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Subclasses of Starlike and Convex Functions Associated with Mittag-Leffler-type Poisson Distribution Series

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Abstract

In this paper, we find the necessary and sufficient conditions, inclusion relations for Mittag-Leffler-type Poisson distribution series belonging to the classes $\mathcal{S}^*(\zeta,\delta)$ and $C^*(\zeta,\delta)$. Further, we consider an integral operator related to Mittag-Leffler-type Poisson distribution series.

Keywords: Analytic functions, Univalent functions, Mittag-Leffler-type Poisson distribution series

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1. Introduction and definitions

Let $\mathcal{A}$ be the class of analytic functions in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad z \in \mathbb{U}. \tag{1.1}$$

We also let $\mathcal{T}$ be the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be starlike of order ζ $(0 \le \zeta < 1)$, if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \zeta \quad (z \in \mathbb{U}).$$

This function class is denoted by $\mathcal{S}^*(\zeta)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of starlike functions with respect to the origin.

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A function $f \in \mathcal{A}$ is said to be convex of order ζ $(0 \leq \zeta < 1)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \zeta \quad (z \in \mathbb{U}).$$

This class is denoted by $C(\gamma)$. Further, $C = C(0)$, the well-known standard class of convex functions.

Let $\mathcal{S}^*(\zeta, \delta)$ be the subclass of $\mathcal{T}$ consisting of functions which satisfy the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1 - 2\zeta} \right| < \delta \quad (z \in \mathbb{U}),$$

where $0 \leq \zeta < 1$ and $0 < \gamma \leq 1$.

Also, let $C^*(\zeta, \delta)$ be the subclass of $\mathcal{T}$ consisting of functions which satisfy the condition

$$\left| \frac{\frac{zf''(z)}{f'(z)}}{\frac{zf''(z)}{f'(z)} + 1 - 2\zeta} \right| < \delta \quad (z \in \mathbb{U}),$$

where $0 \leq \zeta < 1$ and $0 < \delta \leq 1$.

The classes $\mathcal{S}^*(\zeta, \delta)$ and $C^*(\zeta, \delta)$; were introduced and studied by Gupta and Jain [20](see also, [1]). We note that for $\delta = 1$ the classes $\mathcal{S}^*(\zeta, \delta)$ and $C^*(\zeta, \delta)$ reduce to the class of starlike and convex functions of order ζ $(0 \leq \zeta < 1)$ (see [34]).

Now we recall the well Mittag-Leffler function $E_\alpha(z)$ studied by Mittag-Leffler [25] and given by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (z \in \mathbb{C}, \Re(\alpha) > 0).$$

A more general function $E_{\alpha,\beta}$ generalizing $E_\alpha(z)$ was introduced by Wiman [37] and defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (z, \alpha, \beta \in \mathbb{C},\ \Re(\beta) > 0,\ \Re(\alpha) > 0). \tag{1.3}$$

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g. in [2, 3, 4, 5, 12, 18, 19, 22, 36].

Observe that Mittag-Leffler function $E_{\alpha,\beta}$ does not belong to the family $\mathcal{A}$. Therefore, we consider the following normalization of the Mittag-Leffler function (see, [5])

$$\begin{aligned} \mathbb{E}_{\alpha,\beta}(z) &= \Gamma(\beta) z E_{\alpha,\beta}(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} z^n, \end{aligned} \tag{1.4}$$

where $z, \alpha, \beta \in \mathbb{C}; \beta \neq 0, -1, -2, \cdots$ and $\Re(\beta) > 0, \Re(\alpha) > 0$.

Whilst formula (1.4) holds for complex-valued α, β and $z \in \mathbb{C}$, however in this paper, we shall restrict our attention to the case of real-valued α, β and $z \in \mathbb{U}$. Observe that the function $\mathbb{E}_{\alpha,\beta}$ contains many well-known functions as its special case, for example,

$\mathbb{E}_{2,1}(z) = z \cosh \sqrt{z}$, $\mathbb{E}_{2,2}(z) = \sqrt{z} \sinh \sqrt{z}$, $\mathbb{E}_{2,3}(z) = 2[\cosh \sqrt{z} - 1]$ and $\mathbb{E}_{2,4}(z) = 6[\sinh \sqrt{z} - \sqrt{z}]/\sqrt{z}$.

