Montes Taurus J. Pure Appl. Math. 3 (3), 122–134, 2021

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

On Function Spaces with Fractional Wavelet Transform

Muhammed Duman [a], Öznur Kulak [b]

[a] Giresun University, Faculty of Sciences and Letters, Department of Mathematics, Giresun, Turkey
[b] Amasya University, Faculty of Sciences and Letters, Department of Mathematics, Amasya, Turkey

Abstract

Let ω_1 and ω_2 be weight functions on $\mathbb{R}$. In this paper, we define $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ to be the vector space of $f \in L^p_{\omega_1}(\mathbb{R})$ such that the fractional wavelet transform $W^\theta_\psi f$ belongs to $L^q_{\omega_2}(\mathbb{R})$ for $1 \leq p,q < \infty$. We endow this space with a sum norm and show that $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ becomes a Banach space. Also we prove that $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is an essential Banach Module over $L^1_{\omega_1}(\mathbb{R})$ under some conditions. We obtain its approximate identities, dual space and multipliers space. At the end of this paper we discuss the inclusion properties, compact embeddings of these spaces.

Keywords: Fractional wavelet transform, Essential Banach module, Approximate identity, Compact embedding, Multipliers space

2010 MSC: 43A15, 43A22, 43A32

1. Introduction

Throughout this paper $C_c(\mathbb{R})$ and $S(\mathbb{R})$ denote the space of complex-valued continuous functions on $\mathbb{R}$ with compact support and the space of complex-valued continuous functions on $\mathbb{R}$ rapidly decreasing at infinity, respectively. For any function $f : \mathbb{R} \longrightarrow \mathbb{C}$, the translation and dilation operators T_b and D_a are given by $T_b f(t) = f(t-b)$ and $D_a f(t) = \frac{1}{\sqrt{a}} f\left(\frac{t}{a}\right)$ for all $a \in \mathbb{R}^+$, $b \in \mathbb{R}$ respectively. The parameters in wavelet theory are "time" b and "scale" a. Dilation operator D_a preserves the shape of f, but it changes the scale, [2]. The set $L^p(\mathbb{R}), (1 \leq p < \infty)$ denotes the usual Lebesgue space. Let ω be a weight function on $\mathbb{R}$, that is, a measurable, locally bounded and positive real valued function ω satisfying $\omega(x) \geq 1$, $\omega(x+y) \leq \omega(x)\omega(y)$ for all $x, y \in \mathbb{R}$. Let $s \geq 0$. A weight $\omega(x) = (1+|x|)^s$ which is defined on $\mathbb{R}$ is called weight of polynomial type. It is written that the inequality $\omega(xt) \leq \omega(x)\omega(t)$ for $x, t \in \mathbb{R}$, (see, [9]). For $1 \leq p < \infty$, the weighted Lebesgue space is defined by $L^p_\omega(\mathbb{R}) = \{f : f\omega \in L^p(\mathbb{R})\}$. It is well known that $L^p_\omega(\mathbb{R})$ is a Banach space under the norm $\|f\|_{p,\omega} = \|f\omega\|_p$, [13]. Let ω_1 and ω_2 are two weight functions. We write $\omega_1 \prec \omega_2$ if there exists $C > 0$ such that $\omega_1(x) \leq C\omega_2(x)$ for all $x \in \mathbb{R}$. Two weight function ω_1 and ω_2 are called equivalent and write $\omega_1 \approx \omega_2$ if and only if $\omega_1 \prec \omega_2$ and $\omega_2 \prec \omega_1$, [6].

Given any fixed $0 \neq \psi \in L^2(\mathbb{R})$ (called the wavelet function), the classical wavelet transform of a function $f \in L^2(\mathbb{R})$ with respect to ψ is defined by

$$W_\psi f(b,a) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt = \langle f, T_b D_a \psi \rangle$$

†*Article ID:* MTJPAM-D-20-00045

Email addresses: duman_muhammed@hotmail.com (Muhammed Duman), oznur.kulak@amasya.edu.tr (Öznur Kulak)
Received:5 December 2020, Accepted:1 January 2021, Published:25 April 2021
⋆*Corresponding Author:* Öznur Kulak

for $a \in \mathbb{R}^+$, $b \in \mathbb{R}$. We can write the wavelet transform as convolution $W_\psi f(b,a) = f * D_a\psi^*(b)$, where $\psi^*(t) = \overline{\psi(-t)}$. Also the wavelet transform of a function $f \in L^p(\mathbb{R})$ with respect to $0 \neq g \in L^1(\mathbb{R})$ is defined similarly, [2].

Now let's give the definition of fractional wavelet transform, which is a generalization of classical wavelet transform and is very important for time-frequency analysis. The fractional wavelet transform with an angle θ of $f \in L^2(\mathbb{R})$ is defined by

$$W_\psi^\theta f(b,a) = \int_{\mathbb{R}} f(t)\,\overline{\psi_{b,a,\theta}(t)}\,dt,$$

where $\psi_{b,a,\theta}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)e^{-\frac{i}{2}\left(t^2-b^2\right)}\cot\theta$ for $a \in \mathbb{R}^+$, $b \in \mathbb{R}$. Since $\psi_{b,a,\frac{\pi}{2}} = T_bD_a\psi$, the fractional wavelet transform with $\theta = \frac{\pi}{2}$ corresponds to the classical wavelet transform. The fractional convolution of $\psi, \varphi \in L^2(\mathbb{R})$ is defined as

$$(\psi *_\theta \varphi)(t) = \int_{\mathbb{R}} \psi(y)\,\varphi(t-y)\,e^{\frac{-i}{2}\left(t^2-y^2\right)\cot\theta}dy.$$

If the angle $\theta = \frac{\pi}{2}$, then the fractional convolution is equal to the usual convolution. Also we can write the fractional wavelet transform as convolution $W_\psi^\theta f(b,a) = (f *_\theta \psi)(b) = e^{\frac{-i}{2}b^2\cot\theta}\left(e^{\frac{i}{2}(.)^2\cot\theta}f * \overline{\psi_a^*}\right)(b)$, [11, 12]. This transformation, which is similar to fractional Fourier transform (see, [1], [15]), is one of the time-frequency operators that can provide the time-frequency information of a signal in the successful way.

