Montes Taurus J. Pure Appl. Math. 3 (3), 39–94, 2021

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Characterizations of Generalized Topologically Open Sets in Relator Spaces

Themistocles M. Rassias[a], Muwafaq M. Salih[b], Árpád Száz[c]

[a]*Department of Mathematics, National Technical University of Athens, Athens, Greece*
[b]*Department of Mathematics, University of Debrecen, Debrecen, Hungary*
[c]*Department of Mathematics, University of Debrecen, Debrecen, Hungary*

Abstract

A family $\mathcal{R}$ of binary relations on a set X will be called a relator on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ will be called a relator space.

Each generalized topology $\mathcal{T}$ on X can be easily derived from the family $\mathcal{R}_{\mathcal{T}}$ of all Pervin's preorder relations $R_V = V^2 \cup V^c \times X$ with $V \in \mathcal{T}$.

For a subset A of the relator space $X(\mathcal{R})$, we may briefly define

$$A^- = \mathrm{cl}_{\mathcal{R}}(A) = \bigcap \{ R^{-1}[A] : \ R \in \mathcal{R} \},$$

$A^\circ = \mathrm{int}_{\mathcal{R}}(A) = \mathrm{cl}_{\mathcal{R}}(A^c)^c$ and $A^\dagger = \mathrm{res}_{\mathcal{R}}(A) = \mathrm{cl}_{\mathcal{R}}(A) \setminus A$.

Now, we may also naturally define $\mathcal{T}_{\mathcal{R}} = \{ A \subseteq X : \ A \subseteq A^\circ \}$,

$$\mathcal{D}_{\mathcal{R}} = \{ A \subseteq X : \ A^- = X \} \quad \text{and} \quad \mathcal{N}_{\mathcal{R}} = \{ A \subseteq X : \ A^{-\circ} = \emptyset \}.$$

Moreover, following some basic definitions in topological spaces, a subset A of the relator space $X(\mathcal{R})$ may, for instance, be naturally called topologically

(1) regular open if $A = A^{-\circ}$;
(2) preopen if $A \subseteq A^{-\circ}$; (3) semi-open if $A \subseteq A^{\circ -}$;
(4) α–open if $A \subseteq A^{\circ - \circ}$; (5) β–open if $A \subseteq A^{- \circ -}$;
(6) quasi-open if there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A \subseteq V^-$;
(7) pseudo-open if there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq A^-$.

And, the family of all such subsets A of $X(\mathcal{R})$ may, for instance, be naturally denoted by $\mathcal{T}_{\mathcal{R}}^{\kappa}$ with $\kappa = r,\ p,\ s,\ \alpha,\ \beta,\ q$ and ps, respectively.

Here, we shall mainly be interested in the relationships and characterizations of the families $\mathcal{T}_{\mathcal{R}}^{\kappa}$. For instance, we shall prove the following assertions:

(1) If $\mathcal{R}$ is topological, then $\mathcal{T}_{\mathcal{R}}^{s} = \mathcal{T}_{\mathcal{R}}^{q}$. Moreover, $A \in \mathcal{T}_{\mathcal{R}}^{s}$ if and only if $A = V \cup B$ for some $V \in \mathcal{T}_{\mathcal{R}}$ and $B \subseteq V^\dagger$.
(2) If $\mathcal{R}$ is topological, then $\mathcal{T}_{\mathcal{R}}^{p} = \mathcal{T}_{\mathcal{R}}^{ps}$. Moreover, if in addition $\mathcal{R}$ is topologically filtered, then $A \in \mathcal{T}_{\mathcal{R}}^{p}$ if and only if $A = V \cap B$ for some $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{D}_{\mathcal{R}}$;
(3) If $\mathcal{R}$ is topological, then $\mathcal{T}_{\mathcal{R}}^{\alpha} = \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}$. Moreover, if in addition $\mathcal{R}$ is topologically filtered, then $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$ if and only if $A = V \setminus B$ for some $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$.

Set-theoretic properties and some proximal and paratopological counterparts of the families $\mathcal{T}_{\mathcal{R}}^{\kappa}$ will be investigated in some subsequent papers.

Keywords: Generalized uniformities, Interiors and closures, Generalized open sets, Characterization theorems

2010 MSC: 54E15, 54A05

†*Article ID:* MTJPAM-D-20-00046

Email addresses: trassias@math.ntua.gr (Themistocles M. Rassias), muwafaq.salih@science.unideb.hu (Muwafaq M. Salih), szaz@science.unideb.hu (Árpád Száz)

Received:7 December 2020, Accepted:2 January 2021, Published:25 April 2021

⋆*Corresponding Author:* Árpád Száz

1. Introduction

If $\mathcal{T}$ is a family of subsets of a set X such that $\mathcal{T}$ is closed under finite intersections and arbitrary unions, then the family $\mathcal{T}$ is called a *topology* on X, and the ordered pair $X(\mathcal{T}) = (X, \mathcal{T})$ is called a *topological space*.

The members of $\mathcal{T}$ are called the *open subsets* of X. While, the members of $\mathcal{F} = \{A^c : A \in \mathcal{T}\}$, where $A^c = X \setminus A$, are called the *closed subsets* of X. And, the members of $\mathcal{T} \cap \mathcal{F}$ are called the *clopen subsets* of X.

Since $\emptyset = \bigcup \emptyset$ and $X = \bigcap \emptyset$, we necessary have $\{\emptyset, X\} \subseteq \mathcal{T} \cap \mathcal{F}$. If in particular $\mathcal{T} = \{\emptyset, X\}$, then $\mathcal{T}$ is called *minimal (indiscrete)*. While, if $\mathcal{T} \cap \mathcal{F} = \{\emptyset, X\}$, then $\mathcal{T}$ is called *connected* [174, p. 31].

For a subset A of $X(\mathcal{T})$, the sets $A^\circ = \operatorname{int}(A) = \bigcup \mathcal{T} \cap \mathcal{P}(A)$,

$$A^- = \operatorname{cl}(A) = \operatorname{int}(A^c)^c \quad \text{and} \quad A^\dagger = \operatorname{res}(A) = \operatorname{cl}(A) \setminus A$$

are called the *interior, closure and residue of* A, respectively.

Thus, $-$ is a *Kuratowski closure operation* on $\mathcal{P}(X)$. That is, $\emptyset^- = \emptyset$, and $-$ is *extensive, idempotent and additive* in the sense that, for any A, $B \subseteq X$, we have $A \subseteq A^-$, $A^{--} = A^-$ and $(A \cup B)^- = A^- \cup B^-$.

In particular, the members of the families

$$\mathcal{D} = \{A \subseteq X : \ A^- = X\} \quad \text{and} \quad \mathcal{N} = \{A \subseteq X : \ A^{-\circ} = \emptyset\}$$

are called the *dense and rare (nowhere dense) subsets* of $X(\mathcal{T})$, respectively.

In 1922, a subset A of a closure space $X(-)$ was called *regular open* by Kuratowski [87] if $A = A^{-\circ}$. With suitable operations, the family of regular open subsets forms a complete *Boolean agebra* [64, p. 66].

In 1982, a subset A of $X(\mathcal{T})$ was called *preopen* by Mashhour et al. [102] if $A \subseteq A^{-\circ}$. However, by Dontchev [50], preopen sets, under various names, were much earlier studied by several mathematicians.

In 1964, Corson and Michael [26] called a subset A of $X(\mathcal{T})$ *locally dense* if it is a dense subset of some $V \in \mathcal{T}$, i. e., $A \subseteq V \subseteq A^-$. Moreover, they noted that this property is equivalent to the inclusion $A \subseteq A^{-\circ}$.

This equivalence was later also stated by Jun at al. [79]. Moreover, Ganster [59] proved that A is preopen if and only if there exist $V \in \mathcal{T}$ and $B \in \mathcal{N}$ such that $A = V \cap B$. (See also Dontchev [50].)

In 1963, a subset A of $X(\mathcal{T})$ was called *semi-open* by Levine [94] if there exists $V \in \mathcal{T}$ such that $V \subseteq A \subseteq V^-$. First of all, he showed that the set A is semi-open if and only if $A \subseteq A^{\circ -}$.

Moreover, he also proved that if A is a semi-open subset of $X(\mathcal{T})$, then there exist $V \in \mathcal{T}$ and $B \in \mathcal{N}$ such that $A = V \cup B$ and $V \cap B = \emptyset$. In addition, he also noted that the converse statement is false.

Levine's statement closely resembles to a famous theorem of Hyers [73] which says that an ε–approximately additive function of one Banach space to another is the sum of an additive function and an ε–small function.

Analogously to the paper of Hyers, Levine's paper has also attracted the interest of a surprisingly great number of mathematicians. For instance, by the Google Scholar, it has been cited by 2901 works.

Moreover, the above statement of Levine was improved by Dlaska et al. [47] who observed that a subset A of $X(\mathcal{T})$ is semi-open if and only if there exist $V \in \mathcal{T}$ and $B \subseteq \operatorname{res}(V)$ such that $A = V \cup B$.

The latter observation was later reformulated, in a more convenient form, by Duszyńki and Noiri [51] who noted that a subset A of $X(\mathcal{T})$ is semi-open if and only if there exists $B \subseteq \operatorname{res}(A^\circ)$ such that $A = A^\circ \cup B$.

In particular, in 1965 and 1971, Njåstad [110] and Isomichi [75], being not aware of the paper of Levine, studied semi-open sets under the names *β-sets* and *subcondensed sets*, respectively.

Moreover, Njåstad called a subset A of $X(\mathcal{T})$ to be an *α–set* if $A \subseteq A^{\circ-\circ}$. And, he proved that the set A is an α–set if and only if there exist $V \in \mathcal{T}$ and $B \in \mathcal{N}$ such that $A = V \setminus B$.

In 1983, the subset A was called *β–open* by Abd El-Monsef et al. [1] if $A \subseteq A^{-\circ-}$. While, in 1986, Andrijević [8] used the term *semi-preopen* instead of β–open.

Actually, Andrijević called a subset A of $X(\mathcal{T})$ to be semi-preopen if there exists a preopen subset V of $X(\mathcal{T})$ such that $V \subseteq A \subseteq V^-$. And, he showed that this is equivalent to the inclusion $A \subseteq A^{-\circ-}$.

In 1996, a subset A of $X(\mathcal{T})$ was called *b–open* by Andrijević [12] if $A \subseteq A^{\circ-} \cup A^{-\circ}$. He proved that A is b–open if and only if there exist a preopen subset B and a semi-open subset C of $X(\mathcal{T})$ such that $A = B \cup C$.

In the present paper, we shall show that the above definitions and theorems on generalized open sets can be naturally extended not only to generalized topological and closure spaces, but also to relator spaces.

Relator spaces, introduced by the third author in [145, 154], are common generalizations not only of topological, closure and proximity spaces, but also those of ordered sets, context spaces and uniform spaces.

The necessary prerequisites on relators, which are certainly unfamiliar to the reader, will be briefly laid out in the subsequent preparatory sections which will also contain several new observations.

These sections may also be useful for all those readers who are not very much interested in the various generalizations of open sets having been studied recently by a great number of topologists.

2. A Few Basic Facts on Relations

A subset F of a product set $X{\times}Y$ is called a *relation on X to Y*. In particular, a relation on X to itself is called a *relation on X*. And, $\Delta_X = \{(x, x) : \ x \in X\}$ is called the *identity relation of X*.

If F is a relation on X to Y, then by the above definitions we can also state that F is a relation on $X \cup Y$. However, for several purposes, the latter view of the relation F would be quite unnatural.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{ y \in Y : \ (x, y) \in F\}$ and $F[A] = \bigcup\{F(x) : \ x \in A\}$ are called the *images or neighbourhoods* of x and A under F, respectively.

If $(x, y) \in F$, then instead of $y \in F(x)$ we may also write $x\,F\,y$. However, instead of $F[A]$ we cannot write $F(A)$. Namely, it may occur that, in addition to $A \subseteq X$, we also have $A \in X$.

Now, the sets $D_F = \{x \in X : \ F(x) \neq \emptyset\}$ and $R_F = F[X]$ may be called the *domain* and *range* of F, respectively. If in particular $D_F = X$, then we may say that F is a *relation of X to Y*, or that F is a *non-partial relation on X to Y*.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ instead of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation on X*. While, a function $*$ of X^2 to X is called a *binary operation on X*. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*((x, y))$, respectively.

If F is a relation on X to Y, then a function f of D_F to Y is called a *selection function* of F if $f(x) \in F(x)$ for all $x \in D_F$. By using the Axiom of Choice, it can be shown that every relation is the union of its selection functions.

For a relation F on X to Y, we may naturally define two *set-valued functions* φ_F of X to $\mathcal{P}(Y)$ and Φ_F of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $\varphi_F(x) = F(x)$ for all $x \in X$ and $\Phi_F(A) = F[A]$ for all $A \subseteq X$.

Functions of X to $\mathcal{P}(Y)$ can be identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more general objects than relations on X to Y. In [164, 170, 171], they were briefly called *corelations* on X to Y.

However, if R is a relation on X to Y, U is a relation on $\mathcal{P}(X)$ to Y, and V is a relation on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then it is better to say that R is an *ordinary relation*, U is a *super relation*, and V is a *hyper relation* on X to Y.

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\}{\times}F(x)$. Therefore, the images $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *complement* F^c and the *inverse* F^{-1} can be defined such that $F^c(x) = F(x)^c = Y \setminus F(x)$ for all $x \in X$ and $F^{-1}(y) = \{x \in X : \ y \in F(x)\}$ for all $y \in Y$. Thus, we also have $F^c = X{\times}Y \setminus F$.

Moreover, if in addition G is a relation on Y to Z, then the *composition* $G \circ F$ can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subseteq X$.

While, if G is a relation on Z to W, then the *box product* $F \boxtimes G$ can be defined such that $(F \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$. Thus, we have $(F \boxtimes G)[A] = G \circ A \circ F^{-1}$ for all $A \subseteq X{\times}Z$ [162].

Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for any family of relations.

Now, a relation R on X may be briefly defined to be *reflexive* on X if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly defined to be *symmetric* if $R^{-1} \subseteq R$, and *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation*.

For any relation R on X, we may naturally define $R^0 = \Delta_X$ and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may naturally define $R^\infty = \bigcup_{n=0}^\infty R^n$. Thus, R^∞ is the smallest preorder relation on X containing R [65].

For $A \subseteq X$, *Pervin's relation* $R_A = A^2 \cup A^c{\times}X$, with $A^2 = A{\times}A$, is an important preorder on X. While, for a *pseudometric* d on X, *Weil's surrounding* $B_r = \{(x, y) \in X^2 : \ d(x, y) < r\}$, with $r > 0$, is an important tolerance on X. (See [176], [82, pp. 174–175] and [121], [98].)

Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cap (A^c)^2$ is already an equivalence relation on X. And, more generally if $\mathcal{A}$ is a *cover (partition)* of X, then $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is a tolerance (equivalence) relation on X.

As an important generalization of the Pervin relation R_A, for any $A \subseteq X$ and $B \subseteq Y$, we may also naturally consider the *Hunsaker-Lindgren relation* $R_{(A,B)} = A{\times}B \cap A^c{\times}Y$ [71]. Namely, thus we evidently have $R_A = R_{(A,A)}$.

The Pervin relations R_A and the Hunsaker-Lindgren relations $R_{(A,B)}$ were actually first used by Davis [45] and Császár [32, pp. 42 and 351] in some less explicit and convenient forms, respectively.

3. A Few Basic Facts on Relators

A family $\mathcal{R}$ of relations on one set X to another Y is called a *relator on X to Y*, and the ordered pair $(X, Y)(\mathcal{R}) = ((X, Y), \mathcal{R})$ is called a *relator space*. For the origins of this notion, see [145, 154].

If in particular $\mathcal{R}$ is a relator on X to itself, then $\mathcal{R}$ is simply called a *relator on X*. Thus, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ instead of $(X, X)(\mathcal{R})$.

Relator spaces of this simpler type are already substantial generalizations of the various *ordered sets* [44] and *uniform spaces* [58]. However, they are insufficient for several purposes. (See, for instance, [62] and [154].)

A relator $\mathcal{R}$ on X to Y, or the relator space $(X, Y)(\mathcal{R})$, is called *simple* if $\mathcal{R} = \{R\}$ for some relation R on X to Y. Simple relator spaces $(X, Y)(R)$ and $X(R)$ were called *formal contexts* and *gosets* in [62] and [166], respectively.

Moreover, a relator $\mathcal{R}$ on X, or the relator space $X(\mathcal{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathcal{R}$ is reflexive on X. Thus, we may also naturally speak of *preorder, tolerance, and equivalence relators*.

For instance, for a family $\mathcal{A}$ of subsets of X, the family $\mathcal{R}_{\mathcal{A}} = \{R_A : \ A \in \mathcal{A}\}$, where $R_A = A^2 \cup A^c{\times}X$, is an important preorder relator on X. Such relators were first used by Pervin [121] and Levine [98].

While, for a family $\mathcal{D}$ of *pseudo-metrics* on X, the family $\mathcal{R}_{\mathcal{D}} = \{B_r^d : \ r > 0, \ d \in \mathcal{D}\}$, where $B_r^d = \{(x, y) : d(x, y) < r\}$, is an important tolerance relator on X. Such relators were first considered by Weil [176].

Moreover, if $\mathfrak{S}$ is a family of *partitions* of X, then the family $\mathcal{R}_{\mathfrak{S}} = \{S_{\mathcal{A}} : \ \mathcal{A} \in \mathfrak{S}\}$, where $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$, is an equivalence relator on X. Such practically important relators were first investigated by Levine [96].

If $\star$ is a unary operation for relations on X to Y, then for any relator $\mathcal{R}$ on X to Y we may naturally define $\mathcal{R}^\star = \{R^\star : \ R \in \mathcal{R}\}$. However, this plausible notation may cause some confusions whenever, for instance, $\star = c$.

In particular, for any relator $\mathcal{R}$ on X, we may naturally define $\mathcal{R}^\infty = \{R^\infty : \ R \in \mathcal{R}\}$. Moreover, we may also naturally define $\mathcal{R}^\partial = \{S \subseteq X^2 : \ S^\infty \in \mathcal{R}\}$. Namely, thus the operations ∞ and ∂ form a Galois connection [44, p. 155].

While, if $*$ is a binary operation for relations, then for any two relators $\mathcal{R}$ and $\mathcal{S}$ we may naturally define $\mathcal{R} * \mathcal{S} = \{R * S : \ R \in \mathcal{R}, \ S \in \mathcal{S}\}$. However, this plausible notation may again cause some confusions whenever, for instance, $* = \cap$.

Therefore, in general, we rather write $\mathcal{R} \wedge \mathcal{S} = \{R \cap S : \ R \in \mathcal{R}, \ S \in \mathcal{S}\}$. Moreover, for instance, we also write $\mathcal{R} \triangle \mathcal{R}^{-1} = \{R \cap R^{-1} : \ R \in \mathcal{R}\}$. Note that thus $\mathcal{R} \triangle \mathcal{R}^{-1}$ is a symmetric relator such that $\mathcal{R} \triangle \mathcal{R}^{-1} \subseteq \mathcal{R} \wedge \mathcal{R}^{-1}$.

A function $\square$ of the family of all relators on X to Y is called a *direct (indirect) unary operation for relators* if, for every relator $\mathcal{R}$ on X to Y, the value $\mathcal{R}^\square = \square(\mathcal{R})$ is a relator on X to Y (on Y to X).

For instance, c and -1 are "involution operations" for relators. While, ∞ and ∂ are "projection operations" for relators. Moreover, the operation $\square = c$, ∞ or ∂ is inversion compatible in the sense that $\mathcal{R}^{\square\, -1} = \mathcal{R}^{-1\, \square}$.

More generally, a function $\mathfrak{F}$ of the family of all relators on X to Y is called a *structure for relators* if, for every relator $\mathcal{R}$ on X to Y, the value $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}(\mathcal{R})$ is in a power set depending only on X and Y.

For instance, if $\mathrm{cl}_{\mathcal{R}}(B) = \bigcap \{R^{-1}[B] : \ R \in \mathcal{R}\}$ for every relator $\mathcal{R}$ on X to Y and $B \subseteq Y$, then the function $\mathfrak{F}$, defined by $\mathfrak{F}(\mathcal{R}) = \mathrm{cl}_{\mathcal{R}}$, is a structure for relators such that $\mathfrak{F}(\mathcal{R}) \subseteq \mathcal{P}(Y) {\times} X$, and thus $\mathfrak{F}(\mathcal{R}) \in \mathcal{P}(\mathcal{P}(Y) {\times} X)$.

A structure $\mathfrak{F}$ for relators is called *increasing* if $\mathcal{R} \subseteq \mathcal{S}$ implies $\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}}$ for any two relators $\mathcal{R}$ and S on X to Y. And, $\mathfrak{F}$ is called *quasi-increasing* if $R \in \mathcal{R}$ implies $\mathfrak{F}_R \subseteq \mathfrak{F}_{\mathcal{R}}$ for any relator $\mathcal{R}$ on X to Y. Note that here $\mathfrak{F}_R = \mathfrak{F}_{\{R\}}$.

Moreover, the structure $\mathfrak{F}$ is called *union-preserving* if $\mathfrak{F}_{\bigcup_{i\in I}\mathcal{R}_i} = \bigcup_{i\in I} \mathfrak{F}_{\mathcal{R}_i}$ for any family $(\mathcal{R}_i)_{i\in I}$ of relators on X to Y. It can be shown that $\mathfrak{F}$ is union-preserving if and only if $\mathfrak{F}_{\mathcal{R}} = \bigcup_{R\in\mathcal{R}} \mathfrak{F}_R$ for every relator $\mathcal{R}$ on X to Y [164].

In particular, an increasing operation $\Box$ for relators on X to Y is called a *projection or modification operation* for relators if it is idempotent in the sense that $\mathcal{R}^{\Box\Box} = \mathcal{R}^{\Box}$ holds for any relator $\mathcal{R}$ on X to Y.

Moreover, a modification operation $\Box$ for relators on X to Y is called a *closure or refinement operation* for relators if it is extensive in the sense that $\mathcal{R} \subseteq \mathcal{R}^{\Box}$ holds for any relator $\mathcal{R}$ on X to Y.

By using Pataki connections [172], several closure operations for relators can be derived from union-preserving structures. However, more generally, one can also find first the Galois adjoint $\mathfrak{G}$ of such a structure $\mathfrak{F}$, and then take $\Box_{\mathfrak{F}} = \mathfrak{G} \circ \mathfrak{F}$.

Now, for an operation $\Box$ for relators, a relator $\mathcal{R}$ on X to Y may be naturally called $\Box$*–fine* if $\mathcal{R}^{\Box} = \mathcal{R}$. And, for some structure $\mathfrak{F}$ for relators, two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y may be naturally called $\mathfrak{F}$*–equivalent* if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$.

Moreover, for a structure $\mathfrak{F}$ for relators, a relator $\mathcal{R}$ on X to Y may, for instance, be naturally called $\mathfrak{F}$*–simple* if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_R$ for some relation R on X to Y. Thus, in particular singleton relators have to be actually called *properly simple*.

4. Structures Derived from Relators

Definition 4.1. If $\mathcal{R}$ is a relator on X to Y, then for any $A \subseteq X$, $B \subseteq Y$ and $x \in X$, $y \in Y$ we define:

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ if $R[A] \subseteq B$ for some $R \in \mathcal{R}$;

(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ if $R[A] \cap B \neq \emptyset$ for all $R \in \mathcal{R}$;

(3) $x \in \operatorname{int}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(B)$; (4) $x \in \sigma_{\mathcal{R}}(y)$ if $x \in \operatorname{int}_{\mathcal{R}}(\{y\})$;

(5) $x \in \operatorname{cl}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Cl}_{\mathcal{R}}(B)$; (6) $x \in \rho_{\mathcal{R}}(y)$ if $x \in \operatorname{cl}_{\mathcal{R}}(\{y\})$;

(7) $B \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(B) \neq \emptyset$; (8) $B \in \mathcal{D}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(B) = X$.

Remark 4.2. The relations $\operatorname{Int}_{\mathcal{R}}$, $\operatorname{int}_{\mathcal{R}}$ and $\sigma_{\mathcal{R}}$ are called *the proximal, topological and infinitesimal interiors* generated by $\mathcal{R}$, respectively. While, the members of the families, $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$ are called the *fat and dense subsets* of the relator space $(X, Y)(\mathcal{R})$, respectively.

The origins of the relations $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$ go back to Efremović's proximity δ [52] and Smirnov's strong inclusion $\Subset$ [141], respectively. While, the convenient notations $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$, and family $\mathcal{E}_{\mathcal{R}}$, together with its dual $\mathcal{D}_{\mathcal{R}}$, was first explicitly introduced by the third author in [145, 150].

The following theorem shows that the big interior and closure are equivalent tools in a relator space.

Theorem 4.3. *If $\mathcal{R}$ is a relator on X to Y, then for any $B \subseteq Y$ we have*

(1) $\operatorname{Cl}_{\mathcal{R}}(B) = \operatorname{Int}_{\mathcal{R}}(B^c)^c$; *(2)* $\operatorname{Int}_{\mathcal{R}}(B) = \operatorname{Cl}_{\mathcal{R}}(B^c)^c$.

Remark 4.4. By using the notation $C_Y(B) = B^c$, assertion (1) can be expressed in the form that $\operatorname{Cl}_{\mathcal{R}} = (\operatorname{Int}_{\mathcal{R}} \circ C_Y)^c$ or $\operatorname{Cl}_{\mathcal{R}} = (\operatorname{Int}_{\mathcal{R}})^c \circ C_Y$.

From Theorem 4.3, we can easily derive the following

Theorem 4.5. *If $\mathcal{R}$ is a relator on X to Y, then for any $B \subseteq Y$ we have*

(1) $\operatorname{cl}_{\mathcal{R}}(B) = \operatorname{int}_{\mathcal{R}}(B^c)^c$; *(2)* $\operatorname{int}_{\mathcal{R}}(B) = \operatorname{cl}_{\mathcal{R}}(B^c)^c$.

Remark 4.6. By using the notations $B^- = \operatorname{cl}_{\mathcal{R}}(B)$ and $B^\circ = \operatorname{int}_{\mathcal{R}}(B)$, assertion (1) can be expressed in the form that $- = c \circ c$ or $-c = c\circ$.

The small closures and interiors are, in general much weaker tools than the big ones. Namely, we can only prove the following

Theorem 4.7. *If $\mathcal{R}$ is a relator on X to Y, then for any $A \subseteq X$ and $B \subseteq Y$*

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ implies $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$;

(2) $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ implies $A \in \operatorname{Cl}_{\mathcal{R}}(B)$.

Concerning closures and interiors, we can also prove the following two theorems which show that, despite their equivalences, closures are sometimes more convenient tools than interiors.