The probability mass function of the Mittag–Leffler-type Poisson distribution is given by

$$P(X = r) = \frac{m^r}{\Gamma(\alpha k + \beta) \mathbb{E}_{\alpha,\beta}(m)}, \quad r = 0, 1, 2, 3, \cdots, \tag{1.5}$$

where $m > 0$, $\alpha > 0$ and $\beta > 0$. Due the work of Porwal et al. [33], and using the normalized form of Mittag–Leffler function as assumed in (1.4), we define a power series whose coefficients are probabilities of Mittag–Leffler-type Poisson distribution series, as below:

$$\Psi^m_{\alpha,\beta}(z) := z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)} z^n, \qquad z \in \mathbb{U}.$$

Further, we define the series

$$\Phi^m_{\alpha,\beta}(z) := 2z - \Psi^m_{\alpha,\beta}(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)} z^n, \qquad z \in \mathbb{U}. \tag{1.6}$$

Using the concept convolution or Hadamard product of two series, we introduce the convolution operator

$$\mathcal{I}^m_{\alpha,\beta} f(z) = \Psi^m_{\alpha,\beta}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)} a_n z^n, \qquad z \in \mathbb{U},$$

where $*$ denote the convolution or Hadamard product of two series.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, using hypergeometric functions (see for example, [7, 17, 24, 27, 35]), generalized Bessel functions (see for example, [6, 15, 26]), Struve functions (see for example, [8, 21]), Poisson distribution series (see for example, [10, 31, 13, 16, 28, 30]) and Pascal distribution series (see for example, [11, 14, 29]), in this paper we determine the necessary and sufficient conditions for $\Phi^m_{\alpha,\beta}$ to be in the classes $\mathcal{S}^*(\zeta,\delta)$ and $C^*(\zeta,\delta)$. Furthermore, we give sufficient conditions for $\mathcal{I}^m_{\alpha,\beta}(\mathcal{R}^\tau(A,B)) \subset C^*(\zeta,\delta)$. Finally, we give necessary and sufficient conditions the integral operator $\mathcal{G}^m_{\alpha,\beta}(z) = \int_0^z \frac{\Phi^m_{\alpha,\beta}(t)}{t} dt$ to be in the class $C^*(\zeta,\delta)$.

2. Preliminary lemmas

To establish our main results, we need the following Lemmas.

Lemma 2.1. *[20] A function $f \in \mathcal{T}$ of the form (1.2) is in the class $\mathcal{S}^*(\zeta,\delta)$ if and only if*

$$\sum_{n=2}^{\infty} [n(1+\delta) - 1 + \delta(1-2\zeta)]\,|a_n| \leq 2\delta(1-\zeta)\,. \tag{2.1}$$

The result (2.1) is sharp.

Lemma 2.2. *[20] A function $f \in \mathcal{T}$ of the form (1.2) is in the class $C^*(\zeta,\delta)$ if and only if*

$$\sum_{n=2}^{\infty} n[n(1+\delta) - 1 + \delta(1-2\zeta)]\,|a_n| \leq 2\delta(1-\zeta). \tag{2.2}$$

The result (2.2) is sharp.

3. Necessary and sufficient conditions

Firstly, we obtain the necessary and sufficient conditions for $\Phi^m_{\alpha,\beta}$ to be in the class $\mathcal{S}^*(\zeta,\delta)$.

Theorem 3.1. *If $\alpha, m > 0$ and $\beta > 1$, then $\Phi^m_{\alpha,\beta}(z) \in \mathcal{S}^*(\zeta,\delta)$ if and only if*

$$\frac{\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)} \left[\frac{1+\delta}{\alpha} \left(E_{\alpha,\beta-1}(m) - \frac{1}{\Gamma(\beta-1)} \right) + \left(\frac{(1+\delta)(1-\beta)}{\alpha} + 2\delta(1-\zeta) \right) \left(E_{\alpha,\beta}(m) - \frac{1}{\Gamma(\beta)} \right) \right] \leq 2\delta(1-\zeta). \tag{3.1}$$

