then*

$$|c_2 - vc_1^2| \leqq 2\max(1, |2v - 1|).$$

The result is sharp for the functions

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 2.4. *[4] Let $P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ be in $\mathcal{P}$ then for any complex number μ,*

$$\Big|c_2 - \mu\frac{c_1^2}{2}\Big| \leq \max\{2, 2|\mu - 1|\} = \begin{cases} 2, & 0 \leq \mu \leq 2; \\ 2|\mu - 1|, & \textit{elsewhere.} \end{cases}$$

The result is sharp for the functions defined by $P(z) = \frac{1+z^2}{1-z^2}$ or $P(z) = \frac{1+z}{1-z}$.

Theorem 2.5. *Let $\alpha \geqq 0$ and $\lambda \geqq 0$. If $f(z)$ given by (1.1) belongs to $\mathscr{M}_{\alpha,\lambda}(q)$, then*

$$\begin{aligned} |a_2| &\leqq \frac{1}{(1+\alpha)(1+\lambda)}, \\ |a_3| &\leqq \frac{1}{(\alpha+2)(1+2\lambda)} \max\{1, \Big|\Big(\frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2((1+\alpha)(1+\lambda))^2} - \frac{1}{2}\Big)\Big|\}. \end{aligned}$$

These results are sharp.

Proof. If $f \in \mathscr{M}_{\alpha,\lambda}(q)$, then there is a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that

$$\begin{aligned} \frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right] &= q(w(z)) \\ &= w(z) + \sqrt{1 + [w(z)]^2}. \end{aligned} \tag{2.1}$$

Define the function $P(z)$ by

$$P(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots .$$

it is easy to see that

$$\begin{aligned} w(z) &= \frac{P(z) - 1}{P(z) + 1} \\ &= \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right]. \end{aligned} \tag{2.2}$$

Since $w(z)$ is a Schwarz function, we see that $\Re(p_1(z)) > 0$ and $p_1(0) = 1$. Let us define the function $p(z)$ by

$$\begin{aligned} p(z) :&= \frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right] \\ &= 1 + b_1 z + b_2 z^2 + \cdots . \end{aligned} \tag{2.3}$$

In view of the equations (2.1), (2.2), (2.3), we have

$$\begin{aligned} p(z) &= q\left(\frac{P(z) - 1}{P(z) + 1}\right) \\ &= \sqrt{1 + \left(\frac{P(z) - 1}{P(z) + 1}\right)^2} + \frac{P(z) - 1}{P(z) + 1} \\ &= 1 + \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{8}\right)z^2 + \left(\frac{c_3}{2} - \frac{c_1 c_2}{4}\right)z^3 + ... \end{aligned} \tag{2.4}$$

Now by (2.3) and (2.4),

$$b_1 = \frac{c_1}{2} \qquad \text{and} \qquad b_2 = \frac{c_2}{2} - \frac{c_1^2}{8}.$$

For given $f(z)$ of the form (1.1), a computation shows that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 + a_2^3 - 3a_3a_2)z^3 + \cdots .$$

Similarly we have

$$1 + \frac{zf''(z)}{f'(z)} = 1 + 2a_2 z + (6a_3 - 4a_2^2)z^2 + \cdots .$$

An easy computation shows that

$$\begin{aligned} \frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right] &= 1 + (1+\alpha)(1+\lambda)a_2 z + (\alpha+2)(1+2\lambda)a_3 z^2 \\ &+ \left(\frac{\alpha^2+\alpha}{2} - (\alpha+3)\lambda - 1\right)a_2^2 z^2 + \cdots . \end{aligned}$$

In view of the equation (2.3), we see that

$$b_1 = (1+\alpha)(1+\lambda)a_2, \tag{2.5}$$

$$b_2 = (\alpha+2)(1+2\lambda)a_3 + \left(\frac{\alpha^2+\alpha}{2} - (\alpha+3)\lambda - 1\right)a_2^2 \tag{2.6}$$

or equivalently, we have

$$\begin{aligned} a_2 &= \frac{c_1}{2(1+\alpha)(1+\lambda)}, \\ a_3 &= \frac{1}{(\alpha+2)(1+2\lambda)}\left(\frac{c_2}{2} - \frac{1}{8}\left(1 + \frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{((1+\alpha)(1+\lambda))^2}\right)c_1^2\right). \end{aligned} \tag{2.7}$$