Theorem 4.8. *For any relator $\mathcal{R}$ on X to Y, we have*

(1) $\operatorname{Cl}_{\mathcal{R}^{-1}} = \operatorname{Cl}_{\mathcal{R}}^{-1}$; *(2) $\operatorname{Int}_{\mathcal{R}^{-1}} = C_Y \circ \operatorname{Int}_{\mathcal{R}}^{-1} \circ C_X$.*

Theorem 4.9. *If $\mathcal{R}$ is a relator on X to Y, then for any $B \subseteq Y$, we have*

(1) $\operatorname{cl}_{\mathcal{R}}(B) = \bigcap_{R \in \mathcal{R}} R^{-1}[B]$; *(2) $\operatorname{int}_{\mathcal{R}}(B) = \bigcup_{R \in \mathcal{R}} R^{-1}[B^c]^c$.*

From the $B = \{y\}$ particular case of this theorem, we can derive

Corollary 4.10. *For any relator $\mathcal{R}$ on X to Y, we have*

$$\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = \left(\bigcap \mathcal{R}\right)^{-1}.$$

Moreover, by using the $\mathcal{R} = \{R\}$ particular case of Theorem 4.9, we can prove the following

Theorem 4.11. *If R is a relation on X to Y, then for any $A \subseteq X$ and $B \subseteq Y$ we have*

$$A \subseteq \operatorname{int}_R(B) \iff \operatorname{cl}_{R^{-1}}(A) \subseteq B.$$

Remark 4.12. Thus the mappings $A \mapsto \operatorname{cl}_{R^{-1}}(A)$ and $B \mapsto \operatorname{int}_R(B)$ form a Galois connection between the posets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

The above important closure-interior Galois connection, used first in [169], is not independent from the upper and lower bound Galois connection [160].

The following two closely related theorems show that the fat and dense sets are also equivalent tools in a relator space.

Theorem 4.13. *If $\mathcal{R}$ is a relator on X to Y, then for any $B \subseteq Y$ we have*

(1) $B \in \mathcal{D}_{\mathcal{R}} \iff B^c \notin \mathcal{E}_{\mathcal{R}}$; *(2) $B \in \mathcal{E}_{\mathcal{R}} \iff B^c \notin \mathcal{D}_{\mathcal{R}}$.*

Theorem 4.14. *If $\mathcal{R}$ is a relator on X to Y, then for any $B \subseteq Y$ we have*

(1) $B \in \mathcal{D}_{\mathcal{R}}$ if and only if $B \cap E \neq \emptyset$ for all $E \in \mathcal{E}_{\mathcal{R}}$;

(2) $B \in \mathcal{E}_{\mathcal{R}}$ if and only if $B \cap D \neq \emptyset$ for all $D \in \mathcal{D}_{\mathcal{R}}$.

Remark 4.15. By the corresponding definitions, we have $R(x) \in \mathcal{E}_{\mathcal{R}}$, and thus also $R(x)^c \notin \mathcal{D}_{\mathcal{R}}$, for all $x \in X$ and $R \in \mathcal{R}$.

Moreover, by using the notation

$$\mathcal{U}_{\mathcal{R}}(x) = \operatorname{int}_{\mathcal{R}}^{-1}(x) = \{B \subseteq Y : \ x \in \operatorname{int}_{\mathcal{R}}(B)\},$$

we can also note that $\mathcal{E}_{\mathcal{R}} = \bigcup_{x \in X} \mathcal{U}_{\mathcal{R}}(x)$.

By using Definition 4.1, we may naturally introduce several further important definitions. For instance, we may also naturally have

Definition 4.16. If $\mathcal{R}$ is a relator on X to Y, then for any $B \subseteq Y$, we also define

(1) $\operatorname{bnd}_{\mathcal{R}}(B) = \operatorname{cl}_{\mathcal{R}}(B) \setminus \operatorname{int}_{\mathcal{R}}(B)$.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$ we also define

(2) $\operatorname{res}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(A) \setminus A$; (3) $\operatorname{bor}_{\mathcal{R}}(A) = A \setminus \operatorname{int}_{\mathcal{R}}(A)$.

Remark 4.17. Somewhat differently, the *border, boundary and residue of a set* were also introduced by Hausdorff and Kuratowski [87, pp. 4–5]. (See also Elez and Papaz [56] for a recent treatment.)

If in particular $\mathcal{R}$ is a reflexive relator on X, then by Definition 4.1, for any $A \subseteq X$, we have $A^\circ \subseteq A \subseteq A^-$ Therefore,

$$\mathrm{bnd}_{\mathcal{R}}(A) = \mathrm{res}_{\mathcal{R}}(A) \cup \mathrm{bor}_{\mathcal{R}}(A) = \mathrm{res}_{\mathcal{R}}(A) \cup \mathrm{res}_{\mathcal{R}}(A^c).$$

Namely, by using Definition 4.16 and Theorem 4.5, we can easily see that

$$\mathrm{res}_{\mathcal{R}}(A^c) = A^{c\,-} \setminus A^c = A^{c\,-} \cap A^{c\,c} = A^{\circ\,c} \cap A = A \setminus A^\circ = \mathrm{bor}_{\mathcal{R}}(A).$$

Note that if in particular $A \in \mathcal{T}_{\mathcal{R}}$ in the sense that $A \subseteq A^\circ$, then $\mathrm{bor}_{\mathcal{R}}(A) = \emptyset$. Therefore, in this particular case, by the above equality, we can simply state that $\mathrm{bnd}_{\mathcal{R}}(A) = \mathrm{res}_{\mathcal{R}}(A)$.

5. Further Structures Derived from Relators

By using Definition 4.1, we may also naturally introduce the following

Definition 5.1. If in particular $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$ we also define:

(1) $A \in \tau_{\mathcal{R}}$ if $A \in \mathrm{Int}_{\mathcal{R}}(A)$; (2) $A \in \bar{\tau}_{\mathcal{R}}$ if $A^c \notin \mathrm{Cl}_{\mathcal{R}}(A)$;

(3) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subseteq \mathrm{int}_{\mathcal{R}}(A)$; (4) $A \in \mathcal{F}_{\mathcal{R}}$ if $\mathrm{cl}_{\mathcal{R}}(A) \subseteq A$;

(5) $A \in \mathcal{N}_{\mathcal{R}}$ if $\mathrm{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}$; (6) $A \in \mathcal{M}_{\mathcal{R}}$ if $\mathrm{int}_{\mathcal{R}}(A) \in \mathcal{D}_{\mathcal{R}}$.

Remark 5.2. The members of the families, $\tau_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{N}_{\mathcal{R}}$ are called the *proximally open, topologically open and rare (or nowhere dense) subsets* of the relator space $X(\mathcal{R})$, respectively.

The family $\tau_{\mathcal{R}}$ was first introduced by the third author in [150]. While, the notation $\bar{\tau}_{\mathcal{R}}$ was suggested by János Kurdics who first noticed that "connectedness" is a particular case of "well-chainedness" [89, 90, 118, 132].

By using the corresponding results of Section 4, we can easily establish the following theorems.

Theorem 5.3. *If $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$, we have*

(1) $A \in \bar{\tau}_{\mathcal{R}} \iff A^c \in \tau_{\mathcal{R}}$; *(2)* $A \in \tau_{\mathcal{R}} \iff A^c \in \bar{\tau}_{\mathcal{R}}$.

Theorem 5.4. *For any relator $\mathcal{R}$ on X, we have*

(1) $\bar{\tau}_{\mathcal{R}} = \tau_{\mathcal{R}^{-1}}$; *(2)* $\tau_{\mathcal{R}} = \bar{\tau}_{\mathcal{R}^{-1}}$.

Theorem 5.5. *If $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$, we have*

(1) $A \in \mathcal{F}_{\mathcal{R}} \iff A^c \in \mathcal{T}_{\mathcal{R}}$; *(2)* $A \in \mathcal{T}_{\mathcal{R}} \iff A^c \in \mathcal{F}_{\mathcal{R}}$.

Theorem 5.6. *For any relator $\mathcal{R}$ on X, we have*

(1) $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$; *(2)* $\bar{\tau}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$.

Remark 5.7. In particular, for any relation R on X, we have

(1) $\tau_R = \mathcal{T}_R$; (2) $\bar{\tau}_R = \mathcal{F}_R$.

Theorem 5.8. *For any relator $\mathcal{R}$ on X, we have*

(1) $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$; *(2)* $\mathcal{D}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \{X\}$.

Remark 5.9. Hence, by using global complementations, we can easily infer that $\mathcal{F}_{\mathcal{R}} \subseteq (\mathcal{D}_{\mathcal{R}})^c \cup \{X\}$ and $\mathcal{D}_{\mathcal{R}} \subseteq (\mathcal{F}_{\mathcal{R}})^c \cup \{X\}$.

Theorem 5.10. *If $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$ we have*

(1) $A \in \mathcal{E}_{\mathcal{R}}$ *if* $V \subseteq A$ *for some* $V \in \mathcal{T}_R \setminus \{\emptyset\}$;

(2) $A \in \mathcal{D}_{\mathcal{R}}$ *only if* $A \setminus W \neq \emptyset$ *for all* $W \in \mathcal{F}_R \setminus \{X\}$.

Remark 5.11. The fat sets are frequently more convenient tools than the topologically open ones. For instance, if $\leq$ is a relation on X, then $\mathcal{T}_{\leq}$ and $\mathcal{E}_{\leq}$ are the families of all ascending and residual subsets of the goset (generalized ordered set) $X(\leq)$, respectively.

Moreover, if in particular $X = \mathbb{R}$ and $R(x) = \{x-1\} \cup [x, +\infty[$ for all $x \in X$, then R is a reflexive relation on X such that $\mathcal{T}_R = \{\emptyset, X\}$, but $\mathcal{E}_R$ is quite a large family. Namely, the supersets of each $R(x)$ are also contained in $\mathcal{E}_R$.

However, the importance of fat and dense lies mainly in the following

Definition 5.12. If $\mathcal{R}$ is a relator on X to Y, and φ and ψ are functions of a relator space $\Gamma(\mathcal{U})$ to X and Y, respectively, then using the notation

$$(\varphi \boxtimes \psi)(\gamma) = (\varphi(\gamma), \psi(\gamma))$$

for all $\gamma \in \Gamma$, we may also naturally define

(1) $\varphi \in \mathrm{Lim}_{\mathcal{R}}(\psi)$ if $(\varphi \boxtimes \psi)^{-1}[R] \in \mathcal{E}_{\mathcal{U}}$ for all $R \in \mathcal{R}$,

(2) $\varphi \in \mathrm{Adh}_{\mathcal{R}}(\psi)$ if $(\varphi \boxtimes \psi)^{-1}[R] \in \mathcal{D}_{\mathcal{U}}$ for all $R \in \mathcal{R}$.

Moreover, for any $x \in X$, we may also naturally define:

(3) $x \in \mathrm{lim}_{\mathcal{R}}(\psi)$ if $x_{\Gamma} \in \mathrm{Lim}_{\mathcal{R}}(\psi)$, (4) $x \in \mathrm{adh}_{\mathcal{R}}(\psi)$ if $x_{\Gamma} \in \mathrm{Adh}_{\mathcal{R}}(\psi)$,

where x_{Γ} is a function of Γ to X such that $x_{\Gamma}(\gamma) = x$ for all $\gamma \in \Gamma$.

Remark 5.13. Fortunately, the small limit and adherece relations are equivalent to the small closure and interior ones.

However, the big limit and adherence relations, suggested by Efremović and Švarc [53], are usually stronger tools than the big closure and interior ones.

In this respect, it seems convenient to only mention here the following

Theorem 5.14. *If $\mathcal{R}$ is a relator on X to Y, then for any $A \subseteq X$ and $B \subseteq Y$ the following assertions are equivalent:*

(1) $A \in \mathrm{Cl}_{\mathcal{R}}(B)$*;*

(2) there exist functions φ and ψ of the poset $\mathcal{R}(\supseteq)$ to A and B, respectively, such that $\varphi \in \mathrm{Lim}_{\mathcal{R}}(\psi)$;

(3) there exist functions φ and ψ of a relator space $\Gamma(\mathcal{U})$ to A and B, respectively, such that $\varphi \in \mathrm{Lim}_{\mathcal{R}}(\psi)$.

Proof. To prove that (1) $\Longrightarrow$ (2), note that if (1) holds, then for each $R \in \mathcal{R}$, we have $R[A] \cap B \neq \emptyset$. Therefore, there exist $\varphi(R) \in A$ and $\psi(R) \in B$ such that $\psi(R) \in R(\varphi(R))$. Hence, we can see that $(\varphi \boxtimes \psi)(R) = (\varphi(R), \psi(R)) \in R$, and thus $R \in (\varphi \boxtimes \psi)^{-1}[R]$.

Therefore, if $R \in \mathcal{R}$, then for any $S \in \mathcal{R}$, with $R \supseteq S$, we have

$$S \in (\varphi \boxtimes \psi)^{-1}[S] \subseteq (\varphi \boxtimes \psi)^{-1}[R].$$

This shows that $(\varphi \boxtimes \psi)^{-1}[R]$ is a fat subset of $\mathcal{R}(\supseteq)$, and thus $\varphi \in \mathrm{Lim}_{\mathcal{R}}(\psi)$. $\square$

Remark 5.15. Finally, we note that if $\mathcal{R}$ is a relator on X to Y, then according to [155], for any $A \subseteq X$ and $B \subseteq Y$, we may also naturally define

(1) $A \in \mathrm{Lb}_{\mathcal{R}}(B)$ and $B \in \mathrm{Ub}_{\mathcal{R}}(A)$ if $A \times B \subseteq R$ for some $R \in \mathcal{R}$.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$ we may also naturally define

(2) $\mathrm{Min}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \mathrm{Lb}_{\mathcal{R}}(A)$; (3) $\mathrm{Sup}_{\mathcal{R}}(A) = \mathrm{Min}_{\mathcal{R}}(\mathrm{Ub}_{\mathcal{R}}(A))$.

However, the above algebraic structures are not independent of the former topological ones. Namely, by using appropriate complements, it can be easily shown that

$$\mathrm{Lb}_{\mathcal{R}} = \mathrm{Int}_{\mathcal{R}^c} \circ \mathcal{C}_Y \quad \text{and} \quad \mathrm{Int}_{\mathcal{R}} = \mathrm{Lb}_{\mathcal{R}^c} \circ \mathcal{C}_Y.$$

Therefore, in contrast to a common belief, some algebraic and topological structures are just as closely related to each other, by the above equalities, as the exponential and the trigonometric functions are so by the celebrated Euler formulas.