For the wavelet function $\psi \in L^2(\mathbb{R})$, if the following condition

$$C_{\psi,\theta} = \int_{\mathbb{R}} \left|\hat{f}^\theta\left(e^{\frac{-i}{2}(.)^2\cot\theta}\psi\right)(t)\right|^2 \frac{dt}{t} < \infty$$

holds, then it is say that ψ satisfies the admissibility condition, where $\hat{f}^\theta$ denotes the fractional Fourier transform. If the wavelet function $\psi \in L^2(\mathbb{R})$ satisfies the admissibility condition, then for all $f, g \in L^2(\mathbb{R})$

$$\int_0^\infty \int_{\mathbb{R}} W_\psi^\theta f(b,a)\,\overline{W_\psi^\theta g(b,a)}\frac{dbda}{a^2} = 2\pi\sin\theta C_{\psi,\theta}\langle f, g\rangle$$

holds and f is reconstructed from its fractional wavelet transform by

$$f(t) = \frac{1}{2\pi\sin\theta C_{\psi,\theta}}\int_{\mathbb{R}}\int_0^\infty W_\psi^\theta f(b,a)\,\psi_{b,a,\theta}(t)\,\frac{dadb}{a^2},$$

[11, 12]. In this paper we will assume that the wavelet function ψ satisfies the admissibility condition.

2. Main Results

2.1. The Space $\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R})$

Definition 2.1. Let ω_1, ω_2 be weight functions on $\mathbb{R}$ and $1 \leq p, q < \infty$. Assume that $0 \neq \psi \in S(\mathbb{R})$. The space $\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R})$ consists of all $f \in L_{\omega_1}^p(\mathbb{R})$ such that their fractional wavelet transforms $W_\psi^\theta f$ are in $L_{\omega_2}^q(\mathbb{R})$, where the scale a is fixed. We endow this space with the sum norm

$$\|f\|_{\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a} = \|f\|_{p,\omega_1} + \left\|W_\psi^\theta f\right\|_{q,\omega_2}.$$

Theorem 2.2. *$\left(\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R}), \|.\|_{\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a}\right)$ is a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Then $(f_n)_{n\in\mathbb{N}}$ and $\left(W^{\theta}_{\psi}f_n\right)_{n\in\mathbb{N}}$ are Cauchy sequences in $L^p_{\omega_1}(\mathbb{R})$ and $L^q_{\omega_2}(\mathbb{R})$, respectively. Since $L^p_{\omega_1}(\mathbb{R})$ and $L^q_{\omega_2}(\mathbb{R})$ are Banach spaces, there exist $f \in L^p_{\omega_1}(\mathbb{R})$ and $g \in L^q_{\omega_2}(\mathbb{R})$ such that $\|f_n - f\|_{p,\omega_1} \longrightarrow 0$, $\left\|W^{\theta}_{\psi}f_n - g\right\|_{q,\omega_2} \longrightarrow 0$. That means $\left\|W^{\theta}_{\psi}f_n - g\right\|_{q} \longrightarrow 0$. Then $\left(W^{\theta}_{\psi}f_n\right)_{n\in\mathbb{N}}$ has a subsequence $\left(W^{\theta}_{\psi}f_{n_k}\right)_{n_k\in\mathbb{N}}$ that converges pointwise to g almost everywhere. It is easy to show that $\left\|f_{n_k} - f\right\|_p \longrightarrow 0$. Also we have

$$\begin{aligned}\left|W^{\theta}_{\psi}f(b,a) - g(b)\right| &\leq \left|W^{\theta}_{\psi}(f_{n_k} - f)(b,a)\right| + \left|W^{\theta}_{\psi}f_{n_k}(b,a) - g(b)\right| \\ &\leq \frac{1}{\sqrt{a}}\int_{\mathbb{R}}\left|(f_{n_k} - f(t))\,\bar{\psi}\left(\frac{t-b}{a}\right)e^{\frac{i}{2}(t^2-b^2)\cot\theta}\right|dt + \left|W^{\theta}_{\psi}f_{n_k}(b,a) - g(b)\right| \\ &= \frac{1}{\sqrt{a}}\int_{\mathbb{R}}\left|(f_{n_k} - f)(t)\right|\left|T_b\psi_a(t)\right|dt + \left|W^{\theta}_{\psi}f_{n_k}(b,a) - g(b)\right|\end{aligned}$$

for all $b \in \mathbb{R}$. By this inequality and Hölder's inequality, we find

$$\left|W^{\theta}_{\psi}f(b,a) - g(b)\right| \leq a^{\frac{1}{p'}-\frac{1}{2}}\left\|f_{n_k} - f\right\|_p \|\psi\|_{p'} + \left|W^{\theta}_{\psi}f_{n_k}(b) - g(b)\right|$$

for all $b \in \mathbb{R}$. So using last inequality, it is easily seen that $W^{\theta}_{\psi}f = g$ almost everywhere. Since the equivalence classes of $W^{\theta}_{\psi}f$ and g are equal, then $\|f_n - f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} \longrightarrow 0$ and $f \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Hence $\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}), \|.\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}\right)$ is a Banach space. □

Theorem 2.3. ***a)*** *$\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is invariant under translations.*

b) *The mapping $f \longrightarrow T_u f$ is continuous from $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ into $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ for every fixed $u \in \mathbb{R}$.*

Proof. **a)** Take any $f \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Then we have $f \in L^p_{\omega_1}(\mathbb{R})$ and $W^{\theta}_{\psi}f \in L^q_{\omega_2}(\mathbb{R})$. Also since

$$\|T_u f\|_{p,\omega_1} \leq \omega_1(u)\|f\|_{p,\omega_1}$$

[7], we write that $T_u f \in L^p_{\omega_1}(\mathbb{R})$ for all $u \in \mathbb{R}$. On the other hand, it is easy to see that

$$W^{\theta}_{\psi}(T_u f)(.,a) = T_u W^{\theta}_{\psi}f(.,a)$$

for fixed $a \in \mathbb{R}$. Then we have $\left\|W^{\theta}_{\psi}(T_u f)\right\|_{q,\omega_2} \leq \omega_2(u)\left\|W^{\theta}_{\psi}f\right\|_{q,\omega_2}$. Therefore we achieve

$$\|T_u f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} \leq \tilde{\omega}(u)\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a},$$

where $\tilde{\omega}(u) = \max\{\omega_1(u), \omega_2(u)\}$. Finally we say $(T_u f) \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$.

b) Let $f \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ be given. Since $f \longrightarrow T_u f$ is linear, we will prove the theorem for $f = 0$. For any given $\varepsilon > 0$, choose $\delta > 0$ to be $\delta = \frac{\varepsilon}{\tilde{\omega}(u)}$. If $\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} < \delta$, then

$$\|T_u f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} \leq \tilde{\omega}(u)\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} \leq \tilde{\omega}(u)\delta = \varepsilon.$$