Therefore, we have

$$a_3 = \frac{1}{2(\alpha+2)(1+2\lambda)}\left(c_2 - \frac{1}{4}\left(1 + \frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{((1+\alpha)(1+\lambda))^2}\right)c_1^2\right). \tag{2.8}$$

Using the estimate $|c_2 - vc_1^2| \leqq 2\max(1, |2v-1|)$ given in Lemma 2.3, we have

$$\begin{aligned} |a_3| &\leqq \frac{1}{(\alpha+2)(1+2\lambda)}\max\{1, \left|2 \times \frac{1}{4}\left(1 + \frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{((1+\alpha)(1+\lambda))^2}\right) - 1\right|\} \\ &= \frac{1}{(\alpha+2)(1+2\lambda)}\max\{1, \left|\left(\frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2\,((1+\alpha)(1+\lambda))^2}\right) - \frac{1}{2}\right|\}. \end{aligned}$$

To show that the bounds are sharp,we define the functions $K_{\phi_n}(z)$ $\phi_n = q(z^{n-1})(n = 2, 3, \ldots)$ with $K_{\phi_n}(0) = 0 = [K_{\phi_n}]'(0) - 1$, by

$$\frac{z(K_{\phi_n})'(z)}{K_{\phi_n}(z)}\left(\frac{K_{\phi_n}(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{z(K_{\phi_n})''(z)}{(K_{\phi_n})'(z)} - \frac{z(K_{\phi_n})'(z)}{K_{\phi_n}(z)} + \alpha\left(\frac{z(K_{\phi_n})'(z)}{K_{\phi_n}(z)} - 1\right)\right] = q(z^{n-1}).$$

Clearly the functions $K_{q_n} \in \mathscr{M}_{\alpha,\lambda}(q)$. we write $K_q := K_{q_2}$. That is, when

$$n = 2 \text{ we get } \quad q(z) = z + \sqrt{1+z^2} = z + z^2/2 - z^4/8 + z^6/16 + \cdots$$

Here we have $b_1 = 1$ and $b_2 = 1/2$. By using (2.5) we get

$$|a_2| = \frac{1}{(1+\alpha)(1+\lambda)}$$

and again by using (2.6) we have

$$\frac{1}{2} = (\alpha+2)(1+2\lambda)a_3 + \left(\frac{\alpha^2+\alpha}{2} - (\alpha+3)\lambda - 1\right)a_2^2.$$

Substituting for $a_2 = \frac{1}{(1+\alpha)(1+\lambda)}$ simple calculation and taking absolute value gives

$$|a_3| = \frac{1}{(\alpha+2)(1+2\lambda)}\left|\frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2((1+\alpha)(1+\lambda))^2} - \frac{1}{2}\right|.$$

□

Remark 2.6. Let $\alpha = 0$ and $\lambda \geqq 0$. If $f(z)$ given by (1.1) belongs to $\mathscr{M}_{0,\lambda}(q)$, then

$$\begin{aligned} |a_2| &\leqq \frac{1}{1+\lambda}, \\ |a_3| &\leqq \frac{1}{2(1+2\lambda)}\max\{1, |\frac{\lambda^2+8\lambda+3}{2(1+\lambda)^2}|\} = \frac{\lambda^2+8\lambda+3}{4(1+2\lambda)(1+\lambda)^2}. \end{aligned}$$

Remark 2.7. Let $\alpha = 0$ and $\lambda = 0$. If $f(z)$ given by (1.1) belongs to $\mathscr{M}_{0,0}(q)$, then

$$|a_2| \leqq 1, \qquad \text{and} \qquad |a_3| \leqq \frac{1}{2}\max\{1, |\frac{3}{2}|\} = \frac{3}{4}.$$

Remark 2.8. Let $\alpha = 0$ *and* $\lambda = 1$. If $f(z)$ given by (1.1) belongs to $\mathscr{M}_{0,\lambda}(q)$, then

$$|a_2| \leqq \frac{1}{2}, \qquad \text{and} \qquad |a_3| \leqq \frac{1}{6}\max\{1, |\frac{3}{2}|\} = \frac{1}{4}.$$