6. Closure Operations for Relators

Definition 6.1. For any relator $\mathcal{R}$ on X to Y, the relators

$$\mathcal{R}^{*} = \{ S \subseteq X{\times}Y : \quad \exists\ R \in \mathcal{R} : \quad R \subseteq S \} ;$$
$$\mathcal{R}^{\#} = \{ S \subseteq X{\times}Y : \quad \forall\ A \subseteq X : \quad \exists\ R \in \mathcal{R} : \quad R[A] \subseteq S[A] \} ;$$
$$\mathcal{R}^{\wedge} = \{ S \subseteq X{\times}Y : \quad \forall\ x \in X : \quad \exists\ R \in \mathcal{R} : \quad R(x) \subseteq S(x) \} ;$$

and

$$\mathcal{R}^{\triangle} = \{ S \subseteq X{\times}Y : \quad \forall\ x \in X : \ \exists\ u \in X : \ \exists\ R \in \mathcal{R} : \ R(u) \subseteq S(x) \}$$

are called the *uniform, proximal, topological and paratopological closures (or refinements)* of the relator $\mathcal{R}$, respectively.

Remark 6.2. Thus, we evidently have $\mathcal{R} \subseteq \mathcal{R}^{*} \subseteq \mathcal{R}^{\#} \subseteq \mathcal{R}^{\wedge} \subseteq \mathcal{R}^{\triangle}$. Moreover, if in particular $\mathcal{R}$ is a relator on X, then we can also easily prove that $\mathcal{R}^{\infty} \subseteq \mathcal{R}^{*\infty} \subseteq \mathcal{R}^{\infty *} \subseteq \mathcal{R}^{*}$.

Remark 6.3. However, it is now more important to note that, because of Definition 4.1, we also have

$$\mathcal{R}^{\#} = \{ S \subseteq X{\times}Y : \quad \forall\ A \subseteq X : \quad A \in \operatorname{Int}_{\mathcal{R}}(S[A]) \} ,$$
$$\mathcal{R}^{\wedge} = \{ S \subseteq X{\times}Y : \quad \forall\ x \in X : \quad x \in \operatorname{int}_{\mathcal{R}}(S(x)) \} ,$$
$$\mathcal{R}^{\triangle} = \{ S \subseteq X{\times}Y : \quad \forall\ x \in X : \quad S(x) \in \mathcal{E}_{\mathcal{R}} \} .$$

Morover, by using Pataki connections [117, 172, 132], the following equivalences and their corollaries can be proved in a unified way.

Theorem 6.4. $\#$, $\wedge$ *and* $\triangle$ *are closure operations for relators on* X *to* Y *such that, for any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *to* Y, *we have*

(1) $\mathcal{S} \subseteq \mathcal{R}^{\#} \iff \mathcal{S}^{\#} \subseteq \mathcal{R}^{\#} \iff \operatorname{Int}_{\mathcal{S}} \subseteq \operatorname{Int}_{\mathcal{R}} \iff \operatorname{Cl}_{\mathcal{R}} \subseteq \operatorname{Cl}_{\mathcal{S}}$,

(2) $\mathcal{S} \subseteq \mathcal{R}^{\wedge} \iff \mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\wedge} \iff \operatorname{int}_{\mathcal{S}} \subseteq \operatorname{int}_{\mathcal{R}} \iff \operatorname{cl}_{\mathcal{R}} \subseteq \operatorname{cl}_{\mathcal{S}}$,

(3) $\mathcal{S} \subseteq \mathcal{R}^{\triangle} \iff \mathcal{S}^{\triangle} \subseteq \mathcal{R}^{\triangle} \iff \mathcal{E}_{\mathcal{S}} \subseteq \mathcal{E}_{\mathcal{R}} \iff \mathcal{D}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{S}}$.

Corollary 6.5. *For any relator* $\mathcal{R}$ *on* X *to* Y,

(1) $\mathcal{S} = \mathcal{R}^{\#}$ *is the largest relator on* X *to* Y *such that* $\operatorname{Int}_{\mathcal{S}} \subseteq \operatorname{Int}_{\mathcal{R}}$ $(\operatorname{Int}_{\mathcal{S}} = \operatorname{Int}_{\mathcal{R}})$, *or equivalently* $\operatorname{Cl}_{\mathcal{R}} \subseteq \operatorname{Cl}_{\mathcal{S}}$ $(\operatorname{Cl}_{\mathcal{S}} = \operatorname{Cl}_{\mathcal{R}})$;

(2) $\mathcal{S} = \mathcal{R}^{\wedge}$ *is the largest relator on* X *to* Y *such that* $\operatorname{int}_{\mathcal{S}} \subseteq \operatorname{int}_{\mathcal{R}}$ $(\operatorname{int}_{\mathcal{S}} = \operatorname{int}_{\mathcal{R}})$, *or equivalently* $\operatorname{cl}_{\mathcal{R}} \subseteq \operatorname{cl}_{\mathcal{S}}$ $(\operatorname{cl}_{\mathcal{S}} = \operatorname{cl}_{\mathcal{R}})$;

(3) $\mathcal{S} = \mathcal{R}^{\triangle}$ *is the largest relator on* X *to* Y *such that* $\mathcal{E}_{\mathcal{S}} \subseteq \mathcal{E}_{\mathcal{R}}$ $(\mathcal{E}_{\mathcal{S}} = \mathcal{E}_{\mathcal{R}})$, *or equivalently* $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{S}}$ $(\mathcal{D}_{\mathcal{S}} = \mathcal{D}_{\mathcal{R}})$.

Remark 6.6. To prove some similar statements for the operation $*$, the structures $\operatorname{Lim}_{\mathcal{R}}$ and $\operatorname{Adh}_{\mathcal{R}}$ have to be used [145].

Moreover, for instance, to investigate the structures $\operatorname{Lb}_{\mathcal{R}}$ and $\operatorname{Ub}_{\mathcal{R}}$ the compound operation $\circledast = c\,\#\,c$ is needed [165].

Concerning the above basic closure operations, we can also prove the following two theorems.

Theorem 6.7. *For any relator $\mathcal{R}$ on X to Y, we have*

(1) $\mathcal{R}^{\#} = \mathcal{R}^{*\#} = \mathcal{R}^{\#*}$,

(2) $\mathcal{R}^{\wedge} = \mathcal{R}^{\Diamond\wedge} = \mathcal{R}^{\wedge\Diamond}$ *with* $\Diamond = *$ *and* $\#$,

(3) $\mathcal{R}^{\triangle} = \mathcal{R}^{\Diamond\triangle} = \mathcal{R}^{\triangle\Diamond}$ *with* $\Diamond = *$, $\#$ *and* $\wedge$.

Proof. To prove (1), note that, by Remark 6.2 and the closure properties, we have $\mathcal{R}^{\#} \subseteq \mathcal{R}^{\#*} \subseteq \mathcal{R}^{\#\#} = \mathcal{R}^{\#}$ and $\mathcal{R}^{\#} \subseteq \mathcal{R}^{*\#} \subseteq \mathcal{R}^{\#\#} = \mathcal{R}^{\#}$. □

Theorem 6.8. *For any relator $\mathcal{R}$ on X to Y, we have*

(1) $\mathcal{R}^{*-1} = \mathcal{R}^{-1*}$; *(2)* $\mathcal{R}^{\#-1} = \mathcal{R}^{-1\#}$.

Proof. To prove (2), note that by Theorems 4.8 and 6.4 we have $\mathrm{Cl}_{\mathcal{R}^{\#-1}} = \mathrm{Cl}_{\mathcal{R}^{\#}}^{-1} = \mathrm{Cl}_{\mathcal{R}}^{-1} = \mathrm{Cl}_{\mathcal{R}^{-1}}$, and thus in particular $\mathrm{Cl}_{\mathcal{R}^{-1}} \subseteq \mathrm{Cl}_{\mathcal{R}^{\#-1}}$. Hence, by using Theorem 6.4, we can infer that $\mathcal{R}^{\#-1} \subseteq \mathcal{R}^{-1\#}$.

Now, by writing $\mathcal{R}^{-1}$ in place of $\mathcal{R}$, we can see that $\mathcal{R}^{-1\#-1} \subseteq \mathcal{R}^{\#}$, and thus $\mathcal{R}^{-1\#} \subseteq \mathcal{R}^{\#-1}$. Therefore, (2) is also true. □

Remark 6.9. For instance, the elementwise operations c and ∞ are also inversion compatible. Moreover, the operation ∂ is also inversion compatible.

However, unfortunately, the operations $\wedge$ and $\triangle$ are not inversion compatible. Therefore, in addition to Definition 6.1, we must also have the following

Definition 6.10. For any relator $\mathcal{R}$ on X to Y, we define

$$\mathcal{R}^{\vee} = \mathcal{R}^{\wedge-1} \quad \text{and} \quad \mathcal{R}^{\triangledown} = \mathcal{R}^{\triangle-1}.$$

Remark 6.11. The latter operations have very curious properties. For instance, we shall see that if $\mathcal{R}$ is nonvoid, then $\mathcal{R}^{\vee\wedge} = \{\rho_{\mathcal{R}}\}^{\wedge}$. Thus, in particular, $\mathcal{R}^{\vee}$ is topologically simple.

7. Some Further Theorems on the Operations $\wedge$ and $\triangle$

A preliminary form of the following theorem was already proved in [145].

Theorem 7.1. *If $\mathcal{R}$ is nonvoid relator on X to Y, then for any $B \subseteq Y$ we have:*

(1) $\mathrm{Int}_{\mathcal{R}^{\wedge}}(B) = \mathcal{P}(\mathrm{int}_{\mathcal{R}}(B))$; *(2)* $\mathrm{Cl}_{\mathcal{R}^{\wedge}}(B) = \mathcal{P}(\mathrm{cl}_{\mathcal{R}}(B)^c)^c$.

Proof. If $A \in \mathrm{Int}_{\mathcal{R}^{\wedge}}(B)$, then using Theorems 4.7 and 6.4 we can see that $A \subseteq \mathrm{int}_{\mathcal{R}^{\wedge}}(B) = \mathrm{int}_{\mathcal{R}}(B)$, and thus $A \in \mathcal{P}(\mathrm{int}_{\mathcal{R}}(B))$. Therefore, $\mathrm{Int}_{\mathcal{R}^{\wedge}}(B) \subseteq \mathcal{P}(\mathrm{int}_{\mathcal{R}}(B))$.

While, if $A \in \mathcal{P}(\mathrm{int}_{\mathcal{R}}(B))$, then $A \subseteq \mathrm{int}_{\mathcal{R}}(B)$. Therefore, for each $x \in A$, there exists $R_x \in \mathcal{R}$ such that $R_x(x) \subseteq B$. Now, by defining

$$S(x) = R_x(x) \quad \text{for all} \quad x \in A \quad \text{and} \quad S(x) = Y \quad \text{for all} \quad x \in A^c,$$

we can easily see that $S \in \mathcal{R}^{\wedge}$ such that $S[A] \subseteq B$. Therefore, we also have $A \in \mathrm{Int}_{\mathcal{R}^{\wedge}}(B)$. Consequently, $\mathcal{P}(\mathrm{int}_{\mathcal{R}}(B)) \subseteq \mathrm{Int}_{\mathcal{R}^{\wedge}}(B)$, and thus (1) also holds.

Now, by using Theorems 4.3 and 4.5, we can also easily see that

$$\mathrm{Cl}_{\mathcal{R}^{\wedge}}(B) = \mathrm{Int}_{\mathcal{R}^{\wedge}}(B^c)^c = \mathcal{P}(\mathrm{int}_{\mathcal{R}}(B^c))^c = \mathcal{P}(\mathrm{cl}_{\mathcal{R}}(B)^c)^c.$$

□

Remark 7.2. Thus, for any $A \subseteq X$, we have $A \in \mathrm{Cl}_{\mathcal{R}^{\wedge}}(B)$ if and only if $A \cap \mathrm{cl}_{\mathcal{R}}(B) \neq \emptyset$.

From Theorem 7.1, by using Definition 5.1, we can immediately derived

Corollary 7.3. *If $\mathcal{R}$ is a nonvoid relator on X, then*

(1) $\tau_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}$; *(2)* $\tau\!\!\!-_{\mathcal{R}^{\wedge}} = \mathcal{F}_{\mathcal{R}}$.

Remark 7.4. Hence, since $\tau_{\mathcal{R}} = \bigcup_{R\in\mathcal{R}} \tau_R = \bigcup_{R\in\mathcal{R}} \mathcal{T}_R$, we can infer that $\mathcal{T}_{\mathcal{R}} = \bigcup_{R\in\mathcal{R}^{\wedge}} \mathcal{T}_R$.

Unfortunately, in contrast to the structures Int, int, $\mathcal{E}$ and τ, the increasing structure $\mathcal{T}$ is already not union-preserving.

Example 7.5. If $\operatorname{card}(X) > 2$ and $x_1, x_2 \in X$ such that $x_1 \neq x_2$, and

$$R_i = \{x_i\}^2 \cup (X \setminus \{x_i\})^2$$

for all $i = 1, 2$, then $\mathcal{R} = \{R_1, R_2\}$ is an equivalence relator on X such that $\{x_1, x_2\} \in \mathcal{T}_{\mathcal{R}} \setminus (\mathcal{T}_{R_1} \cup \mathcal{T}_{R_2})$, and thus $\mathcal{T}_{\mathcal{R}} \nsubseteq \mathcal{T}_{R_1} \cup \mathcal{T}_{R_1}$.

From Corollary 7.3, by using Theorem 6.7, we can also derive

Corollary 7.6. *If $\mathcal{R}$ is a nonvoid relator on X, then*

(1) $\tau_{\mathcal{R}^{\triangle}} = \mathcal{T}_{\mathcal{R}^{\triangle}}$; *(2)* $\tau\!\!\!-_{\mathcal{R}^{\triangle}} = \mathcal{F}_{\mathcal{R}^{\triangle}}$.