□

Theorem 2.4. *Let ω_2 be a weight function of polynomial type. Then*

a) *$C_c(\mathbb{R})$ is dense in $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$.*

b) *$\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is dense in $L^p_{\omega_1}(\mathbb{R})$.*

Proof. **a)** Take any $f \in C_c(\mathbb{R})$. Since $C_c(\mathbb{R}) \subset L^p_{\omega_1}(\mathbb{R})$, we have $f \in L^p_{\omega_1}(\mathbb{R})$. Also since ω_2 is a weight function of polynomial type, we write $\omega_2(st) \leq \omega_2(s)\omega_2(t)$. So if we say that $\frac{-t}{a} = u$, then

$$\begin{aligned} \left\|\psi_a^*\right\|_{1,\omega_2} &= \int_{\mathbb{R}} \left|\psi_a^*(t)\right| \omega_2(t)\,dt = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \left|\psi\left(\frac{-t}{a}\right)\right| \omega_2(t)\,dt \\ &= \frac{1}{\sqrt{a}} \int_{\mathbb{R}} |\psi(u)|\,\omega_2(-au)\,du = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} |\psi(u)|\,\omega_2(au)\,du \\ &\leq \frac{1}{\sqrt{a}} \int_{\mathbb{R}} |\psi(u)|\,\omega_2(a)\,\omega_2(u)\,du = \frac{1}{\sqrt{a}}\omega_2(a)\|\psi\|_{1,\omega_2} < \infty. \end{aligned} \tag{2.1}$$

Also by [7], $L^q_{\omega_2}(\mathbb{R})$ is a Banach convolution module over $L^1_{\omega_2}(\mathbb{R})$. Thus if we use the equality $W^\theta_\psi f(b,a) = e^{\frac{-i}{2}b^2\cot\theta}\left(e^{\frac{i}{2}(.)^2\cot\theta} f * \overline{\psi_a^*}\right)(b)$ in [11] and the inequality (2.1), we have

$$\begin{aligned} \left\|W^\theta_\psi f\right\|_{q,\omega_2} &= \left\|e^{\frac{-i}{2}b^2\cot\theta}\left(e^{\frac{i}{2}(.)^2\cot\theta} f * \overline{\psi_a^*}\right)\right\|_{q,\omega_2} = \left\|e^{\frac{i}{2}(.)^2\cot\theta} f * \overline{\psi_a^*}\right\|_{q,\omega_2} \\ &\leq \left\|e^{\frac{-i}{2}(.)^2\cot\theta} f\right\|_{q,\omega_2} \left\|\overline{\psi_a^*}\right\|_{1,\omega_2} \\ &\leq \frac{1}{\sqrt{a}}\omega_2(a)\|f\|_{q,\omega_2}\|\psi\|_{1,\omega_2} < \infty. \end{aligned} \tag{2.2}$$

Then by (2.2), $W^\theta_\psi f \in L^q_{\omega_2}(\mathbb{R})$ is written. Therefore we find $C_c(\mathbb{R}) \subset \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$.

Now take any $f \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Then we have $f \in L^p_{\omega_1}(\mathbb{R})$ and $W^\theta_\psi f \in L^q_{\omega_2}(\mathbb{R})$. Since $\overline{C_c}(\mathbb{R}) = L^p_{\omega_1}(\mathbb{R})$, there exists $(g_n)_{n\in\mathbb{N}} \subset C_c(\mathbb{R})$ such that

$$\|f - g_n\|_{p,\omega_1} \to 0. \tag{2.3}$$

Also again since $\overline{C_c(\mathbb{R})} = L^q_{\omega_2}(\mathbb{R})$, there exists $(h_n)_{n\in\mathbb{N}} \subset C_c(\mathbb{R})$ such that

$$\left\|W^\theta_\psi f - h_n\right\|_{q,\omega_2} \to 0. \tag{2.4}$$

This implies $\left\|W^\theta_\psi f - h_n\right\|_q \to 0$. So $(h_n)_{n\in\mathbb{N}}$ has a subsequence $(h_{n_k})_{n_k\in\mathbb{N}}$ which converges pointwise to $W^\theta_\psi f$ almost everywhere. From (2.4), it is easy to show that

$$\left\|W^\theta_\psi f - h_{n_k}\right\|_{q,\omega_2} \to 0. \tag{2.5}$$

Also by Hölder's inequality, we have for all $b \in \mathbb{R}$

$$\begin{aligned} \left|h_{n_k}(b) - W^\theta_\psi g_n(b,a)\right| &\leq \left|h_{n_k}(b) - W^\theta_\psi f(b,a)\right| + \left|W^\theta_\psi f(b,a) - W^\theta_\psi g_n(b,a)\right| \\ &\leq \left|W^\theta_\psi f(b,a) - h_{n_k}(b)\right| + \frac{1}{\sqrt{a}} \int_{\mathbb{R}} |(f - g_n)(t)|\,|T_a\psi_a(t)|\,dt \\ &\leq \left|W^\theta_\psi f(b,a) - h_{n_k}(b)\right| + \frac{1}{\sqrt{a}}\|g_n - f\|_p \|T_a\psi_a(t)\|_{p'} \\ &\leq \left|W^\theta_\psi f(b,a) - h_{n_k}(b)\right| + \|g_n - f\|_p\, a^{\frac{1}{p'}-\frac{1}{2}}\|\psi\|_{p'} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. By last inequality, it is easily seen $h_{n_k} = W^\theta_\psi g_n$. Then by (2.5), we have

$$\left\|W^\theta_\psi f - W^\theta_\psi g_n\right\|_{q,\omega_2} \to 0. \tag{2.6}$$

Therefore combining (2.3) and (2.6) , we obtain

$$\begin{aligned} \|f - g_n\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} &= \|f - g_n\|_{p,\omega_1} + \left\|W^\theta_\psi(f - g_n)\right\|_{q,\omega_2} \\ &= \|f - g_n\|_{p,\omega_1} + \left\|W^\theta_\psi f - W^\theta_\psi g_n\right\|_{q,\omega_2} \to 0. \end{aligned}$$

That means $\overline{C_c(\mathbb{R})} = \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$.

b) Take any $f \in C_c(\mathbb{R})$. By (a), we know that $C_c(\mathbb{R}) \subset \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Also since $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}) \subset L^p_{\omega_1}(\mathbb{R})$, we write the following inclusion

$$C_c(\mathbb{R}) \subset \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}) \subset L^p_{\omega_1}(\mathbb{R}). \tag{2.7}$$

Therefore since $\overline{C_c(\mathbb{R})} = L^p_{\omega_1}(\mathbb{R})$ and by (2.7), we obtain $\overline{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})} = L^p_{\omega_1}(\mathbb{R})$. □