Theorem 2.9. *Let* $0 \leqq \mu \leqq 1,\ \alpha \geqq 0$ *and* $\lambda \geqq 0$. *If* $f(z)$ *given by* (1.1) *belongs to* $\mathscr{M}_{\alpha,\lambda}(q)$, *then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \frac{1}{2\xi}\left(1 - \frac{\gamma}{2\tau^2}\right), & \text{if } \mu \leqq \sigma_1, \\ \frac{1}{\xi}, & \text{if } \sigma_1 \leqq \mu \leqq \sigma_2, \\ \frac{1}{2\xi}\left(-1 + \frac{\gamma}{2\tau^2}\right), & \text{if } \mu \geqq \sigma_2, \end{cases}$$

where, for convenience,

$$\sigma_1 = \frac{-\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi}; \sigma_2 = \frac{3\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi}; \sigma_3 = \frac{\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi},$$

$$\gamma := \rho - 2(\alpha+3)\lambda + 2\mu\xi, \tag{2.9}$$

$$\rho := \alpha^2 + \alpha - 2, \tag{2.10}$$

$$\xi := (\alpha+2)(1+2\lambda), \tag{2.11}$$

and

$$\tau := (1+\alpha)(1+\lambda). \tag{2.12}$$

Further, if $\sigma_1 \leqq \mu \leqq \sigma_3$, *then*

$$|a_3 - \mu a_2^2| + \frac{\tau^2}{2\xi}\left(1 + \frac{\gamma}{\tau^2}\right)|a_2|^2 \leqq \frac{1}{\xi}.$$

If $\sigma_3 \leqq \mu \leqq \sigma_2$, *then*

$$|a_3 - \mu a_2^2| + \frac{\tau^2}{2\xi}\left(3 - \frac{\gamma}{\tau^2}\right)|a_2|^2 \leqq \frac{1}{\xi}.$$

These results are sharp.

Proof. Now by making use of (2.7) and (2.8) , we get

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{(\alpha+2)(1+2\lambda)}\left(\frac{c_2}{2} - \frac{c_1^2}{8}\left(\frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda+2\mu(\alpha+2)(1+2\lambda)}{8\left((1+\alpha)(1+\lambda)\right)^2}\right)c_1^2\right) \\ &= \frac{1}{2(\alpha+2)(1+2\lambda)}\left(c_2 - \frac{c_1^2}{2}\left(\frac{1}{2} + \frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda+2\mu(\alpha+2)(1+2\lambda)}{2\left((1+\alpha)(1+\lambda)\right)^2}\right)\right) \\ &= \frac{1}{2(\alpha+2)(1+2\lambda)}\left(c_2 - vc_1^2\right) \end{aligned}$$

where

$$v := \frac{1}{2}\left(\frac{1}{2} + \frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda+2\mu(\alpha+2)(1+2\lambda)}{2\left((1+\alpha)(1+\lambda)\right)^2}\right).$$

The assertion of Theorem 2.9 now follows by an application of Lemma 2.1.

To show that the bounds are sharp, we define the functions the functions F_η and G_η $(0 \leqq \eta \leqq 1)$, respectively, with $F_\eta(0) = 0 = F'_\eta(0) - 1$ and $G_\eta(0) = 0 = G'_\eta(0) - 1$ by

$$\frac{z(F_\eta)'(z)}{F_\eta(z)}\left(\frac{F_\eta(z)}{z}\right)^\alpha + \lambda\left[1 + \frac{z(F_\eta)''(z)}{(F_\eta)'(z)} - \frac{z(F_\eta)'(z)}{F_\eta(z)} + \alpha\left(\frac{z(F_\eta)'(z)}{F_\eta(z)} - 1\right)\right] = \Upsilon\left(\frac{z(z+\eta)}{1+\eta z}\right),$$

and

$$\frac{z(G_\eta)'(z)}{G_\eta(z)}\left(\frac{G_\eta(z)}{z}\right)^\alpha + \lambda\left[1 + \frac{z(G_\eta)''(z)}{(G_\eta)'(z)} - \frac{z(G_\eta)'(z)}{G_\eta(z)} + \alpha\left(\frac{z(G_\eta)'(z)}{G_\eta(z)} - 1\right)\right] = \Upsilon\left(-\frac{z(z+\eta)}{1+\eta z}\right),$$

respectively.