Concerning the operation $\triangle$, we can also prove the following

Theorem 7.7. *If $\mathcal{R}$ is a nonvoid relator on X to Y, then for any $B \subseteq Y$ we have:*

(1) $\operatorname{Int}_{\mathcal{R}^{\triangle}}(B) = \{\emptyset\}$ *if* $B \notin \mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{Int}_{\mathcal{R}^{\triangle}}(B) = \mathcal{P}(X)$ *if* $B \in \mathcal{E}_{\mathcal{R}}$;

(2) $\operatorname{Cl}_{\mathcal{R}^{\triangle}}(B) = \emptyset$ *if* $B \notin \mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{Cl}_{\mathcal{R}^{\triangle}}(B) = \mathcal{P}(X) \setminus \{\emptyset\}$ *if* $B \in \mathcal{D}_{\mathcal{R}}$.

Proof. If $A \in \operatorname{Int}_{\mathcal{R}^{\triangle}}(B)$, then there exists $S \in \mathcal{R}^{\triangle}$ such that $S[A] \subseteq B$. Therefore, if $A \neq \emptyset$, then there exists $x \in X$ such that $S(x) \subseteq B$. Hence, since $S(x) \in \mathcal{E}_{\mathcal{R}}$, it follows that $B \in \mathcal{E}_{\mathcal{R}}$. Therefore, the first part of (1) is true.

To prove the second part of (1), it is enough to note only that if $B \in \mathcal{E}_{\mathcal{R}}$, then $R = X \times B \in \mathcal{R}^{\triangle}$ such that $R[A] \subseteq B$, and thus $A \in \operatorname{Int}_{\mathcal{R}^{\triangle}}(B)$ for all $A \subseteq X$.

Assertion (2) can again be derived from (1) by using Theorem 4.3. □

From this theorem, by Definition 4.1, it is clear that we also have

Corollary 7.8. *If $\mathcal{R}$ is nonvoid relator on X to Y, then for any $B \subseteq Y$:*

(1) $\operatorname{cl}_{\mathcal{R}^{\triangle}}(B) = \emptyset$ *if* $B \notin \mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{cl}_{\mathcal{R}^{\triangle}}(B) = X$ *if* $B \in \mathcal{D}_{\mathcal{R}}$;

(2) $\operatorname{int}_{\mathcal{R}^{\triangle}}(B) = \emptyset$ *if* $B \notin \mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{int}_{\mathcal{R}^{\triangle}}(B) = X$ *if* $B \in \mathcal{E}_{\mathcal{R}}$.

Hence, by using Definitions 4.1 and 5.1, we can immediately derive

Corollary 7.9. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathcal{T}_{\mathcal{R}^{\triangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$; *(2)* $\mathcal{F}_{\mathcal{R}^{\triangle}} = (\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}) \cup \{X\}$.

Remark 7.10. Note that if in particular $\mathcal{R} = \emptyset$, then $\mathcal{E}_{\mathcal{R}} = \emptyset$. Moreover, $\mathcal{R}^{\triangle} = \emptyset$ if $X \neq \emptyset$, and $\mathcal{R}^{\triangle} = \{\emptyset\}$ if $X = \emptyset$. Therefore, $\mathcal{T}_{\mathcal{R}^{\triangle}} = \{\emptyset\}$, and thus (1) is still true.

Now, since $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ if $\mathcal{R}$ is non-partial, we can also state

Corollary 7.11. *If $\mathcal{R}$ is a non-partial relator on X, then*

(1) $\mathcal{E}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\triangle}} \setminus \{\emptyset\}$, *(2)* $\mathcal{D}_{\mathcal{R}} = (\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}^{\triangle}}) \cup \{X\}$.

8. Projection Operations for Relators

By using the basic properties of the operation ∞, in addition to a particular case of Theorem 6.4, we can also prove the following

Theorem 8.1. ∞ *is a closure operation for relations on X such that, for any two relations R and S on X, we have*

$$S \subseteq R^{\infty} \iff S^{\infty} \subseteq R^{\infty} \iff \tau_R \subseteq \tau_S \iff \tau\!\!\!-_R \subseteq \tau\!\!\!-_S .$$

Proof. To prove that $\tau_R \subseteq \tau_S \iff S \subseteq R^\infty$, note that if $x \in X$, then because of the inclusion $R \subseteq R^\infty$ and the transitivity of R^∞ we have

$$R[R^\infty(x)] \subseteq R^\infty[R^\infty(x)] = (R^\infty \circ R^\infty)(x) \subseteq R^\infty(x).$$

Thus, by the definition of τ_R, we have $R^\infty(x) \in \tau_R$. Now, if $\tau_R \subseteq \tau_S$ holds, then we can see that $R^\infty(x) \in \tau_S$, and thus $S[R^\infty(x)] \subseteq R^\infty(x)$. Hence, by using the reflexivity of R^∞, we can already infer that $S(x) \subseteq R^\infty(x)$. Therefore, $S \subseteq R^\infty$ also holds.

While, if $A \in \tau_R$, then by the definition of τ_R we have $R[A] \subseteq A$. Hence, by induction, we can see that $R^n[A] \subseteq A$ for all $n \in \mathbb{N}$. Now, since $R^0[A] = \Delta_X[A] = A$ also holds, we can already state that

$$R^\infty[A] = \Big(\bigcup_{n=0}^{\infty} R^n\Big)[A] = \bigcup_{n=0}^{\infty} R^n[A] \subseteq \bigcup_{n=0}^{\infty} A = A.$$

Therefore, if $S \subseteq R^\infty$ holds, then we have $S[A] \subseteq R^\infty[A] \subseteq A$, and thus $A \in \tau_S$ also holds. □

Now, analogously to Corollary 6.5, we can also state

Corollary 8.2. *For any relation R on X, $S = R^\infty$ is the largest relation on X such that $\tau_R \subseteq \tau_S$ $(\tau_R = \tau_S)$, or equivalently $\tau_R \subseteq \tau_S$ $(\tau_R = \tau_S)$.*

Remark 8.3. Preliminary forms the above theorem and its corollary were first proved by Mala [99].

Moreover, he also proved that $R^\infty(x) = \bigcap\{A \in \tau_R : x \in A\}$ for all $x \in X$, and thus $R^\infty = \bigcap\{R_A : A \in \tau_R\}$.

By using Theorem 8.1, as an analogue of Theorem 6.4, we can also prove

Theorem 8.4. *$\#\partial$ is a closure operation for relators on X such that, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have*

$$\mathcal{S} \subseteq \mathcal{R}^{\#\partial} \iff \mathcal{S}^{\#\partial} \subseteq \mathcal{R}^{\#\partial} \iff \tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}};$$

Thus, analogously to Corollary 6.5, we can also state

Corollary 8.5. *For any relator $\mathcal{R}$ on X, $\mathcal{S} = \mathcal{R}^{\#\partial}$ is the largest relator on X such that $\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}}$ $(\tau_{\mathcal{S}} = \tau_{\mathcal{R}})$, or equivalently $\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}}$ $(\tau_{\mathcal{S}} = \tau_{\mathcal{R}})$.*

By using the basic properties of the operation ∂, Theorem 8.4 can be reformulated in the following more convenient form.

Theorem 8.6. *$\#\infty$ is a projection operation for relators on X such that, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have*

$$\mathcal{S}^\infty \subseteq \mathcal{R}^\# \iff \mathcal{S}^{\#\infty} \subseteq \mathcal{R}^{\#\infty} \iff \tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}} \iff \tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}}.$$

Remark 8.7. Moreover, it can be shown that the inclusions $\mathcal{S}^\infty \subseteq \mathcal{R}^\#$, $\mathcal{S}^{\#\infty} \subseteq \mathcal{R}^\#$ and $\mathcal{S}^{\infty\#} \subseteq \mathcal{R}^{\infty\#}$ are also equivalent.

Now, analogously to our former corollaries, we can also state

Corollary 8.8. *For any relator $\mathcal{R}$ on X, $\mathcal{S} = \mathcal{R}^{\#\infty}$ is the largest preorder relator on X such that $\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}}$ $(\tau_{\mathcal{S}} = \tau_{\mathcal{R}})$, or equivalently $\tau_{\mathcal{S}} \subseteq \tau_{\mathcal{R}}$ $(\tau_{\mathcal{S}} = \tau_{\mathcal{R}})$.*

Remark 8.9. The advantage of the projection operation $\#\infty$ over the closure operation $\#\partial$ lies mainly in the fact that, in contrast to $\#\partial$, it is *stable* in the sense $\{X^2\}^{\#\infty} = \{X^2\}$.

Since the structure $\mathcal{T}$ is not union-preserving, by using some parts of the theory of Pataki connections [117, 172, 132], we can only prove the following

Theorem 8.10. *$\wedge\partial$ is a preclosure operation for relators such that, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have*

$$\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}} \implies \mathcal{S}^\wedge \subseteq \mathcal{R}^{\wedge\partial} \implies \mathcal{S}^{\wedge\partial} \subseteq \mathcal{R}^{\wedge\partial}.$$

Remark 8.11. If $\operatorname{card}(X) > 2$, then by using the equivalence relator $\mathcal{R} = \{X^2\}$ Mala [99, Example 5.3] proved that there does not exist a largest relator $\mathcal{S}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$.

Moreover, Pataki [117, Example 7.2] proved that $\mathcal{T}_{\mathcal{R}^{\wedge\partial}} \not\subseteq \mathcal{T}_{\mathcal{R}}$ and $\wedge\partial$ is not idempotent. (Actually, it can be proved that $\mathcal{R}^{\wedge\partial\wedge} \not\subseteq \mathcal{R}^{\wedge\partial}$ also holds [157, Example 10.11].)

Fortunately, as an analogue of Theorem 8.6, we can also prove

Theorem 8.12. *$\wedge\infty$ is a projection operation for relators on X such that, for any two nonvoid relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have*

$$\mathcal{S}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge} \iff \mathcal{S}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge\infty} \iff \mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}}.$$

Thus, in particular, we can also state

Corollary 8.13. *For any nonvoid relator $\mathcal{R}$ on X, $\mathcal{S} = \mathcal{R}^{\wedge\infty}$ is the largest preorder relator on X such that $\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{R}}$ $(\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}})$, or equivalently $\mathcal{F}_{\mathcal{S}} \subseteq \mathcal{F}_{\mathcal{R}}$ $(\mathcal{F}_{\mathcal{S}} = \mathcal{F}_{\mathcal{R}})$.*

Remark 8.14. In the light of the several disadvantages of the structure $\mathcal{T}$, it is rather curious that most of the works in general topology and abstract analysis have been based on open sets suggested by Tietze [175] and Alexandroff [3], and standardized by Bourbaki [18] and Kelley [82]. (See Thron [174, p. 18].)

Moreover, it also a very striking fact that, despite the results of Davis [45], Pervin [122], Hunsaker and Lindgren [71] and the third author [150, 159], generalized proximities and closures, minimal structures, generalized topologies and stacks (ascending systems) are still intensively investigated by a great number of mathematicians without using generalized uniformities.

9. Reflexive, Non-Partial and Non-Degenerated Relators

Definition 9.1. A relator $\mathcal{R}$ on X is called *reflexive* if each member R of $\mathcal{R}$ is a reflexive relation on X.

Remark 9.2. Thus, the following assertions are equivalent:

(1) $\mathcal{R}$ is reflexive;

(2) $x \in R(x)$ for all $x \in X$ and $R \in \mathcal{R}$;

(3) $A \subseteq R[A]$ for all $A \subseteq X$ and $R \in \mathcal{R}$.

The importance of reflexive relators is also apparent from the following two obvious theorems.

Theorem 9.3. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\rho_{\mathcal{R}}$ is reflexive; *(2) $\mathcal{R}$ is reflexive;*

(3) $A \subseteq \operatorname{cl}_{\mathcal{R}}(A)$ for all $A \subseteq X$; *(4) $\operatorname{int}_{\mathcal{R}}(A) \subseteq A$ for all $A \subseteq X$.*

Proof. To prove the equivalence of (1) and (2), recall that by Corollary 4.10 we have $\rho_{\mathcal{R}} = (\bigcap \mathcal{R})^{-1}$. □

Theorem 9.4. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is reflexive;

(2) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ implies $A \subseteq B$ for all $A, B \subseteq X$;

(3) $A \cap B \neq \emptyset$ implies $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ for all $A, B \subseteq X$.

Remark 9.5. In addition to the above two theorems, it is also worth mentioning that if $\mathcal{R}$ is a reflexive ordinary relator on X, then

(1) $\operatorname{Int}_{\mathcal{R}}$ is transitive;

(2) $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ implies $\mathcal{P}(X) = \operatorname{Cl}_{\mathcal{R}}(A)^c \cup \operatorname{Cl}_{\mathcal{R}}^{-1}(B)$;

(3) $\operatorname{int}_{\mathcal{R}}(\operatorname{bor}_{\mathcal{R}}(A)) = \emptyset$ and $\operatorname{int}_{\mathcal{R}}(\operatorname{res}_{\mathcal{R}}(A)) = \emptyset$ for all $A \subseteq X$.

Thus, for instance, for any $A \subseteq X$ we have $\operatorname{res}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}$ if and only if $A \in \mathcal{F}_{\mathcal{R}}$.

Analogously to Definition 9.1, we may also naturally have the following

Definition 9.6. A relator $\mathcal{R}$ on X to Y is called *non-partial* if each member R of $\mathcal{R}$ is a non-partial relation on X to Y.

Remark 9.7. Thus, the following assertions are equivalent:

(1) $\mathcal{R}$ is non-partial;

(2) $R^{-1}[\,Y\,] = X$ for all $R \in \mathcal{R}$;

(3) $R(x) \neq \emptyset$ for all $x \in X$ and $R \in \mathcal{R}$.