Theorem 2.5. *The translation mapping $k \to T_k f$ is continuous from $\mathbb{R}$ into $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$.*

Proof. Let $f \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ be given. It is known that the translation mapping is continuous from $\mathbb{R}$ into $L^p_{\omega_1}(\mathbb{R})$, [7]. So for any given $\varepsilon > 0$, there exists $\delta_1(\varepsilon) > 0$ such that if $\|k-u\| < \delta_1$ for $k,u \in \mathbb{R}$, then

$$\|T_k f - T_u f\|_{p,\omega_1} < \frac{\varepsilon}{2}. \tag{2.8}$$

Also again since the translation mapping is continuous from $\mathbb{R}$ into $L^q_{\omega_2}(\mathbb{R})$, so for the some $\varepsilon > 0$, there exists $\delta_2(\varepsilon) > 0$ such that if $\|k-u\| < \delta_2$ for all $k,u \in \mathbb{R}$, then

$$\left\|W_\psi(T_k f - T_u f)\right\|_{q,w_2} = \left\|T_k W_\psi f - T_u W_\psi f\right\|_{q,w_2} < \frac{\varepsilon}{2}. \tag{2.9}$$

We set $\delta = \min\{\delta_1, \delta_2\}$. By (2.8) and (2.9), if $\|k-u\| < \delta$ for all $k,u \in \mathbb{R}$, then

$$\|T_k f - T_u f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} = \|T_k f - T_u f\|_{p,\omega_1} + \left\|W_\psi(T_k f - T_u f)\right\|_{q,\omega_2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore the proof is completed. □

Theorem 2.6. *$\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is a Banach function space.*

Proof. Let $f \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ be given. We take a compact subset $A \subset \mathbb{R}$. Since $\mu(A) < \infty$ and $p \geq 1$, there exists $B > 0$ such that

$$\int_A |f(x)|\,dx \leq B\,\|f\|_{p,\omega_1}. \tag{2.10}$$

Using the inequality (2.10), we obtain

$$\int_A |f(x)|\,dx \leq B\left\{\|f\|_{p,\omega_1} + \left\|W^\theta_\psi f\right\|_{q,\omega_2}\right\} = B\,\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}.$$

Also since $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is a Banach space, we obtain that $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is a Banach function space. □

Lemma 2.7. *Let $f,g \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Then the equality*

$$W^\theta_\psi(f*g)(u,a) = \left(g * W^\theta_\psi f\right)(u,a)$$

is satisfied for all $u \in \mathbb{R}$ and fixed $a \in \mathbb{R}$.

Proof. Take any $f,g \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ and $u \in \mathbb{R}$. We have

$$\begin{aligned} W^\theta_\psi(f*g)(u,a) &= ((f*g)*_\theta \psi_a)(u) = \int_{\mathbb{R}} (f*g)(\xi)\,\psi_a(u-\xi)\,e^{-\frac{i}{2}(u^2-\xi^2)\cot\theta}d\xi \\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(z)\,g(\xi-z)\,dz\right)\psi_a(u-\xi)\,e^{-\frac{i}{2}(u^2-\xi^2)\cot\theta}d\xi \end{aligned}$$

for fixed $a \in \mathbb{R}$. If we set $\xi - z = y$, then by Fubini Theorem

$$\begin{aligned} W_\psi^\theta (f * g)(u,a) &= \int_{\mathbb{R}} g(y) \left(\int_{\mathbb{R}} T_y f(\xi) \psi_a (u-\xi) e^{-\frac{i}{2}(u^2-\xi^2)\cot\theta} d\xi \right) dy \\ &= \int_{\mathbb{R}} g(y) \left(T_y f *_\theta \psi_a \right)(u)\, dy \\ &= \int_{\mathbb{R}} g(y) W_\psi^\theta T_y f(u,a)\, dy \end{aligned} \tag{2.11}$$

is written. So using the equality (2.11) and the equality $W_\psi^\theta T_y f(u,a) = T_y W_\psi^\theta f(u,a)$, we obtain

$$W_\psi^\theta (f * g)(u,a) = \int_{\mathbb{R}} g(y) T_y W_\psi^\theta f(u,a)\, dy = \int_{\mathbb{R}} g(y) W_\psi^\theta f(u-y,a)\, dy = \left(g * W_\psi^\theta f \right)(u,a).$$

□

Theorem 2.8. *Assume that $\omega_2 \prec \omega_1$. Then $\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R})$ is an essential Banach module over $L_{\omega_1}^1(\mathbb{R})$.*

Proof. Let $f \in \left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R})$ and $h \in \left(L_{\omega_1}^1\right)(\mathbb{R})$ be given. Since $L_{\omega_1}^p(\mathbb{R})$ is a Banach convolution module over $L_{\omega_1}^1(\mathbb{R})$, we can say

$$\|f * h\|_{p,\omega_1} \le \|f\|_{p,\omega_1} \|h\|_{1,\omega_1}. \tag{2.12}$$

Then by Lemma 2.7 and assumption $\omega_2 \prec \omega_1$, we have

$$\begin{aligned} \left\| W_\psi^\theta (f * g) \right\|_{q,\omega_2} &= \left\| h * W_\psi^\theta f \right\|_{q,\omega_2} \le \|h\|_{1,\omega_2} \left\| W_\psi^\theta f \right\|_{q,\omega_2} \\ &\le \|h\|_{1,\omega_1} \left\| W_\psi^\theta f \right\|_{q,\omega_2} < \infty. \end{aligned} \tag{2.13}$$

So $W_\psi^\theta f \in L_{w_2}^q(\mathbb{R})$ is obtained. Using the inequalities (2.12) and (2.13), we achieve

$$\begin{aligned} \|f * h\|_{\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a} &= \|f * h\|_{p,\omega_1} + \left\| W_\psi^\theta (f * h) \right\|_{q,\omega_2} \\ &\le \|h\|_{1,\omega_1} \left(\|f\|_{p,\omega_1} + \left\| W_\psi^\theta f \right\|_{q,\omega_2} \right) \\ &= \|h\|_{1,\omega_1} \|f\|_{\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a}. \end{aligned}$$

Thus we say that $\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R})$ is Banach module over $L_{\omega_1}^1(\mathbb{R})$.