Proof. Since

$$\Phi^m_{\alpha,\beta}(z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{\Gamma(\alpha(n-1)+\beta)E_{\alpha,\beta}(m)} z^n,$$

by virtue of Lemma 2.1 and (2.1) it suffices to show that

$$\sum_{n=2}^{\infty}[n(1+\delta) - 1 + \delta(1-2\zeta)] \frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)} \leq 2\delta(1-\zeta).$$

Now,

$$\begin{aligned}
& \sum_{n=2}^{\infty}[n(1+\delta) - 1 + \delta(1-2\zeta)] \frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)} \\
= & \sum_{n=1}^{\infty}[(n+1)(1+\delta) - 1 + \delta(1-2\zeta)] \frac{\Gamma(\beta)m^{n}}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)} \\
= & \sum_{n=1}^{\infty}[n(1+\delta) + 2\delta(1-\zeta)] \frac{\Gamma(\beta)m^{n}}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)} \\
= & \left[\frac{1+\delta}{\alpha}\sum_{n=1}^{\infty}[(\alpha n+\beta-1)+(1-\beta)] \frac{\Gamma(\beta)m^{n}}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)} + 2\delta(1-\alpha)\sum_{n=1}^{\infty}\frac{\Gamma(\beta)m^{n}}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)}\right] \\
= & \frac{\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\left[\frac{1+\delta}{\alpha}\sum_{n=1}^{\infty}\frac{m^n}{\Gamma(\alpha n+\beta-1)} + \left(\frac{(1+\delta)(1-\beta)}{\alpha} + 2\delta(1-\zeta)\right)\sum_{n=1}^{\infty}\frac{m^n}{\Gamma(\alpha n+\beta)}\right] \\
= & \frac{\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\left[\frac{1+\delta}{\alpha}\left(E_{\alpha,\beta-1}(m) - \frac{1}{\Gamma(\beta-1)}\right) + \left(\frac{(1+\delta)(1-\beta)}{\alpha} + 2\delta(1-\zeta)\right)\left(E_{\alpha,\beta}(m) - \frac{1}{\Gamma(\beta)}\right)\right] \\
\leq & 2\delta(1-\zeta),
\end{aligned}$$

by the given hypothesis (3.1). This complete the proof of Theorem 3.1. □

Theorem 3.2. *If* $\alpha, m > 0$ *and* $\beta > 2$, *then* $\Phi^m_{\alpha,\beta}(z) \in C^*(\zeta,\delta)$ *if and only if*

$$\begin{aligned}
& \frac{\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\left[\frac{1+\delta}{\alpha^2}\left(E_{\alpha,\beta-2}(m) - \frac{1}{\Gamma(\beta-2)}\right) + \left(\frac{(1+\delta)(3-2\beta)+\alpha(1+\delta(3-2\zeta))}{\alpha^2}\right)\left(E_{\alpha,\beta-1}(m) - \frac{1}{\Gamma(\beta-1)}\right)\right. \\
& \left. + \left(\frac{(1+\delta)(1-\beta)^2}{\alpha^2} + \frac{(1+\delta(3-2\zeta))(1-\beta)}{\alpha} + 2\delta(1-\zeta)\right)\left(E_{\alpha,\beta}(m) - \frac{1}{\Gamma(\beta)}\right)\right] \\
\leq & 2\delta(1-\zeta). \qquad (3.2)
\end{aligned}$$

Proof. In view of Lemma 2.2 and (2.2) it suffices to show that

$$\sum_{n=2}^{\infty} n[n(1+\delta) - 1 + \delta(1-2\zeta)] \frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)} \leq 2\delta(1-\zeta).$$