The importance of non-partial relators is apparent from the following

Theorem 9.8. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is non-partial;*

(2) $\emptyset \notin \mathcal{E}_{\mathcal{R}}$; *(3)* $\mathcal{D}_{\mathcal{R}} \neq \emptyset$; *(4)* $Y \in \mathcal{D}_{\mathcal{R}}$; *(5)* $\mathcal{E}_{\mathcal{R}} \neq \mathcal{P}(Y)$.

In addition to Definition 9.6, it is also worth introducing the following

Definition 9.9. A relator $\mathcal{R}$ on X to Y is called *non-degerated* if $X \neq \emptyset$ and $\mathcal{R} \neq \emptyset$.

Thus, analogously to Theorem 9.8, we can also easily establish

Theorem 9.10. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is non-degenerated;*

(2) $\emptyset \notin \mathcal{D}_{\mathcal{R}}$; *(3)* $\mathcal{E}_{\mathcal{R}} \neq \emptyset$; *(4)* $Y \in \mathcal{E}_{\mathcal{R}}$; *(5)* $\mathcal{D}_{\mathcal{R}} \neq \mathcal{P}(Y)$.

Remark 9.11. In addition to Theorems 9.8 and 9.10, it is also worth mentioning that if a relator $\mathcal{R}$ on X to Y is paratopologically simple in the sense that $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_S$, or equivalently $\mathcal{R}^{\triangle} = \{S\}^{\triangle}$, for some relation S on X to Y, then the stack $\mathcal{E}_{\mathcal{R}}$ has a base $\mathcal{B}$ with $\operatorname{card}(\mathcal{B}) \leq \operatorname{card}(X)$. (See [116, Theorem 5.9] of Pataki.)

The existence of a non-paratopologically simple (actually finite equivalence) relator, proved first by Pataki [116, Example 5.11], shows that in our definitions of the relations $\operatorname{Lim}_{\mathcal{R}}$ and $\operatorname{Adh}_{\mathcal{R}}$ we cannot restrict ourselves to functions of gosets (generalized ordered sets) without a substantial loss of generality.

In this respect, it is noteworthy that, in addition to certain simple relator spaces called "physical continuums", some multi-ordered sets were also considered by Riesz [130]. Thus, he could be the inventor of uniform spaces.

Moreover, to define "mathematical continuums", he used cluster points [131] instead of the adherence ones. Thus, he could not be the inventor of closure and proximity spaces, despite that he spent much time with the investigation of spatial notions.

10. Topological and Quasi-Topological Relators

The following improvement of [146, Definition 2.1] was first considered in [148].

Definition 10.1. A relator $\mathcal{R}$ on X is called:

(1) *quasi-topological* if $x \in \operatorname{int}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(R(x)))$ for all $x \in X$ and $R \in \mathcal{R}$;

(2) *topological* if for any $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$.

The appropriateness of these definitions is already quite obvious from the following four theorems.

Theorem 10.2. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topological;*

(2) $\operatorname{int}_{\mathcal{R}}(R(x)) \in \mathcal{T}_{\mathcal{R}}$ *for all* $x \in X$ *and* $R \in \mathcal{R}$;

(3) $\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}$ $(\operatorname{int}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}})$ *for all* $A \subseteq X$.

Theorem 10.3. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent :*

(1) $\mathcal{R}$ is topological ; *(2) $\mathcal{R}$ is reflexive and quasi-topological .*

Remark 10.4. By Theorem 10.2, the relator $\mathcal{R}$ may be called *weakly (strongly) quasi-topological* if $\rho_{\mathcal{R}}(x) \in \mathcal{F}_{\mathcal{R}}$ $(R(x) \in \mathcal{T}_{\mathcal{R}})$ for all $x \in X$ and $R \in \mathcal{R}$.

Moreover, by Theorem 10.3, the relator $\mathcal{R}$ may be called *weakly (strongly) toplogical* if it is reflexive and weakly (strongly) quasi-topological.

Theorem 10.5. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent :*

(1) $\mathcal{R}$ is topological ;

(2) $\operatorname{int}_{\mathcal{R}}(A) = \bigcup \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$ *for all* $A \subseteq X$*;*

(3) $\operatorname{cl}_{\mathcal{R}}(A) = \bigcap \mathcal{F}_{\mathcal{R}} \cap \mathcal{P}^{-1}(A)$ *for all* $A \subseteq X$.

Now, as an immediate consequence of this theorem, we can also state

Corollary 10.6. *If $\mathcal{R}$ is topological relator on X, then for any $A \subset X$, we have*

(1) $A \in \mathcal{E}_{\mathcal{R}}$ if and only if there exists $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such that $V \subseteq A$;

(2) $A \in \mathcal{D}_{\mathcal{R}}$ if and only if for all $W \in \mathcal{F}_{\mathcal{R}} \setminus \{X\}$ we have $A \setminus W \neq \emptyset$.

However, it is now more important to note that we can also prove

Theorem 10.7. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent :*

(1) $\mathcal{R}$ is topological ;

(2) $\mathcal{R}$ is topologically equivalent to $\mathcal{R}^{\wedge\infty}$;

(3) $\mathcal{R}$ is topologically equivalent to a preorder relator on X.

Proof. To prove the implication (1) $\Longrightarrow$ (3), note that if (1) holds, then by Definition 10.1, for any $x \in X$ and $R \in \mathcal{R}$, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$. Hence, by considering the Pervin relator

$$\mathcal{S} = \mathcal{R}_{\mathcal{T}_{\mathcal{R}}} = \{ R_V : \quad V \in \mathcal{T}_{\mathcal{R}} \}, \quad \text{with} \quad R_V = V^2 \cup V^c \times X,$$

we can note that $\mathcal{R} \subseteq \mathcal{S}^{\wedge}$, and thus $\mathcal{R}^{\wedge} \subseteq \mathcal{S}^{\wedge\wedge} = \mathcal{S}^{\wedge}$. Moreover, since

$$R_V(x) = V \quad \text{if} \quad x \in V \quad \text{and} \quad R_V(x) = X \quad \text{if} \quad x \in V^c,$$

we can also note that $\mathcal{S} \subseteq \mathcal{R}^{\wedge}$, and thus $\mathcal{S}^{\wedge} \subseteq \mathcal{R}^{\wedge\wedge} = \mathcal{R}^{\wedge}$. Therefore, we actually have $\mathcal{R}^{\wedge} = \mathcal{S}^{\wedge}$, and thus $\mathcal{R}$ is topologically equivalent to $\mathcal{S}$. Hence, since $\mathcal{S}$ is a preorder relator on X, we can already see that (3) also holds. □

In addition to Theorems 10.2 and 10.3, it is also worth proving

Theorem 10.8. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent :*

(1) $\mathcal{R}$ is quasi-topological;

(2) $\mathcal{R} \subseteq (\mathcal{R}^{\wedge} \circ \mathcal{R})^{\wedge}$*;* *(3)* $\mathcal{R}^{\wedge} \subseteq (\mathcal{R}^{\wedge} \circ \mathcal{R}^{\wedge})^{\wedge}$.

Remark 10.9. By [146], a relator $\mathcal{R}$ on X may be naturally called topologically transitive if, for each $x \in X$ and $R \in \mathcal{R}$ there exist $S, T \in \mathcal{R}$ such that $T[S(x)] \subseteq R(x)$.

This property can also be reformulated in the more concise form that $\mathcal{R} \subseteq (\mathcal{R} \circ \mathcal{R})^{\wedge}$. Thus, the equivalence (1) and (3) can be expressed by saying that $\mathcal{R}$ is quasi-topological if and only if $\mathcal{R}^{\wedge}$ is topologically transitive.

11. Proximal and Quasi-Proximal Relators

Analogously to Definition 10.1, we may also naturally have the following

Definition 11.1. A relator $\mathcal{R}$ on X is called

(1) *quasi-proximal* if $A \in \operatorname{Int}_{\mathcal{R}}[\,\tau_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{R}}(R[\,A\,])\,]$ for all $A \subseteq X$ and $R \in \mathcal{R}$;

(2) *proximal* if for any $A \subseteq X$ and $R \in \mathcal{R}$ there exists $V \in \tau_{\mathcal{R}}$ such that $A \subseteq V \subseteq R[\,A\,]$.

Remark 11.2. Thus, the relator $\mathcal{R}$ is quasi-proximal if and only if, for any $A \subseteq X$ and $R \in \mathcal{R}$, there exists $V \in \tau_{\mathcal{R}}$ such that $A \in \operatorname{Int}_{\mathcal{R}}(V)$ and $V \in \operatorname{Int}_{\mathcal{R}}(R[\,A\,])$.

Now, by using the corresponding definitions, we can also easily prove the following analogues of Theorems 10.3 and 10.5.

Theorem 11.3. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximal; *(2) $\mathcal{R}$ is reflexive and quasi-proximal.*

Proof. To prove the implication (1) $\Longrightarrow$ (2), note that if (1) holds, then for any $A \subseteq X$ and $R \in \mathcal{R}$, there exists $V \in \tau_{\mathcal{R}}$ such that $A \subseteq V \subseteq R[\,A\,]$. Hence, since A may be $\{x\}$ for any $x \in X$, and $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$, we can at once see that $\mathcal{R}$ is not only reflexive, but also topological.

Moreover, since $V \in \tau_{\mathcal{R}}$, we can also note $V \in \operatorname{Int}_{\mathcal{R}}(V)$. Hence, by using that $A \subseteq V$ and $V \subseteq R[\,A\,]$, we can already infer that the inclusions $A \in \operatorname{Int}_{\mathcal{R}}(V)$ and $V \in \operatorname{Int}_{\mathcal{R}}(R[\,A\,])$ are also true. Therefore, by Remark 11.2, $\mathcal{R}$ is quasi-proximal, and thus (2) also holds. □

Remark 11.4. Note that if $\mathcal{R}$ is only a weakly proximal relator on X in the sense that, for any $x \in X$ and $R \in \mathcal{R}$, there exists $V \in \tau_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$, then because of $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$ we can already state that $\mathcal{R}$ is topological.

Theorem 11.5. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximal;

(2) $\operatorname{Int}_{\mathcal{R}}(A) = \bigcup\{\mathcal{P}(V):\ A \supseteq V \in \tau_{\mathcal{R}}\}$ for all $A \subseteq X$;

(3) $\operatorname{Cl}_{\mathcal{R}}(A) = \bigcap\{\mathcal{P}(X) \setminus \mathcal{P}(W^c):\ A \subseteq W \in \tau_{\mathcal{R}}\}$ for all $A \subseteq X$.

Proof. To check the equivalence of (1) and (2), note that, for any $A,\,B \subseteq X$, we have $B \in \bigcup\{\mathcal{P}(V):\ A \supseteq V \in \tau_{\mathcal{R}}\}$ if and only if there exists $V \in \tau_{\mathcal{R}}$ such that $V \subseteq A$ and $B \subseteq V$. Thus, $\bigcup\{\mathcal{P}(V):\ A \supseteq V \in \tau_{\mathcal{R}}\} \subseteq \operatorname{Int}_{\mathcal{R}}(A)$ is always true.

Moreover, if $A \subseteq X$ and $R \in \mathcal{R}$, then because of $R[\,A\,] \subseteq R[\,A\,]$, we always have $A \in \operatorname{Int}_{\mathcal{R}}(R[\,A\,])$. Therefore, if the essential part of (2) holds, then there exists $V \in \tau_{\mathcal{R}}$ such that $V \subseteq R[\,A\,]$ and $A \subseteq V$, and thus (1) also holds. □

Now, since $\mathcal{P}(A) = \{B:\ B \subseteq A\}$ and $\mathcal{P}(A) = \operatorname{Int}_{\Delta_X}(A)$, we can also state

Corollary 11.6. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximal;

(2) $\operatorname{Int}_{\mathcal{R}}(A) = \mathcal{P}[\,\tau_{\mathcal{R}} \cap \mathcal{P}(A)]$ for all $A \subseteq X$;

(3) $\operatorname{Int}_{\mathcal{R}}(A) = \operatorname{Int}_{\Delta_X}[\,\tau_{\mathcal{R}} \cap \operatorname{Int}_{\Delta_X}(A)]$ for all $A \subseteq X$.

However, it is now more important to note that we also have the following

Theorem 11.7. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximal;

(2) $\mathcal{R}$ is proximally equivalent to $\mathcal{R}^{\infty}$ or $\mathcal{R}^{\#\infty}$;

(3) $\mathcal{R}$ is proximally equivalent to a preorder relator on X.

In principle, each theorem on topological and quasi-topological relators can be immediately derived from a corresponding theorem on proximal and quasi-proximal relators by using the following two theorems.

Theorem 11.8. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent :*

(1) $\mathcal{R}$ is quasi-topological ; *(2) $\mathcal{R}^{\wedge}$ is quasi-proximal .*

Proof. To prove the implication (1) $\Longrightarrow$ (2), assume that (1) holds, and moreover $A \subseteq X$ and $S \in \mathcal{R}^{\wedge}$. Define $V = \operatorname{int}_{\mathcal{R}}(S [A])$. Then, if $\mathcal{R} \neq \emptyset$, by Theorem 10.2 and Corollary 7.3, we have $V \in \mathcal{T}_{\mathcal{R}} = \tau_{\mathcal{R}^{\wedge}}$. Moreover, since $V \subseteq \operatorname{int}_{\mathcal{R}}(S [A])$, by Theorem 7.1 we also have $V \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(S [A])$. Therefore, $V \in \tau_{\mathcal{R}^{\wedge}} \cap \operatorname{Int}_{\mathcal{R}^{\wedge}}(S [A])$.