Clearly the functions $K_{\Upsilon_n} = q(z^{n-1}), F_\eta, G_\eta \in \mathscr{M}_{\alpha,\lambda}(q)$. In addition,we write $K_{\Upsilon_2} := K_\Upsilon(z) = z + \sqrt{1+z^2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_Υ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is $K_{\Upsilon_3} = q(z^2) = z^2 + \sqrt{1+z^4}$, that is $q(z^2) = z^2 + \sqrt{1+z^4} = z^2 + z^4/2 - z^8/8 + \cdots$. or one of its rotations. Here we note that $b_1 = 0; b_2 = 1$using these values in (2.5) and (2.6)one can get sharp results. If $\mu = \sigma_1$ then the equality holds if and only if f is F_η or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_η or one of its rotations. □

By making use of Lemma 2.3, we immediately obtain the following:

Theorem 2.10. *Let* $0 \leqq \alpha \leqq 1,$ *and* $0 \leqq \lambda \leqq 1$. *If* $f \in \mathscr{M}_{\alpha,\lambda}(q)$, *then for complex* μ, *we have*

$$|a_3 - \mu a_2^2| = \frac{2}{(\alpha+2)(1+2\lambda)}\max\left\{1, \frac{1}{2}\left|-1 + \frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda+2\mu(\alpha+2)(1+2\lambda)}{\left((1+\alpha)(1+\lambda)\right)^2}\right|\right\}.$$

The result is sharp.

Remark 2.11.

1. For the choice $\alpha = 0$, and $\lambda = 1$, Theorem 2.9, coincides with the result obtained for the class $\mathscr{C}(q)$ [16].

2. For the choices $\alpha = 0$, and $\lambda = 0$, Theorem 2.9 reduces to the result for the class $\mathscr{S}^*(q)$ [14].

3. For the choices $\alpha = 0$, Theorem 2.9, coincides with the result obtained for the class $\mathscr{M}_\lambda(q)$ [15].

3. Coefficient Inequalities for the Function f^{-1}

Theorem 3.1. *If $f \in \mathscr{M}_{\alpha,\lambda}(q)$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ is the inverse function of f with $|w| < r_0$ where r_0 is greater than the radius of the Koebe domain of the class $f \in \mathscr{M}_{\alpha,\lambda}(q)$, then for any complex number μ, we have*

$$| d_3 - \mu d_2^2 | \leq \frac{1}{\xi} max \left\{1, \left| \frac{\tau^2 + \rho - 2(\alpha+3)\lambda + (4-2\mu)\xi}{2\tau^2} - 1 \right| \right\}. \tag{3.1}$$

Proof. As

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n \tag{3.2}$$

is the inverse function of f, it can be seen that

$$f^{-1}(f(z)) = f\{f^{-1}(z)\} = z. \tag{3.3}$$

From equations (1.1) and (3.3), it can be reduced to

$$f^{-1}(z + \sum_{n=2}^{\infty} a_n z^n) = z. \tag{3.4}$$

From (3.3) and (3.4), one can obtain

$$z + (a_2 + d_2)z^2 + (a_3 + 2a_2 d_2 + d_3)z^3 + ... = z. \tag{3.5}$$

By comparing the coefficients of z and z^2 from relation (3.5), it can be seen that

$$d_2 = -a_2 \tag{3.6}$$

$$d_3 = 2a_2^2 - a_3. \tag{3.7}$$

From relations (2.7),(2.8),(3.6) and (3.7)

$$d_2 = -\frac{c_1}{2(1+\alpha)(1+\lambda)}; \tag{3.8}$$

$$\begin{aligned} d_3 &= -\frac{1}{2(\alpha+2)(1+2\lambda)} \left(\frac{((1+\alpha)(1+\lambda))^2 + 4(1+\alpha)(1+2\lambda) + \alpha^2 + \alpha - 2(\alpha+3)\lambda - 2}{4\,((1+\alpha)(1+\lambda))^2} c_1^2 - c_2 \right); \\ &= -\frac{1}{2\xi} \left(\frac{\tau^2 + 4\xi + \rho - 2(\alpha+3)\lambda}{4\tau^2} c_1^2 - c_2 \right); \\ &= \frac{1}{2\xi} \left(c_2 - \frac{\tau^2 + 4\xi + \rho - 2(\alpha+3)\lambda}{4\tau^2} c_1^2 \right). \end{aligned} \tag{3.9}$$