Now,

$$
\begin{aligned}
&\sum_{n=2}^{\infty} n[n(1+\delta)-1+\delta(1-2\zeta)]\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)}\\
=\;& \sum_{n=1}^{\infty}(n+1)[(n+1)(1+\delta)-1+\delta(1-2\zeta)]\frac{\Gamma(\beta)m^{n}}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)}\\
=\;& \sum_{n=1}^{\infty}(1+\delta)n^2+(1+\delta(3-2\zeta))\,n+2\delta(1-\zeta)]\frac{\Gamma(\beta)m^{n}}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)}\\
=\;& \Bigg[\frac{1+\delta}{\alpha^2}\sum_{n=1}^{\infty}\Big[(\alpha n+\beta-1)(\alpha n+\beta-2)+(3-2\beta)(\alpha n+\beta-1)+(1-\beta)^2\Big]\frac{\Gamma(\beta)m^{n}}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)}\\
&+\frac{1+\delta(3-2\zeta)}{\alpha}\sum_{n=1}^{\infty}[(\alpha n+\beta-1)+(1-\beta)]\frac{\Gamma(\beta)m^{n}}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)}+2\delta(1-\zeta)\sum_{n=1}^{\infty}\frac{\Gamma(\beta)m^{n}}{\Gamma(\alpha n+\beta)\mathbb{E}_{\alpha,\beta}(m)}\Bigg]\\
=\;& \frac{\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\Bigg[\frac{1+\delta}{\alpha^2}\sum_{n=1}^{\infty}\frac{m^n}{\Gamma(\alpha n+\beta-2)}+\left(\frac{(1+\delta)(3-2\beta)+\alpha(1+\delta(3-2\zeta))}{\alpha^2}\right)\sum_{n=1}^{\infty}\frac{m^n}{\Gamma(\alpha n+\beta-1)}\\
&+\left(\frac{(1+\delta)(1-\beta)^2}{\alpha^2}+\frac{(1+\delta(3-2\zeta))(1-\beta)}{\alpha}+2\delta(1-\zeta)\right)\sum_{n=1}^{\infty}\frac{m^n}{\Gamma(\alpha n+\beta)}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
=\;& \frac{\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\Bigg[\frac{1+\delta}{\alpha^2}\left(E_{\alpha,\beta-2}(m)-\frac{1}{\Gamma(\beta-2)}\right)+\left(\frac{(1+\delta)(3-2\beta)+\alpha(1+\delta(3-2\zeta))}{\alpha^2}\right)\left(E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right)\\
&+\left(\frac{(1+\delta)(1-\beta)^2}{\alpha^2}+\frac{(1+\delta(3-2\zeta))(1-\beta)}{\alpha}+2\delta(1-\zeta)\right)\left(E_{\alpha,\beta}(m)-\frac{1}{\Gamma(\beta)}\right)\Bigg]\\
\le\;& 2\delta(1-\zeta),
\end{aligned}
$$

by the given hypothesis (3.2). This complete the proof of Theorem 3.2. □

4. Inclusion Properties

Making use of Lemma 4.1 (below) we will study the action of the Mittag–Leffler-type Poisson distribution series on the class $C^*(\zeta,\delta)$.

Lemma 4.1. *[9] If $f\in\mathcal{R}^\tau(A,B)$ is of the form (1.1) , then*

$$|a_n|\le (A-B)\frac{|\tau|}{n},\qquad n\in\mathbb{N}-\{1\}.$$

The result is sharp for the function

$$f(z)=\int_0^z(1+(A-B)\frac{\tau t^{n-1}}{1+Bt^{n-1}})dt,\qquad (z\in\mathbb{U};n\in\mathbb{N}-\{1\}).$$

Theorem 4.2. *Let $\alpha,m>0$ and $\beta>1$. If $f\in\mathcal{R}^\tau(A,B)$ and the inequality*

$$\frac{(A-B)\Gamma(\beta)\,|\tau|}{\mathbb{E}_{\alpha,\beta}(m)}\left[\frac{1+\delta}{\alpha}\left(E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right)+\left(\frac{(1+\delta)(1-\beta)}{\alpha}+2\delta(1-\zeta)\right)\left(E_{\alpha,\beta}(m)-\frac{1}{\Gamma(\beta)}\right)\right]\le 2\delta(1-\zeta)\quad (4.1)$$

is satisfied then $\mathcal{I}^m_{\alpha,\beta}f\in C^*(\zeta,\delta)$.