On the other hand, since $S \in \mathcal{R}^{\wedge}$ and $S [A] \subseteq S [A]$, we can also note that $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(S [A])$. Hence, by Theorem 7.1, we can infer that $A \subseteq \operatorname{int}_{\mathcal{R}}(S [A]) = V$. Moreover, since $V \in \tau_{\mathcal{R}^{\wedge}}$, we can also note that $V \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(V)$. Hence, since $A \subseteq V$, we can infer that $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(V)$. Therefore, since $V \in \tau_{\mathcal{R}^{\wedge}} \cap \operatorname{Int}_{\mathcal{R}^{\wedge}}(S [A])$, we also have $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}} [\tau_{\mathcal{R}^{\wedge}} \cap \operatorname{Int}_{\mathcal{R}^{\wedge}} (S [A])]$. This shows that (2) also holds.

Thus, to complete the proof of the implication (1) $\Longrightarrow$ (2), it remains only to note that if $\mathcal{R} = \emptyset$, then $\mathcal{R}$ is topological. Moreover, $\mathcal{R}^{\wedge} = \emptyset$ if $X \neq \emptyset$ and $\mathcal{R}^{\wedge} = \{\emptyset\}$ if $X = \emptyset$. Thus, $\mathcal{R}^{\wedge}$ is proximal. $\square$

Remark 11.9. If assertion (2) holds, then $\mathcal{R}^{\wedge}$ is semi-proximal in the sense that $A \in \operatorname{Int}_{\mathcal{R}^{\wedge}} [\operatorname{Int}_{\mathcal{R}^{\wedge}} (S [A])]$ for all $A \subseteq X$ and $S \in \mathcal{R}^{\wedge}$.

Moreover, if in particular $\{x\} \in \operatorname{Int}_{\mathcal{R}^{\wedge}} [\operatorname{Int}_{\mathcal{R}^{\wedge}} (R(x))]$ for all $x \in X$ and $R \in \mathcal{R}$, then we can already prove that assertion (1) also holds.

From Theorem 11.8, by using Theorems 10.3 and 11.3, we can easily derive

Theorem 11.10. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent :*

(1) $\mathcal{R}$ is topological , *(2) $\mathcal{R}^{\wedge}$ is proximal .*

Remark 11.11. By the corresponding definitions, it is clear that the relator $\mathcal{R}^{\wedge}$ is reflexive if and only if $\mathcal{R}$ is reflexive.

However, if $\mathcal{R} \not\subseteq \{ X^2 \}$, then there exists $R \in \mathcal{R}$ such that $R \neq X^2$. Therefore, there exist $x, y \in X$ such that $x \notin R (y)$. Thus, $S = \{x\} \times R (y) \cup \{x\}^c \times X$ is a non-reflexive relation on X such that $S \in \mathcal{R}^{\vartriangle}$. Therefore, $\mathcal{R}^{\vartriangle}$ cannot be reflexive.

Note that if $\mathcal{R} = \emptyset$, then by Definition 6.1 we have $\mathcal{R}^{\vartriangle} = \emptyset$ if $X \neq \emptyset$, and $\mathcal{R}^{\vartriangle} = \{\emptyset\}$ if $X = \emptyset$. Moreover, if $\mathcal{R} = \{ X^2 \}$, then we have $\mathcal{R}^{\vartriangle} = \{ X^2 \}$. Therefore, in these very particular cases, the relator $\mathcal{R}^{\vartriangle}$ is also reflexive.

12. A Few Basic Facts on Filtered Relators

Intersection properties of relators were also first investigated in [146, 148] .

Definition 12.1. A relator $\mathcal{R}$ on X to Y is called

(1) *properly filtered* if for any $R, S \in \mathcal{R}$ we have $R \cap S \in \mathcal{R}$;
(2) *uniformly filtered* if for any $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T \subseteq R \cap S$;
(3) *proximally filtered* if for any $A \subseteq X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T [A] \subseteq R [A] \cap S [A]$;
(4) *topologically filtered* if for any $x \in X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T (x) \subseteq R (x) \cap S (x)$.

Remark 12.2. By using the binary operation $\wedge$ and the unary operations $*$, $\#$ and $\wedge$, the above properties can be reformulated in some more concise forms.

For instance, we can see that $\mathcal{R}$ is topologically filtered if and only if any one of the properties $\mathcal{R} \wedge \mathcal{R} \subseteq \mathcal{R}^{\wedge}$, $(\mathcal{R} \wedge \mathcal{R})^{\wedge} = \mathcal{R}^{\wedge}$ and $\mathcal{R}^{\wedge} \wedge \mathcal{R}^{\wedge} = \mathcal{R}^{\wedge}$ holds.

However, in general, we only have $(R \cap S)[A] \subseteq R [A] \cap S [A]$. Therefore, the corresponding proximal filteredness properties are, unfortunataly, not equivalent.

Despite this, we can easily prove the following theorem which shows the appropriateness of the above proximal filteredness property.

Theorem 12.3. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) *$\mathcal{R}$ is proximally filtered;*

(2) $\mathrm{Cl}_{\mathcal{R}}(A \cup B) = \mathrm{Cl}_{\mathcal{R}}(A) \cup \mathrm{Cl}_{\mathcal{R}}(B)$ *for all* $A, B \subseteq Y$*;*

(3) $\mathrm{Int}_{\mathcal{R}}(A \cap B) = \mathrm{Int}_{\mathcal{R}}(A) \cap \mathrm{Int}_{\mathcal{R}}(B)$ *for all* $A, B \subseteq Y$.

Proof. To prove the implication (3) $\Longrightarrow$ (1), note that if $A \subseteq X$ and $R, S \in \mathcal{R}$, then by the definition of $\mathrm{Int}_{\mathcal{R}}$ we trivially have $A \in \mathrm{Int}_{\mathcal{R}}(R[A])$ and $A \in \mathrm{Int}_{\mathcal{R}}(S[A])$. Therefore, if (3) holds, then we also have $A \in \mathrm{Int}_{\mathcal{R}}(R[A] \cap S[A])$. Thus, by the definition of $\mathrm{Int}_{\mathcal{R}}$, there exists $T \in \mathcal{R}$ such that $T[A] \subseteq R[A] \cap S[A]$. □

Now, as an immediate consequence of this theorem, we can also state

Corollary 12.4. *If $\mathcal{R}$ is a proximally filtered relator on X, then the families $\tau_{\mathcal{R}}$ and $\tau_{\mathcal{R}}$ are closed under binary unions and intersections, respectively.*

From Theorem 12.3, we can also easily derive the following

Theorem 12.5. *For a relator $\mathcal{R}$ on X to Y, the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topologically filtered;*

(2) $\mathrm{cl}_{\mathcal{R}}(A \cup B) = \mathrm{cl}_{\mathcal{R}}(A) \cup \mathrm{cl}_{\mathcal{R}}(B)$ *for all* $A, B \subseteq Y$*;*

(3) $\mathrm{int}_{\mathcal{R}}(A \cap B) = \mathrm{int}_{\mathcal{R}}(A) \cap \mathrm{int}_{\mathcal{R}}(B)$ *for all* $A, B \subseteq Y$.

Thus, in particular, we can also state the following

Corollary 12.6. *If $\mathcal{R}$ is a topologically filtered relator on X, then the families $\mathcal{F}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{R}}$ are closed under binary unions and intersections, respectively.*

The following example shows that, for a non-topological relator $\mathcal{R}$, the converse of the above corollary need not be true.

Example 12.7. If $X = \{1, 2, 3\}$ and R_i is relation on X, for each $i = 1, 2$, such that

$$R_i(1) = \{1, i+1\} \quad \text{and} \quad R_i(2) = R_i(3) = \{2, 3\},$$

then $\mathcal{R} = \{R_1, R_2\}$ is reflexive relator on X such that $\mathcal{T}_{\mathcal{R}}$ is closed under arbitrary intersections, but $\mathcal{R}$ is still not topologically filtered.

By the corresponding definitions, it is clear that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, \{2, 3\}, X\}$. Moreover, we can note that $R_i(1) \not\subseteq R_1(1) \cap R_2(1)$ for each $i = 1, 2$, and thus by Definition 12.1 the relator $\mathcal{R}$ is not topologically filtered.

13. A Few Basic Facts on Quasi-Filtered Relators

Since $R \subseteq R^{\infty}$ for every relation R on X, in addition to Definition 12.1, we may also naturally introduce the following

Definition 13.1. A relator $\mathcal{R}$ on X is called

(1) *quasi-uniformly filtered* if for any $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T \subseteq R^{\infty} \cap S^{\infty}$;

(2) *quasi-proximally filtered* if for any $A \subseteq X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T[A] \subseteq R^{\infty}[A] \cap S^{\infty}[A]$;

(3) *quasi-topologically filtered* if for any $x \in X$ and $R, S \in \mathcal{R}^{\wedge}$ there exists $T \in \mathcal{R}$ such that $T(x) \subseteq R^{\infty}(x) \cap S^{\infty}(x)$.

Remark 13.2. Analogously to Remark 12.2, the above quasi-filteredness properties can also be reformulated in some more concise forms.

For instance, we can see that $\mathcal{R}$ is quasi-topologically filtered if and only if $\mathcal{R}^{\wedge\infty} \wedge \mathcal{R}^{\wedge\infty} \subseteq \mathcal{R}^{\wedge}$, $(\mathcal{R}^{\wedge\infty} \wedge \mathcal{R}^{\wedge\infty})^{\wedge\infty} = \mathcal{R}^{\wedge\infty}$ or $\mathcal{R}^{\wedge\infty} \wedge \mathcal{R}^{\wedge\infty} = \mathcal{R}^{\wedge\infty}$.

However, it is now more important to note that, by using some former results, we can also prove the following two theorems which show the appropriateness of the above quasi-proximal and quasi-topological filteredness properties.

Theorem 13.3. *For any relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is a quasi-proximally filtered;

(2) $\tau_{\mathcal{R}}$ is closed under binary unions;

(3) $\tau_{\mathcal{R}}$ is closed under binary intersections.

Theorem 13.4. *For any relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is a quasi-topologically filtered;

(2) $\mathcal{F}_{\mathcal{R}}$ is closed under binary unions;

(3) $\mathcal{T}_{\mathcal{R}}$ is closed under binary intersections.

Remark 13.5. In this respect it is also worth mentioning that if $\mathcal{R}$ is a relator on X to Y, then the family $\mathcal{E}_{\mathcal{R}}$ is closed under binary intersections if and only if $\mathcal{R}$ is *quasi-directed* in the sense that for any $x, y \in X$ and $R, S \in \mathcal{R}$ we have $R(x) \cap S(y) \in \mathcal{E}_{\mathcal{R}}$.

From the above two theorems, by using Corollaries 12.4 and 12.6, we can derive

Corollary 13.6. *If $\mathcal{R}$ is a proximally (topologically) filtered relator on X, then $\mathcal{R}$ is also quasi-proximally (quasi-topologically) filtered.*

Now, by using Theorem 13.3, we can also easily prove the following

Theorem 13.7. *If $\mathcal{R}$ is a quasi-proximally filtered, proximal relator on X, then $\mathcal{R}$ is proximally filtered.*

Proof. Suppose that $A \subseteq X$ and $R, S \in \mathcal{R}$. Then, by Definition 11.1, there exist $U, V \in \tau_{\mathcal{R}}$ such that $A \subseteq U \subseteq R[A]$ and $A \subseteq V \subseteq S[A]$.

Moreover, by Theorem 13.3, we can state that $U \cap V \in \tau_{\mathcal{R}}$. Therefore, by the definition of $\tau_{\mathcal{R}}$, there exists $T \in \mathcal{R}$ such that $T[U \cap V] \subseteq U \cap V$. Hence, we can see that $T[A] \subseteq T[U \cap V] \subseteq U \cap V \subseteq R[A] \cap S[A]$. □

Moreover, by using Theorem 13.4, we can quite similarly prove

Theorem 13.8. *If $\mathcal{R}$ is a quasi-topologically filtered, topological relator on X, then $\mathcal{R}$ is topologically filtered.*

Remark 13.9. Our former Example 12.7 shows that even a quasi-proximally filtered, reflexive relator need not be topologically filtered.

Namely, if X and $\mathcal{R}$ are as in Example 12.7, then by the corresponding definitions it is clear that $\tau_{\mathcal{R}} = \{\emptyset, \{2, 3\}, X\}$, and thus by Theorem 13.3 the relator $\mathcal{R}$ is quasi-proximally filtered.

14. Some Further Theorems on Topologically Filtered Relators

Now, by using the arguments of Kuratowski [88, pp. 39, 45], we shall prove some more particular theorems on the relation $\operatorname{cl}_{\mathcal{R}}$.

To make the forthcoming proofs much shorter and more readable, we shall use the convenient notations

$$A^- = \operatorname{cl}_{\mathcal{R}}(A), \qquad A^\circ = \operatorname{int}_{\mathcal{R}}(A) \quad \text{and} \quad A^\dagger = \operatorname{res}_{\mathcal{R}}(A).$$

Theorem 14.1. *If $\mathcal{R}$ is a topologically filtered relator on X to Y, then for any $A, B \subseteq Y$ we have*

$$\operatorname{cl}_{\mathcal{R}}(A) \setminus \operatorname{cl}_{\mathcal{R}}(B) = \operatorname{cl}_{\mathcal{R}}(A \setminus B) \setminus \operatorname{cl}_{\mathcal{R}}(B).$$

Proof. By using Theorem 12.5, we can see that

$$A^- \cup B^- = (A \cup B)^- = \big((A \setminus B) \cup B\big)^- = (A \setminus B)^- \cup B^- .$$

Hence, we can already infer that

$$A^- \setminus B^- = (A^- \cup B^-) \setminus B^- = \big((A \setminus B)^- \cup B^-\big) \setminus B^- = (A \setminus B)^- \setminus B^- .$$

□

Thus, in particular, we can also state the following

Corollary 14.2. *If $\mathcal{R}$ is a topologically filtered relator on X to Y, then for any $A, B \subseteq Y$ we have*

$$\mathrm{cl}_{\mathcal{R}}(A) \setminus \mathrm{cl}_{\mathcal{R}}(B) \subseteq \mathrm{cl}_{\mathcal{R}}(A \setminus B) .$$

This corollary will already allow us to easily prove the following

Theorem 14.3. *If $\mathcal{R}$ is a topologically filtered relator on X, then for any $A \subseteq X$ and $U \in \mathcal{T}_{\mathcal{R}}$ we have*

$$\mathrm{cl}_{\mathcal{R}}(A) \cap U = \mathrm{cl}_{\mathcal{R}}(A \cap U) \cap U .$$

Proof. By Definition 5.1 and Theorem 4.5, we have $U \subseteq U^{\circ} = U^{c-c}$. Hence, by using Corollary 14.2, we can already infer that

$$A^- \cap U \subseteq A^- \cap U^{c-c} = A^- \setminus U^{c-} \subseteq (A \setminus U^c)^- = (A \cap U)^- .$$

Therefore, $A^- \cap U = A^- \cap U \cap U \subseteq (A \cap U)^- \cap U$.