For any complex number μ, consider

$$d_3 - \mu d_2^2 = -\frac{1}{2\xi}\Big(c_2 - \frac{\tau^2 + \rho - 2(\alpha+3)\lambda + (4-2\mu)\xi}{4\tau^2} c_1^2 \Big) \tag{3.10}$$

Taking modulus on both sides and by applying Lemma 2.3 on the right hand side of (3.10), one can obtain the result as in (3.1). Hence this completes the proof. □

Remark 3.2. Suitably specializing the parameters in Theorem 3.1 one can easily state above result for the function classes

$$\mathscr{M}_{0,\lambda}(q) \equiv \mathscr{M}_{\lambda}(q); \quad \mathscr{M}_{0,0}(q) \equiv \mathscr{S}^*(q) \quad \text{and} \quad \mathscr{M}_{0,1}(q) \equiv \mathscr{C}(q).$$

4. Application to Functions Defined by Poisson Distribution

For the application of the results given in the previous section, we define the class $\mathscr{M}^{\delta}_{\alpha,\lambda}(q)$, which requires the following definition. A variable $\mathcal{X}$ is said to be Poisson distributed if it takes the values $0, 1, 2, 3, \cdots$ with probabilities $e^{-m}, m\frac{e^{-m}}{1!}, m^2\frac{e^{-m}}{2!}, m^3\frac{e^{-m}}{3!}, ...$ respectively, where m is called the parameter. Thus

$$P(\mathcal{X} = r) = \frac{m^r e^{-m}}{r!},\ r = 0, 1, 2, 3, \cdots .$$

In [12], Porwal introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathcal{K}(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \qquad z \in \mathcal{U},$$

where $m > 0$. By ratio test the radius of convergence of above series is infinity. Using the Hadamard product, Porwal[12] (see also, [1, 8, 10, 13] introduced a new linear operator $\mathcal{I}^m(z) : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned} \mathcal{I}^m f = \mathcal{K}(m, z) * f(z) &= z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, \\ &= z + \sum_{n=2}^{\infty} \psi_m a_n z^n, \qquad z \in \Delta, \end{aligned}$$

where $\psi_n = \frac{m^{n-1}}{(n-1)!} e^{-m}$, and $*$ denote the convolution or Hadamard product of two series. We define the class $\mathscr{M}^{m}_{\alpha,\lambda}(q)$ in the following way:

$$\mathscr{M}^{m}_{\alpha,\lambda}(q) := \{f \in \mathscr{A} \quad \text{and} \quad \mathcal{I}^m f \in \mathscr{M}_{\alpha,\lambda}(q)\}$$

where $\mathscr{M}_{\alpha,\lambda}(q)$ is given by Definition 1.2.

Let $\varphi(z) = z + \sum_{n=2}^{\infty} \varphi_n z^n$, $(\varphi_n > 0)$. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{M}^{\varphi}_{\alpha,\lambda}(q)$ then $\mathcal{F}(z) = (f * \varphi)(z) = z + \sum_{n=2}^{\infty} \varphi_n a_n z^n \in \mathscr{M}_{\alpha,\lambda}(q)$.

In this section we obtain the coefficient estimate for functions in the class $\mathscr{M}^{\varphi}_{\alpha,\lambda}(q)$, from the corresponding estimate for functions in the class $\mathscr{M}_{\alpha,\lambda}(q)$. Applying Theorem 2.9 for the function

$$\mathcal{F}(z) = (f * \varphi)(z) = z + \varphi_2 a_2 z^2 + \varphi_3 a_3 z^3 + \cdots \tag{4.1}$$

we get the following Theorems 4.1 and 4.2 after an obvious change of the parameter μ.