Now we will show that $\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R})$ is an essential Banach module over $L_{\omega_1}^1(\mathbb{R})$. Since $\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R})$ is a Banach module over $L_{\omega_1}^1(\mathbb{R})$, there exists the inclusion

$$L_{\omega_1}^1(\mathbb{R}) * \left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R}) \subset \left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R}).$$

For the proof, it suffies to prove that $L_{\omega_1}^1(\mathbb{R}) * \left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R})$ is dense in $\left(FW_{\omega_1,\omega_2}^{\theta,p,q}\right)_a(\mathbb{R})$. We know that $L_{\omega_1}^1(\mathbb{R})$ has a bounded approximate identity [8]. We suppose that L is a compact neighbourhood of $0 \in \mathbb{R}$. Then we can choose an approximate identity $(b_\alpha)_{\alpha\in I}$ that is positive bounded and satisfies $\sup pb_\alpha \subset L$, $\|b_\alpha\|_1 = 1$ for all $\alpha \in I$. On the other hand, we fix $\alpha_0 \in I$ and take any $g \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Then we find

$$\begin{aligned}\left\|b_{\alpha_0} * g - g\right\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} &= \left\|\int_{\mathbb{R}} b_{\alpha_0}(u)\, T_u g(y)\, du - \int_{\mathbb{R}} b_{\alpha_0}(u)\, g(y)\, du\right\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} \\ &= \left\|\int_{\mathbb{R}} b_{\alpha_0}(u)\left(T_u g(y) - g(y)\right) du\right\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} \\ &\le \int_{\mathbb{R}} b_{\alpha_0}(u)\, \|(T_u g - g)\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}\, du.\end{aligned}$$

By Theorem 2.5, we know that the translation mapping $u \to T_u f$ is continuous from $\mathbb{R}$ into $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. So for given $\varepsilon > 0$, we can write

$$\|T_u g - g\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} \le \varepsilon \left\|b_{\alpha_0}\right\|_1 = \varepsilon.$$

In other words, we say that $L^1_{\omega_1}(\mathbb{R}) * \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is dense in $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Therefore by the Module Factorization Theorem in [16], we obtain

$$L^1_{\omega_1}(\mathbb{R}) * \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}) \subset \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}).$$

Hence the proof is completed. □

Corollary 2.9. *Suppose that $(b_\alpha)_{\alpha\in I}$ is an approximate identity in $L^1_{\omega_1}(\mathbb{R})$ and $\omega_2 \prec \omega_1$. Then $(b_\alpha)_{\alpha\in I}$ is an appoximate identity of $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$.*

Proof. Let $(b_\alpha)_{\alpha\in I}$ be an approximate identity in $L^1_{\omega_1}(\mathbb{R})$ and let $\omega_2 \prec \omega_1$. Assume that L is a compact neighbourhood of the unit element of $\mathbb{R}$. Also we suppose that this approximate identity $(b_\alpha)_{\alpha\in I}$ is positive bounded and satisfies $\sup p b_\alpha \subset L$, $\|b_\alpha\|_1 = 1$ for all $\alpha \in I$. Then for all $\alpha \in I$, we have

$$\begin{aligned}\|b_\alpha\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} &= \|b_\alpha\|_{p,\omega_1} + \left\|W^\theta_\psi(b_\alpha)\right\|_{q,\omega_2} \\ &= \|b_\alpha\|_{p,\omega_1} + \left\|e^{-\frac{i}{2}b^2\cot\theta}\left(e^{\frac{i}{2}(.)^2\cot\theta} b_\alpha * \overline{\psi^*_a}\right)\right\|_{q,\omega_2} \\ &\le \|b_\alpha\|_{p,\omega_1} + \left\|e^{\frac{i}{2}(.)^2\cot\theta} b_\alpha\right\|_{1,\omega_2}\left\|\overline{\psi^*_a}\right\|_{q,\omega_2} \\ &= \|b_\alpha\|_{p,\omega_1} + \|b_\alpha\|_{1,\omega_2}\left\|\psi^*_a\right\|_{q,\omega_2} < \infty.\end{aligned}$$

Then we achieve $(b_\alpha)_{\alpha\in I} \subset \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Also since $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is an essential Banach module over $L^1_{w_1}(\mathbb{R})$, we obtain $\lim_{\alpha\to 0} b_\alpha * f = f$, [3]. □

2.2. Dual and Multipliers Spaces of $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$

Definition 2.10. Take the mapping H from $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ into $L^p_{\omega_1}(\mathbb{R}) \times L^q_{\omega_2}(\mathbb{R})$ defined by $H(f) = \left(f, W_\psi f\right)$. H is a linear isometry of $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ into $L^p_{\omega_1}(\mathbb{R}) \times L^q_{\omega_2}(\mathbb{R})$ with the norm

$$\|H(f)\| = \left\|\left(f, W_\psi f\right)\right\| = \|f\|_{p,\omega_1} + \left\|W_\psi f\right\|_{q,\omega_2}$$

for all $f \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Assume that $A = H\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})\right)$. We define the set M to be

$$M = \left\{\begin{array}{c}(\lambda,\beta) \in L^{p'}_{\omega_1^{-1}}(\mathbb{R}) \times L^{q'}_{\omega_2^{-1}}(\mathbb{R}) \mid \int_{\mathbb{R}} f(y)\,\lambda(y)\,dy + \int_{\mathbb{R}} W^\theta_\psi f(x,\omega)\beta(x,\omega)\,dx = 0, \\ \forall\left(f, W^\theta_\psi f\right) \in A\end{array}\right\}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Theorem 2.11. *Let* $\frac{1}{p}+\frac{1}{p'}=1$ *and* $\frac{1}{q}+\frac{1}{q'}=1$. *The dual space* $\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})\right)^*$ *is isomorphic to* $\left(L^{p'}_{\omega_1^{-1}}(\mathbb{R})\times L^{q'}_{\omega_2^{-1}}(\mathbb{R})\right)/M$.