Proof. In view of Lemma 2.2 and (2.2) it suffices to show that

$$\sum_{n=2}^{\infty} n[n(1+\delta)-1+\delta(1-2\zeta)]\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)}\,|a_n| \leq 2\delta(1-\zeta).$$

Since $f \in \mathcal{R}^{\tau}(A,B)$, using Lemma 4.1 we have

$$|a_n| \leq \frac{(A-B)\,|\tau|}{n},\ n \in \mathbb{N}\setminus\{1\},$$

therefore, it is enough to show that

$$(A-B)\,|\tau|\left[\sum_{n=2}^{\infty}[n(1+\delta)-1+\delta(1-2\zeta)]\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)}\right] \leq 2\delta(1-\zeta).$$

By a similar proof like those of Theorem 3.1 we get that $\mathcal{I}^m_{\alpha,\beta}f \in C^*(\zeta,\delta)$ if (4.1) holds. □

5. An integral operator

Theorem 5.1. *If $\alpha, m > 0$ and $\beta > 1$, then the integral operator*

$$\mathcal{G}^m_{\alpha,\beta}(z) := \int_0^z \frac{\Phi^m_{\alpha,\beta}(t)}{t}dt,\ z \in \mathbb{U}, \tag{5.1}$$

is in the class $C^*(\zeta,\delta)$ if and only if the inequality (3.1) holds.

Proof. According to (1.6) it follows that

$$\mathcal{G}^m_{\alpha,\beta}(z) = z - \sum_{n=2}^{\infty}\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)}\frac{z^n}{n}.$$

Using Lemma 2.2 and (2.2), the integral operator $\mathcal{G}^m_{\alpha,\beta}(z)$ belongs to $C^*(\zeta,\delta)$ if and only if

$$\sum_{n=2}^{\infty}[n(1+\delta)-1+\delta(1-2\zeta)]\frac{\Gamma(\beta)m^{n-1}}{\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)} \leq 2\delta(1-\zeta).$$

By a similar proof like those of Theorem 3.1 we get that $\mathcal{G}^m_{\alpha,\beta} \in C^*(\zeta,\delta)$ if and only if (3.1) holds. □

6. Corollaries and consequences

By specializing the parameter $\delta = 1$ in Theorems 3.1-5.1, we obtain the following results.

Corollary 6.1. *If $\alpha, m > 0$ and $\beta > 1$, then $\Phi^m_{\alpha,\beta}(z) \in \mathcal{S}^*(\zeta)$ if and only if*

$$\frac{\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\left[\frac{1}{\alpha}\left(E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right)+\left(\frac{1-\beta}{\alpha}+2(1-\zeta)\right)\left(E_{\alpha,\beta}(m)-\frac{1}{\Gamma(\beta)}\right)\right] \leq 2(1-\zeta). \tag{6.1}$$

Corollary 6.2. *If $\alpha, m > 0$ and $\beta > 2$, then $\Phi^m_{\alpha,\beta}(z) \in C^*(\zeta)$ if and only if*

$$\begin{aligned} &= \frac{\Gamma(\beta)}{\mathbb{E}_{\alpha,\beta}(m)}\left[\frac{1}{\alpha^2}\left(E_{\alpha,\beta-2}(m)-\frac{1}{\Gamma(\beta-2)}\right)\left(\frac{(3-2\beta)+\alpha(2-\zeta))}{\alpha^2}\right)\left(E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right)\right. \\ &\quad \left.+\left(\frac{(1-\beta)^2}{\alpha^2}+\frac{(2-\zeta)(1-\beta)}{\alpha}+(1-\zeta)\right)\left(E_{\alpha,\beta}(m)-\frac{1}{\Gamma(\beta)}\right)\right] \\ &\leq 2(1-\zeta). \end{aligned} \tag{6.2}$$

Corollary 6.3. *Let $\alpha, m > 0$ and $\beta > 1$. If $f \in \mathcal{R}^{\tau}(A, B)$ and the inequality*

$$= \frac{(A-B)\Gamma(\beta)\,|\tau|}{\mathbb{E}_{\alpha,\beta}(m)}\left[\frac{1}{\alpha}\left(E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right)+\left(\frac{1-\beta}{\alpha}+(1-\zeta)\right)\left(E_{\alpha,\beta}(m)-\frac{1}{\Gamma(\beta)}\right)\right] \leq 2(1-\zeta) \qquad (6.3)$$

is satisfied then $\mathcal{I}^{m}_{\alpha,\beta}f \in C^{*}(\zeta)$.

Corollary 6.4. *If $\alpha, m > 0$ and $\beta > 1$, then the integral operator given by (5.1) is in the class $C^{*}(\zeta)$ if and only if the inequality (6.1) holds.*

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