Theorem 4.1. *Let $0 \leqq \alpha \leqq 1$, and $0 \leqq \lambda \leqq 1$. If $f \in \mathscr{M}^{\varphi}_{\alpha,\lambda}(q)$, then for complex μ, we have*

$$|a_3 - \mu a_2^2| = \frac{2}{(\alpha+2)(1+2\lambda)\varphi_3} \max\left\{1, \frac{1}{2}\left|-1 + \frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda}{((1+\alpha)(1+\lambda))^2} + \frac{2\mu(\alpha+2)(1+2\lambda)\varphi_3}{((1+\alpha)(1+\lambda)\varphi_2)^2}\right|\right\}.$$

Our main result is the following:

Proof. For $f(z) \in M^{\varphi}_{\alpha,\lambda}(q)$, let $\mathcal{F}(z) = (f * \varphi)(z)$ given by (4.1) we can state the condition as

$$\begin{aligned} p(z) : &= \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\left(\frac{\mathcal{F}(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} - \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + \alpha\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - 1\right)\right] \\ &= 1 + b_1 z + b_2 z^2 + \cdots . \end{aligned} \tag{4.2}$$

Proceeding as in Theorem 2.5 we get

$$p(z) = 1 + (1+\alpha)(1+\lambda)\varphi_2 a_2 z + (\alpha+2)(1+2\lambda)\varphi_3 a_3 z^2 + \left(\frac{\alpha^2+\alpha}{2} - (\alpha+3)\lambda - 1\right)\varphi_2^2 a_2^2 z^2 + \cdots . \tag{4.3}$$

From 2.5- 2.8 and from this equation (4.3), we obtain

$$a_2 = \frac{c_1}{2(1+\alpha)(1+\lambda)\varphi_2} \tag{4.4}$$

$$a_3 = \frac{1}{(\alpha+2)(1+2\lambda)\varphi_3}\left(\frac{c_2}{2} - \frac{1}{8}\left(1 + \frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{((1+\alpha)(1+\lambda)\varphi_2)^2}\right)c_1^2\right). \tag{4.5}$$

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{(\alpha+2)(1+2\lambda)\varphi_3}\left(\frac{c_2}{2} - \frac{1}{8}\left(1 + \frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{((1+\alpha)(1+\lambda)\varphi_2)^2}\right)c_1^2\right) - \mu\frac{c_1^2}{4(1+\alpha)^2(1+\lambda)^2\varphi_2^2} \\ &= \frac{1}{(\alpha+2)(1+2\lambda)\varphi_3}\left(\frac{c_2}{2} - \frac{c_1^2}{8}\left(1 + \frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{((1+\alpha)(1+\lambda)\varphi_2)^2} - \mu\frac{2(\alpha+2)(1+2\lambda)\varphi_3}{(1+\alpha)^2(1+\lambda)^2\varphi_2^2}\right)\right). \end{aligned}$$

Therefore our result now follows by an application of Lemma 2.3. The result is sharp for the function defined by

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\left(\frac{\mathcal{F}(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} - \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + \alpha\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - 1\right)\right] = q(z)$$

and

$$\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\left(\frac{\mathcal{F}(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} - \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + \alpha\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} - 1\right)\right] = q(z^2)$$

□

Theorem 4.2. *Let* $0 \leqq \mu \leqq 1,\ \alpha \geqq 0,\quad \lambda \geqq 0$ *and* $\varphi_n > 0$. *If* $f(z)$ *given by* (1.1) *belongs to* $\mathscr{M}^{\varphi}_{\alpha,\lambda}(q)$, *then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{1}{2\xi\varphi_3}\left(1 - \dfrac{\gamma_2}{2\tau^2}\right), & \text{if} \quad \mu \leqq \sigma_1, \\ \dfrac{1}{\xi\varphi_3}, & \text{if} \quad \sigma_1 \leqq \mu \leqq \sigma_2, \\ \dfrac{1}{2\xi\varphi_3}\left(-1 + \dfrac{\gamma_2}{2\tau^2}\right), & \text{if} \quad \mu \geqq \sigma_2, \end{cases}$$

where, for convenience,

$$\begin{aligned} \sigma_1 &:= \frac{\varphi_2^2}{\varphi_3}\left[\frac{2(\alpha+3)\lambda - \rho - \tau^2}{2\xi}\right], \qquad \sigma_2 = \frac{\varphi_2^2}{\varphi_3}\left[\frac{3\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi}\right], \\ \sigma_3 &:= \frac{\varphi_2^2}{\varphi_3}\left[\frac{\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi}\right], \qquad \gamma_2 := \rho - 2(\alpha+3)\lambda + 2\mu\frac{\varphi_3}{\varphi_2^2}\xi, \end{aligned}$$

and ρ, ξ, τ *are as defined in* (2.10), (2.11) *and* (2.12).