Proof. Since the mapping H is surjective isometry, the space $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is homeomorphic to $H\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})\right)=A$. Also the space $A=H\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})\right)$ is Banach space. So the space A is closed. Then by Duality Theorem in [10], we have

$$A^*\cong\left(L^{p'}_{\omega_1^{-1}}(\mathbb{R})\times L^{q'}_{\omega_2^{-1}}(\mathbb{R})\right)/M.$$

If we use the fact that H is an isometry, then

$$\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})\right)^*\cong A^*.$$

Therefore we achieve

$$\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})\right)^*\cong\left(L^{p'}_{\omega_1^{-1}}(\mathbb{R})\times L^{q'}_{\omega_2^{-1}}(\mathbb{R})\right)/M.$$

□

Theorem 2.12. *Let* $\omega_2\prec\omega_1$. *Then the spaces* $Hom_{L^1_{\omega_1}}\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}),L^\infty_{\omega_1^{-1}}(\mathbb{R})\right)$ *and* $\left(L^{p'}_{\omega_1^{-1}}(\mathbb{R})\times L^{q'}_{\omega_2^{-1}}(\mathbb{R})\right)/M$ *are algebrically isomarphic and topologically homeomorphic.*

Proof. From Theorem 2.8, the space $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is an essential Banach module over $L^1_{\omega_1}(\mathbb{R})$. Then we can write

$$L^1_{\omega_1}(\mathbb{R})\times\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})=\left(F^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}).$$

So by Theorem 2.11 and 1.4 Theorem in [14], we obtain

$$Hom_{L^1_{\omega_1}}\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}),L^\infty_{\omega_1^{-1}}(\mathbb{R})\right)=Hom_{L^1_{\omega_1}}\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}),L^1_{\omega_1}(\mathbb{R})^*\right)$$
$$\cong\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})*L^1_{\omega_1}(\mathbb{R})\right)^*=\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})\right)^*\cong\left(L^{p'}_{\omega_1^{-1}}(\mathbb{R})\times L^{q'}_{\omega_2^{-1}}(\mathbb{R})\right)/M$$

□

Definition 2.13. Let $\omega_2\prec\omega_1$ and let $(b_\alpha)_{\alpha\in I}$ be a bounded approximate identiy in $L^1_{\omega_1}(\mathbb{R})$. The relative completion $\left(\widetilde{FW}^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ of $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is defined by

$$\left(\widetilde{FW}^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})=\left\{\begin{array}{c} f\in L^1_{\omega_1}(\mathbb{R})\mid f*b_\alpha\in\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})\ \text{for all}\ \alpha\in I\\ \text{and}\ \sup\|f*b_\alpha\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}<\infty\end{array}\right\}.$$

The space $\left(\widetilde{FW}^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is a Banach space with the norm

$$\|f\|_{\left(\widetilde{FW}^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}=\sup_{\alpha\in I}\|f*b_\alpha\|_{\left(F^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}$$

for all $f\in\left(\widetilde{FW}^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. This space doesn't depend on the approximate identity [4].

The following theorem is proved easily by Theorem 2.6 in [4].

Theorem 2.14. *Assume that* $\omega_2\prec\omega_1$. *Then the spaces* $M\left(L^1_{\omega_1}(\mathbb{R}),\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})\right)$ *and* $\left(\widetilde{FW}^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ *are algebrically isomorphic and topologically homeomorphic.*

2.3. *Inclusion Properies of* $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$

Theorem 2.15. *Let ω_1, ω_2, ω_3 and ω_4 be weight functions. If $\widetilde{\omega} = \max\{\omega_1,\omega_3\}$ and $\widetilde{\mu} = \max\{\omega_2,\omega_4\}$, then the equality*

$$\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}) \cap \left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R}) = \left(FW^{\theta,p,q}_{\widetilde{\omega},\widetilde{\mu}}\right)_a(\mathbb{R})$$

holds.

Proof. Let $f \in \left(FW^{\theta,p,q}_{\widetilde{\omega},\widetilde{\mu}}\right)_a(\mathbb{R})$ be given. Then

$$\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} = \|f\|_{p,\omega_1} + \left\|W^\theta_\psi f\right\|_{q,\omega_2} \le \|f\|_{p,\widetilde{\omega}} + \left\|W^\theta_\psi f\right\|_{q,\widetilde{\mu}} < \infty.$$

So we have $f \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Similarly we find $f \in \left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R})$. Hence we achive

$$\left(FW^{\theta,p,q}_{\widetilde{\omega},\widetilde{\mu}}\right)_a(\mathbb{R}) \subset \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}) \cap \left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R}).$$

Now, conversely we take any $f \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}) \cap \left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R})$. Since $\widetilde{\omega} = \max\{\omega_1,\omega_3\}$ and $\widetilde{\mu} = \max\{\omega_2,\omega_4\}$, it is clear that

$$\|f\|_{\left(FW^{\theta,p,q}_{\widetilde{\omega},\widetilde{\mu}}\right)_a(\mathbb{R})} = \|f\|_{p,\widetilde{\omega}} + \left\|W^\theta_\psi f\right\|_{q,\widetilde{\mu}} < \infty.$$

So we have $f \in \left(FW^{\theta,p,q}_{\widetilde{\omega},\widetilde{\mu}}\right)_a(\mathbb{R})$. Then we can write the inclusion

$$\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}) \cap \left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R}) \subset \left(FW^{\theta,p,q}_{\widetilde{\omega},\widetilde{\mu}}\right)_a(\mathbb{R}).$$

Therefore we obtain

$$\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}) \cap \left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R}) = \left(FW^{\theta,p,q}_{\widetilde{\omega},\widetilde{\mu}}\right)_a(\mathbb{R}).$$

□

Theorem 2.16. *Assume that $\omega_i\,(i = 1,2,3,4)$ are weight functions. If $\omega_3 \prec \omega_1$ and $\omega_4 \prec \omega_2$, then the inclusion*

$$\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}) \subset \left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R})$$

holds.

Proof. Let $\omega_3 \prec \omega_1$ and $\omega_4 \prec \omega_2$. Then there exist $A_1, A_2 > 0$ such that $\omega_3(b) \le A_1\omega_1(b)$ and $\omega_4(b) \le A_2\omega_2(b)$ for all $b \in \mathbb{R}$. Assume that $f \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Then we write $f \in L^p_{\omega_1}(\mathbb{R})$ and $W^\theta_\psi f \in L^q_{\omega_2}(\mathbb{R})$. So we have

$$\|f\|_{p,\omega_3} \le A_1\|f\|_{p,\omega_1} < \infty$$

and

$$\left\|W^\theta_\psi f\right\|_{q,\omega_4} \le A_2\left\|W^\theta_\psi f\right\|_{q,\omega_2} < \infty.$$

Therefore we obtain $f \in \left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R})$. Finaly we can write

$$\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}) \subset \left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R}).$$

□

Lemma 2.17. *Assume that $\omega_i\,(i = 1,2,3,4)$ are weight functions. If $\left(FW^{\theta,p,q}_{\omega_1,\omega_3}\right)_a(\mathbb{R}) \subset \left(FW^{\theta,p,q}_{\omega_2,\omega_4}\right)_a(\mathbb{R})$, then $\left(FW^{\theta,p,q}_{\omega_1,\omega_3}\right)_a(\mathbb{R})$ is a Banach space under the norm*

$$\|f\|_{FW} = \|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_3}\right)_a} + \|f\|_{\left(FW^{\theta,p,q}_{\omega_2,\omega_4}\right)_a}.$$

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_3}\right)_a(\mathbb{R}), \|.\|_{FW}\right)$. Then $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_3}\right)_a(\mathbb{R}), \|.\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_3}\right)_a}\right)$ and $\left(\left(FW^{\theta,p,q}_{\omega_2,\omega_4}\right)_a(\mathbb{R}), \|.\|_{\left(FW^{\theta,p,q}_{\omega_2,\omega_4}\right)_a(\mathbb{R})}\right)$. Since these spaces are Banach spaces, there exist $f\in\left(FW^{\theta,p,q}_{\omega_1,\omega_3}\right)_a(\mathbb{R})$ and $g\in\left(FW^{\theta,p,q}_{\omega_2,\omega_4}\right)_a(\mathbb{R})$ such that

$$\|f_n - f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_3}\right)_a(\mathbb{R})}\to 0$$

and

$$\|f_n - g\|_{\left(FW^{\theta,p,q}_{\omega_2,\omega_4}\right)_a(\mathbb{R})}\to 0.$$

On the other hand since $\|.\|_p\le\|.\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_3}\right)_a(\mathbb{R})}$ and $\|.\|_p\le\|.\|_{\left(FW^{\theta,p,q}_{\omega_2,\omega_4}\right)_a(\mathbb{R})}$, we find $\|f_n-f\|_p\to 0$ and $\|f_n-g\|_p\to 0$. So we have

$$0\le\|f-g\|_p\le\|f_n-f\|_p+\|f_n-g\|_p\to 0.$$

That means $\|f-g\|_p=0$, and so $f=g$. Therefore $\|f_n-f\|_{FW}\to 0$ and $f\in\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_3}\right)_a(\mathbb{R}), \|.\|_{FW}\right)$. This completes the proof. □

Theorem 2.18. *Suppose that $\omega_i\ (i=1,2,3,4)$ are weight functions. If $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})\subset\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R})$, then there exists an $A>0$ such that*

$$\|f\|_{\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a}\le A\,\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}$$

for all $f\in\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$.

Proof. Let endow the space $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ with the norm

$$\|.\|_{FW}=\|.\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}+\|.\|_{\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a}.$$

From Lemma 2.17, The space $\left(\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}), \|.\|_{FW}\right)$ is Banach space. Then using the closed graph theorem, there exists $A>0$ such that

$$\|f\|_{\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R})}\le A\,\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})}$$

for all $f\in\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. □

Theorem 2.19. *For every $0\ne f\in\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ there exists $A(f)>0$ such that*

$$A(f)\,\omega_1(u)\le\|T_uf\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}\le(\omega_1(u)+\omega_2(u))\,\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}.$$

Proof. Take $0\ne f\in\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Then there exists $A(f)>0$ such that

$$A(f)\,\omega_1(u)\le\|T_uf\|_{p,\omega_1}\le\omega_1(u)\,\|f\|_{p,\omega_1},$$

[7]. Then we have

$$\begin{aligned}
A(f)\,\omega_1(u) &\le \|T_uf\|_{p,\omega_1}\le\|T_uf\|_{p,\omega_1}+\left\|W^\theta_\psi(T_uf)\right\|_{q,\omega_2}\\
&\le \omega_1(u)\,\|f\|_{p,\omega_1}+\left\|T_uW^\theta_\psi f\right\|_{q,\omega_2}\\
&\le \omega_1(u)\,\|f\|_{p,\omega_1}+\omega_2(u)\left\|W^\theta_\psi f\right\|_{q,\omega_2}\\
&\le \omega_1(u)\,\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}+\omega_2(u)\,\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}\\
&= (\omega_1(u)+\omega_2(u))\,\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}
\end{aligned}$$

for all $f\in\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. □

Corollary 2.20. *Let $\omega_i\,(i=1,2)$ be weight functions. For every of $f\in\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$, there exists $k(f)>0$ such that*

$$k(f)\,\omega_1(u)\leq\|T_uf\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}\leq\widetilde{\omega}(u)\,\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a},$$

where $\widetilde{\omega}=\max\{\omega_1,\omega_2\}$.

Theorem 2.21. *Let $\omega_i\,(i=1,2,3,4)$ be weight functions. If $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})\subset\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R})$, then $\omega_3\prec\omega_1+\omega_2$.*

Proof. Let $f\in\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ be given . Then by assumption, we have $f\in\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R})$. From Theorem 2.19, there exist $c_1,c_2,c_3,c_4>0$ such that

$$c_1\omega_1(u)\leq\|T_uf\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}\leq(\omega_1(u)+\omega_2(u))\,c_2\tag{2.14}$$

and

$$c_3\omega_3(u)\leq\|T_uf\|_{\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a}\leq(\omega_3(u)+\omega_4(u))\,c_4\tag{2.15}$$

Also by Lemma 2.17, the space $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is a Banach space under the norm $\|f\|_{FW}=\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}+\|f\|_{\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a}$. Then from the closed graph theorem, there exists $c>0$ such that

$$\|f\|_{\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a}\leq c\,\|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}\tag{2.16}$$

for all $f\in\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. Also since the space $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ is invariant under translations, we have

$$T_uf\in\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})\subset\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R}).$$

So by (2.16) , we write

$$\|T_uf\|_{\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a}\leq c\,\|T_uf\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}.\tag{2.17}$$

Thus combining (2.14), (2.15) and (2.17), we obtain

$$c_3\omega_3(u)\leq\|T_uf\|_{\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a}\leq c\,\|T_uf\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}\leq cc_2\,(\omega_1(u)+\omega_2(u)).$$

Hence, $\omega_3(u)\leq\frac{cc_2}{c_3}(\omega_1(u)+\omega_2(u))$. Then we have $\omega_3(u)\leq k(\omega_1(u)+\omega_2(u))$ where $k=\frac{cc_2}{c_3}$. That means $\omega_3\prec\omega_1+\omega_2$. □

Corollary 2.22. *Assume that $\omega_3\approx k_1$, $\omega_4\approx k_2$, where k_1,k_2 are constant numbers. Then $\left(FW^{\theta,p,q}_{\omega_1,\omega_3}\right)_a(\mathbb{R})\subset\left(FW^{\theta,p,q}_{\omega_2,\omega_4}\right)_a(\mathbb{R})$ if and only if $\omega_2\prec\omega_1$*

Proof. By Theorem 2.16 and Theorem 2.21, the proof is made easly. □

2.4. *Compact Embeddings of the space* $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$