Proof. Using (4.4), (4.5) and proceeding as in Theorem 2.9 and Theorem 4.1 we get desired result. □

Now,we obtain the coefficient estimate for $f \in \mathscr{M}^m_{\alpha,\lambda}(q)$, from the corresponding estimate for $f \in \mathscr{M}_{\alpha,\lambda}(q)$. Applying Theorem 2.9 for the function

$$\mathcal{I}^m f = z + \psi_2 a_2 z^2 + \psi_3 a_3 z^3 + \cdots$$

, we get the following Theorems 4.3 and 4.4 after an obvious change of the parameter μ as in above theorems.

Since, $\mathcal{I}^m f = z + \sum_{n=2}^{\infty} \psi_m a_n z^n$, where $\psi_n = \frac{m^{n-1}}{(n-1)!} e^{-m}$, we have

$$\psi_2 = me^{-m} \tag{4.6}$$

and

$$\psi_3 = \frac{m^2}{2} e^{-m}. \tag{4.7}$$

For ψ_2 and ψ_3 given by (4.6) and (4.7), Theorems 4.1 and 4.2 yields to the following results by taking $\varphi_2 = me^{-m}$ and $\varphi_3 = \frac{m^2}{2} e^{-m}$.

Theorem 4.3. *Let* $0 \leqq \alpha \leqq 1$, *and* $0 \leqq \lambda \leqq 1$. *If* $f \in \mathscr{M}^m_{\alpha,\lambda}(q)$, *then for complex* μ, *we have*

$$|a_3 - \mu a_2^2| = \frac{4}{(\alpha+2)(1+2\lambda)m^2 e^{-m}} \max\left\{1, \frac{1}{2}\left|-1 + \frac{\alpha^2 + \alpha - 2 - 2(\alpha+3)\lambda}{((1+\alpha)(1+\lambda))^2} + \frac{\mu(\alpha+2)(1+2\lambda)}{((1+\alpha)(1+\lambda))^2 e^{-m}}\right|\right\}.$$

Theorem 4.4. *Let* $0 \leqq \mu \leqq 1,\ \alpha \geqq 0,\ \lambda \geqq 0$ *and* $\psi_n > 0$. *If* $f(z)$ *given by* (1.1) *belongs to* $\mathscr{M}^m_{\alpha,\lambda}(q)$, *then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \frac{1}{\xi m^2 e^{-m}}\left(1 - \frac{\gamma_2}{2\tau^2}\right), & \text{if} \quad \mu \leqq \sigma_1, \\ \frac{2}{\xi m^2 e^{-m}}, & \text{if} \quad \sigma_1 \leqq \mu \leqq \sigma_2, \\ \frac{1}{\xi m^2 e^{-m}}\left(-1 + \frac{\gamma_2}{2\tau^2}\right), & \text{if} \quad \mu \geqq \sigma_2, \end{cases}$$

where, for convenience,

$$\sigma_1 := e^{-m}\frac{(2(\alpha+3)\lambda - \rho) - \tau^2}{2\xi}, \qquad \sigma_2 = e^{-m}\frac{3\tau^2 + (2(\alpha+3)\lambda - \rho)}{2\xi},$$

$$\sigma_3 := e^{-m}\frac{\tau^2 - (\rho - 2(\alpha+3)\lambda)}{2\xi}, \quad \gamma_2 := \rho - 2(\alpha+3)\lambda + \frac{\mu}{e^{-m}}\xi,$$

and ρ, ξ, τ *are as defined in* (2.10), (2.11) *and* (2.12).

Remark 4.5. Suitably specializing the parameters in Theorems 4.3 and 4.4 one can easily state the results for the function classes associated with Poisson distribution as listed below:

1. $\mathscr{M}^m_{0,\lambda}(q) \equiv \mathscr{M}^m_{\lambda}(q)$
2. $\mathscr{M}^m_{0,0}(q) \equiv \mathscr{S}^*_m(q)$ and
3. $\mathscr{M}^m_{0,1}(q) \equiv \mathscr{C}^m(q)$

which are new and not been studied so far.

Acknowledgements

This paper is dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th Birthday.

I would like to record to all the reviewers for their careful readings and insightful suggestions such that the original manuscript for this article is greatly improved into present form.