Lemma 2.23. *Assume that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. If $(f_n)_{n\in\mathbb{N}}$ converges to zero in $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$, then $\int_{\mathbb{R}}f_n(x)\,k(x)\,dx\longrightarrow0$ as $n\to\infty$ for all $k\in C_c(\mathbb{R})$.*

Proof. Let $k\in C_c(\mathbb{R})$ be given and let $\frac{1}{p}+\frac{1}{p'}=1$. Then we can write

$$\begin{aligned}\left|\int_{\mathbb{R}}f_n(x)\,k(x)\,dx\right| &\leq \|k\|_{p'}\,\|f_n\|_p\\ &\leq \|k\|_{p'}\,\|f_n\|_{p,\omega_1}\leq\|k\|_{p'}\,\|f_n\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}.\end{aligned}$$

So by last inequality and assumption, we achieve $\int_{\mathbb{R}}f_n(x)\,k(x)\,dx\longrightarrow0$ as $n\to\infty$ for all $k\in C_c(\mathbb{R})$. □

Theorem 2.24. *Let ω_i $(i=1,2)$ be weight functions of polynomial type on $\mathbb{R}$ and let ϑ be a weight function on $\mathbb{R}$. If $\vartheta \prec \omega_1$ and $\frac{\vartheta(u)}{\omega_1(u)+\omega_2(u)} \nrightarrow 0$ for every fixed a and for $u \to \infty$, then the embedding of the space $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ into $L^p_\vartheta(\mathbb{R})$ is never compact.*

Proof. By assumption $\vartheta \prec \omega_1$, there exists $c_1 > 0$ such that $\vartheta(u) \le c_1\omega_1(u)$ for all $u \in \mathbb{R}$. We can write

$$\|f\|_{p,\vartheta} \le c_1 \|f\|_{p,\omega_1} \le c_1 \left(\|f\|_{p_1,\omega_1} + \left\|W^\theta_\psi f\right\|_{q,\omega_2}\right) = c_1 \|f\|_{\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a}.$$

That means $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}) \subset L^p_\vartheta(\mathbb{R})$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence with $t_n \to \infty$ as $n \to \infty$ in $\mathbb{R}$. Since $\frac{\vartheta(u)}{\omega_1(u)+\omega_2(u)}$ does not tend to zero as $u \to \infty$, then there exists $\delta > 0$ such that $\frac{\vartheta(u)}{\omega_1(u)+\omega_2(u)} \ge \delta > 0$ for $u \to \infty$. Take any fixed $f \in \left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. We define a sequence $(f_n)_{n\in\mathbb{N}}$ by

$$f_n = (\omega_1(t_n) + \omega_2(t_n))^{-1} T_{t_n} f.$$

Also $(f_n)_{n\in\mathbb{N}}$ is bounded in $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$. We may write

$$\begin{aligned} \|f_n\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} &= \left\|(\omega_1(t_n) + \omega_2(t_n))^{-1} T_{t_n} f\right\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} \\ &= (\omega_1(t_n) + \omega_2(t_n))^{-1} \left\|T_{t_n} f\right\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}. \end{aligned}$$

So by last inequality and Theorem 2.19, we obtain

$$\|f_n\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} \le (\omega_1(t_n) + \omega_2(t_n))^{-1} (\omega_1(t_n) + \omega_2(t_n)) \|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a} = \|f\|_{\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a}.$$

Then, we have

$$\begin{aligned} \left|\int_{\mathbb{R}} f_n(u)\, k(u)\, du\right| &\le \frac{1}{(\omega_1(t_n) + \omega_2(t_n))} \int_{\mathbb{R}} \left|(T_{t_n} f)(u)\, k(u)\right| du \\ &\le \frac{1}{(\omega_1(t_n) + \omega_2(t_n))} \left\|T_{t_n} f\right\|_p \|k\|_{p'} \\ &= \frac{1}{(\omega_1(t_n) + \omega_2(t_n))} \|f\|_p \|k\|_{p'}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ for all $k \in C_c(\mathbb{R})$. Since the right hand side of last inequality tends to zero as $n \to \infty$, then we have $\int_{\mathbb{R}} f_n(x)\, k(x)\, dx \longrightarrow 0$. Hence by Lemma 2.23, the only possible limit of $(f_n)_{n\in\mathbb{N}}$ in $L^p_\vartheta(\mathbb{R})$ is zero. On the other hand, since there exist $c_1, c_2 > 0$ such that

$$c_1\vartheta(t_n) \le \left\|T_{t_n} f\right\|_{p,\vartheta} \le c_2\vartheta(t_n),$$

then we achieve

$$\|f_n\|_{p,\vartheta} = (\omega_1(t_n) + \omega_2(t_n))^{-1} \left\|T_{t_n} f\right\|_{p,\vartheta} \ge c_1 (\omega_1(t_n) + \omega_2(t_n))^{-1} \vartheta(t_n).$$

Using the fact that $\frac{\vartheta(t_n)}{\omega_1(t_n)+\omega_2(t_n)} \ge \delta > 0$ for all t_n and by the last inequality, we obtain

$$\|f_n\|_{p,\vartheta} \ge c_1 (\omega_1(t_n) + \omega_2(t_n))^{-1} \vartheta(t_n) \ge \delta c_1 > 0.$$

Therefore, we say that there can not exist a norm convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ in $L^p_\vartheta(\mathbb{R})$. This implies that the embedding of the space $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ into $L^p_\vartheta(\mathbb{R})$ is never compact. □

Corollary 2.25. *Assume that ω_i $(i=1,2)$ are weight functions of polynomial type on $\mathbb{R}$. Also, assume that ω_i $(i=3,4)$ are any weight functions on $\mathbb{R}$. If $\omega_3 \prec \omega_1$, $\omega_4 \prec \omega_2$ and $\frac{\omega_3(u)}{\omega_1(u)+\omega_2(u)} \nrightarrow 0$ as $u \to \infty$, then the embedding of the space $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ into $\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R})$ is never compact.*

Proof. Since the assumptions $\omega_3 < \omega_1$ and $\omega_4 < \omega_2$, by Theorem 2.16 we can write $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R}) \subset \left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R})$. Suppose that the unit map is compact. Let a bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\left(FW^{\theta,p,q}_{\omega_1,\omega_2}\right)_a(\mathbb{R})$ be given. If there exists a convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ in $\left(FW^{\theta,p,q}_{\omega_3,\omega_4}\right)_a(\mathbb{R})$, this sequence also converges in $L^p_{\omega_3}(\mathbb{R})$. But this is not possible by Theorem 2.24. So the proof is completed. □

Acknowledgements

This paper is dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th Birthday.

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