References

[1] S. M. El-Deeb and T. Bulboacă, *Fekete-Szegö inequalities for certain class of analytic functions connected with q-anlogue of Bessel function*, J. Egyptian. Math. Soc. **27 (42)**, 1-11, 2019, https://doi.org/10.1186/s42787-019-0049-2.

[2] D. Guo and M-S. Liu, *On certain subclass of Bazilevič functions*, J. Inequal. Pure Appl. Math. **8 (1)**, Article 12, 1-11, 2007.

[3] U. Grenander and G.Szegö, *Toeplitz Forms and Their Applications, California Monographs in Mathematical Sciences*, University of California Press, Berkeley, 1958.

[4] F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20**, 8-12, 1969.

[5] B. Kowalczyk, A. Lecko and H. M. Srivastava, *A note on the Fekete-Szeg problem for close-to-convex functions with respect to convex functions*, Publ. Inst. Math. (Beograd) (Nouvelle Sr.) **101 (115)**, 143-149, 2017.

[6] R. J. Libera and E. J. Zlotkiewicz, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc. **85 (2)**, 225-230, 1982.

[7] W. Ma and D. Minda, *A Unified treatment of some special classes of univalent functions*, *in: Proceedings of the Conference on Complex Analysis*, Z. Li, F. Ren, L. Yang, and S. Zhang(Eds.), Int. Press, 157-169, 1994.

[8] G. Murugusundaramoorthy, *Subclasses of starlike and convex functions involving Poisson distribution series*, Afr. Mat. **28**, 1357-1366, 2017.

[9] G. Murugusundaramoorthy and T. Bulboacă, *Hankel determinants for new subclasses of analytic functions related to a shell shaped region*, Mathematics **8**, 1041, 2020.

[10] G. Murugusundaramoorthy, K. Vijaya and S. Porwal, *Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series*, Hacettepe J. Math. Stat. **45 (4)**, 1101-1107, 2016.

[11] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math. **39**, 1057-1077, 1987.

[12] S. Porwal, *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal. Art. ID 984135, 1-3, 2014.

[13] S. Porwal and M. Kumar, *A unified study on starlike and convex functions associated with Poisson distribution series*, Afr. Mat. **27**, 1021-1027, 2016.

[14] R. K. Raina and J. Sokol, *On coeffcient estimates for a certain class of starlike functions*, Hacettepe Journal of Math. and Stat. **44 (6)**, 1427-1433, 2015.

[15] R. B. Sharma and M. Haripriya, *On a class of a-convex functions subordinate to a shell shaped region*, J. Anal. **25**, 93-105, 2016, DOI10.1007/s41478-017-0031-z.

[16] J. Sokol and D. K. Thomas, *Further results on a class starlike functions related to the Bernoulli lemniscate*, Houston J. Math. **44**, 83-95, 2018.

[17] H. M. Srivastava, *Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis*, Iran. J. Sci. Technol. Trans. A: Sci. **44**, 327-344, 2020.

[18] H. M. Srivastava, S. Hussain, A. Raziq and M. Raza, *The Fekete-Szeg functional for a subclass of analytic functions associated with quasi-subordination*, Carpathian J. Math. **34**, 103-113, 2018.

[19] H. M. Srivastava, N. Khan, M. Darus, S. Khan, Q. Z. Ahmad and S. Hussain, *Fekete-Szeg type problems and their applications for a subclass of q-starlike functions with respect to symmetrical points*, Mathematics **8**, 842, 1-18, 2020.

[20] H. M. Srivastava, A. K. Mishra and M. K. Das, *The Fekete-Szegö problem for a subclass of close-to-convex functions*, Complex Variables Theory Appl. **44 (2)**, 145-163, 2001.

[21] H. M. Srivastava, A. O. Mostafa, M. K. Aouf and H. M. Zayed, *Basic and fractional q-calculus and associated Fekete-Szeg problem for p-valently q-starlike functions and p-valently q-convex functions of complex order*, Miskolc Math. Notes **20**, 489-509, 2019.

[22] H. M. Srivastava, N. Raza, E. S. A. AbuJarad, G. Srivastava and M. H. AbuJarad, *Fekete-Szeg inequality for classes of* (p,q)*-starlike and* (p,q)*-convex functions*, Rev. Real Acad. Cienc. Exactas Fs. Natur. Ser. A Mat. (RACSAM) **113**, 3563-3584, 2019.