Moreover, by using the increasingness of $-$, we can see that $(A \cap U)^- \subseteq A^-$, and thus $(A \cap U)^- \cap U \subseteq A^- \cap U$ is always true. Therefore, we actually have $A^- \cap U = (A \cap U)^- \cap U$. □

Thus, in particular, we can also state the following important corollary which can also be easily proved directly.

Corollary 14.4. *If $\mathcal{R}$ is a topologically filtered relator on X, then for any $A \subseteq X$ and $U \in \mathcal{T}_{\mathcal{R}}$ we have*

$$\mathrm{cl}_{\mathcal{R}}(A) \cap U \subseteq \mathrm{cl}_{\mathcal{R}}(A \cap U) .$$

Remark 14.5. The importance of the closure space counterpart of Corollary 14.4 was also recognized Császár [33, 34, 35, 36, 38, 39] and Sivagami [140] who assumed it as an axiom for an increasing set-to-set function γ.

Moreover, it is also worth noticing that, by using Theorem 4.5, Corollary 14.4 can be reformulated in the dual form that if $\mathcal{R}$ is a topologically filtered relator on X, then for any $A \subseteq X$ and $V \in \mathcal{F}_{\mathcal{R}}$ we have $\mathrm{int}_{\mathcal{R}}(A \cup V) \subseteq \mathrm{int}_{\mathcal{R}}(A) \cup V$.

15. Some More Particular Theorems on Topologically Filtered Relators

By using Corollary 14.4, we can also easily prove the following

Theorem 15.1. *If $\mathcal{R}$ is a topologically filtered, topological relator on X, then for any $A \subseteq X$ and $U \in \mathcal{T}_{\mathcal{R}}$ we have*

$$\mathrm{cl}_{\mathcal{R}}(A \cap U) = \mathrm{cl}_{\mathcal{R}}(\mathrm{cl}_{\mathcal{R}}(A) \cap U) .$$

Proof. By Corollary 14.4, we have $A^- \cap U \subseteq (A \cap U)^-$. Moreover, by Theorems 10.3 and 10.2, we have $(A \cap U)^- \in \mathcal{F}_{\mathcal{R}}$. Hence, by using the increasingness of $-$ and the definition $\mathcal{F}_{\mathcal{R}}$, we can infer that

$$(A^- \cap U)^- \subseteq (A \cap U)^{--} \subseteq (A \cap U)^- .$$

On the other hand, by Theorems 10.3 and 9.3, we have $A \subseteq A^-$, and thus also $A \cap U \subseteq A^- \cap U$. Hence, by using the increasingness of $-$, we can infer that $(A \cap U)^- \subseteq (A^- \cap U)^-$. Therefore, the required equality $(A \cap U)^- = (A^- \cap U)^-$ is also true. □

From this theorem, we can immediate derive

Corollary 15.2. *If $\mathcal{R}$ is a topologically filtered, topological relator on X, then for any $A \in \mathcal{D}_{\mathcal{R}}$ and $U \in \mathcal{T}_{\mathcal{R}}$ we have*

$$\mathrm{cl}_{\mathcal{R}}(U) = \mathrm{cl}_{\mathcal{R}}(A \cap U).$$

Proof. By Definition 4.1 and Theorem 15.1, we evidently have

$$U^- = (X \cap U)^- = (A^- \cap U)^- = (A \cap U)^-.$$

□

Now, by modifying an argument of Levine [95], we can also prove

Theorem 15.3. *If $\mathcal{R}$ is a nonvoid, topological relator on X and $A \subseteq X$ such that $\mathrm{cl}_{\mathcal{R}}(U) = \mathrm{cl}_{\mathcal{R}}(A \cap U)$ for all $U \in \mathcal{T}_{\mathcal{R}}$, then $A \in \mathcal{D}_{\mathcal{R}}$.*

Proof. Assume on the contrary that $A \notin \mathcal{D}_{\mathcal{R}}$. Then, by Definition 4.1, there exists $x \in X$ such that $x \notin A^-$. Thus, by Definition 4.1, there exists $R \in \mathcal{R}$ such that $A \cap R(x) = \emptyset$. Moreover, by Definition 10.1, there exists $U \in \mathcal{T}_{\mathcal{R}}$ such that $x \in U \subseteq R(x)$. Thus, in particular we also have $A \cap U = \emptyset$.

Hence, by using the assumptions of the theorem, we can infer that

$$U^- = (A \cap U)^- = \emptyset^- = \emptyset.$$

Moreover, from the inclusion $x \in U$, by using Theorems 10.3 and 9.3 and the increasingness of $-$, we can infer that $x \in \{x\}^- \subseteq U^-$, and thus $U^- \neq \emptyset$. This contradiction proves that $A \in \mathcal{D}_{\mathcal{R}}$. □

Remark 15.4. If $\mathcal{R}$ is a nonvoid, reflexive relator on X and $A \subseteq X$ such that $\mathrm{cl}_{\mathcal{R}}(R(x)) = \mathrm{cl}_{\mathcal{R}}(A \cap R(x))$ for all $x \in X$ and $R \in \mathcal{R}$, then we can even more easily prove that $A \in \mathcal{D}_{\mathcal{R}}$.

In addition to Theorem 15.3, we can also prove the following

Theorem 15.5. *If $\mathcal{R}$ is a topologically filtered, topological relator on X, then for any $U \in \mathcal{T}_{\mathcal{R}}$ we have*

$$\mathrm{res}_{\mathcal{R}}(U) \in \mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}.$$

Proof. By Theorem 5.5, we have $U^c \in \mathcal{F}_{\mathcal{R}}$. Moreover, by Theorems 10.3 and 10.2, we also have $U^- \in \mathcal{F}_{\mathcal{R}}$. Hence, by using Corollary 12.6, we can already infer that

$$U^\dagger = U^- \setminus U = U^- \cap U^c \in \mathcal{F}_{\mathcal{R}}.$$

Moreover, by using Theorems 12.5 and 9.3 and the increasingness of $-$, we can also see that

$$U^{\dagger\circ} = (U^- \setminus U)^\circ = (U^- \cap U^c)^\circ = U^{-\circ} \cap U^{c\circ} = U^{-\circ} \cap U^{-c} \subseteq U^- \cap U^{-c} = \emptyset,$$

and thus $U^{\dagger\circ} = \emptyset$. Therefore, $U^\dagger \notin \mathcal{E}_{\mathcal{R}}$, and thus $U^\dagger \in \mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}$. □

Now, as an immediate consequence of this theorem, we can also state

Corollary 15.6. *If $\mathcal{R}$ is a topologically filtered, topological relator on X, then $\mathrm{res}_{\mathcal{R}}(U) \in \mathcal{N}_{\mathcal{R}}$ for all $U \in \mathcal{T}_{\mathcal{R}}$.*

Remark 15.7. Note that if $\mathcal{R}$ is a topological relator on X and $U \in \mathcal{T}_{\mathcal{R}}$, then by Theorems 10.3, 10.2 and 9.3 we have $U = U^\circ$. Therefore, under the notation $U^\ddagger = \mathrm{bnd}_{\mathcal{R}}(U)$, we also have $U^\dagger = U^- \setminus U = U^- \setminus U^\circ = U^\ddagger$.

Moreover, in Theorem 15.5 and Corollary 15.6, it is also enough to assume only that $\mathcal{R}$ is a quasi-topologically filtered, topological relator on X. Namely, in this case, $\mathcal{R}$ is already topologically filtered by Theorem 13.8.

16. A Few Basic Facts on Simple Relators

Definition 16.1. A relator $\mathcal{R}$ on X to Y is called *properly simple* if it is a singleton relator. That is, there exists a relation R on X to Y such that $\mathcal{R} = \{R\}$.

Much more generally, the relator $\mathcal{R}$ is called $\mathfrak{F}$*–simple*, for some structure $\mathfrak{F}$ for relators on X to Y, if it is $\mathfrak{F}$–equivalent to a singleton relator. That is, there exists a relation R on X to Y such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_R$.

Thus, by Theorems 6.4 and 8.12, for instance, we can at once state the following two theorems.

Theorem 16.2. *For a relator $\mathcal{R}$ on X on to Y, the following assertions are equivalent:*
(1) $\mathcal{R}$ is $\wedge$–simple; *(2) $\mathcal{R}$ is* cl*–simple;* *(3) $\mathcal{R}$ is* int*–simple.*

Theorem 16.3. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\wedge\infty$–simple; *(2) $\mathcal{R}$ is $\mathcal{F}$–simple;* *(3) $\mathcal{R}$ is $\mathcal{T}$–simple.*

Remark 16.4. Thus, for instance, the relator $\mathcal{R}$ may be naturally called *topologically simple* if it is $\wedge$–simple.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then $\mathcal{R}$ may be naturally called *quasi-topologically simple* if it is $\wedge\infty$–simple.

Now, for instance, we can also easily prove the following

Theorem 16.5. *If $\mathcal{R}$ is a relator on X on to Y, then under the notation $R = \bigcap \mathcal{R}$ the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically simple; *(2) $R \in \mathcal{R}^{\wedge}$;* *(3) $\mathcal{R}^{\wedge} = \{R\}^{\wedge}$.*

Proof. If (1) holds, then by the corresponding definitions there exists a relation S on X to Y such that $\mathcal{R}^{\wedge} = \{S\}^{\wedge}$. Hence, by using Corollary 4.10 and Theorem 6.4, we can infer that

$$R^{-1}(y) = \rho_{\mathcal{R}}(y) = \mathrm{cl}_{\mathcal{R}}(\{y\}) = \mathrm{cl}_S(\{y\}) = \rho_S(y) = S^{-1}(y)$$

for all $y \in Y$. Therefore, $R^{-1} = S^{-1}$, and thus also $R = S$. Consequently, we have $\mathcal{R}^{\wedge} = \{S\}^{\wedge} = \{R\}^{\wedge}$, and thus (3) also holds.

Now, since, (3) trivially implies (1), we need only show that (2) and (3) are also equivalent. For this, note that by Definition 6.1 we always have $R \in \{R\}^{\wedge}$ and $\mathcal{R} \subseteq \{R\}^{\wedge}$. Hence, by the corresponding properties of $\wedge$, it is clear that $\mathcal{R}^{\wedge} \subseteq \{R\}^{\wedge\wedge}$, and thus $\mathcal{R}^{\wedge} \subseteq \{R\}^{\wedge}$. Moreover, we can also note that

$$(2) \implies \{R\} \subseteq \mathcal{R}^{\wedge} \implies \{R\}^{\wedge} \subseteq \mathcal{R}^{\wedge\wedge} \implies \{R\}^{\wedge} \subseteq \mathcal{R}^{\wedge}.$$

□

Remark 16.6. Note that, by Definition 6.1, for any relation R on X to Y we actually have $\{R\}^{\wedge} = \{R\}^{\#} = \{R\}^{*}$.

While, for any relation S on X to Y, we have $S \in \{R\}^{\triangle}$ if and only if for each $x \in X$ there exists $\varphi(x) \in X$ such that $R(\varphi(x)) \subseteq S(x)$. That is, there exists a function φ of X to itself such that $R \circ \varphi \subseteq S$. Therefore, $\{R\}^{\triangle} = (R \circ X^X)^{*}$.

In addition to Theorem 16.5, we can also easily prove the following

Theorem 16.7. *If $\mathcal{R}$ is a nonvoid relator on X to Y, then under the notation $R = \bigcap \mathcal{R}$, we have*

$$\mathcal{R}^{\vee\wedge} = \{R^{-1}\}^{\wedge}.$$

Proof. By Definitions 4.1 and 6.10 and Theorems 4.8, 7.1 and 4.9 and Corollary 4.10, for any $y \in Y$ and $A \subseteq X$, we have

$$y \in \mathrm{cl}_{\mathcal{R}^{\vee}}(A) \iff \{y\} \in \mathrm{Cl}_{\mathcal{R}^{\vee}}(A) \iff \{y\} \in \mathrm{Cl}_{\mathcal{R}^{\wedge -1}}(A) \iff \{y\} \in \mathrm{Cl}_{\mathcal{R}^{\wedge}}^{-1}(A) \iff A \in \mathrm{Cl}_{\mathcal{R}^{\wedge}}(\{y\}) \iff$$
$$A \cap \mathrm{cl}_{\mathcal{R}}(\{y\}) \neq \emptyset \iff A \cap \rho_{\mathcal{R}}(y) \neq \emptyset \iff A \cap R^{-1}(y) \neq \emptyset \iff y \in R[A] \iff y \in \mathrm{cl}_{R^{-1}}(A).$$

Therefore, $\mathrm{cl}_{\mathcal{R}^{\vee}} = \mathrm{cl}_{R^{-1}}$, and thus by Theorem 6.4 the required equality is true. □

Now, as an immediate consequence of this theorem, we can also state

Corollary 16.8. *For any nonvoid relator $\mathcal{R}$ on X to Y, the relator $\mathcal{R}^\vee$ is topologically simple.*

Remark 16.9. Note that if in particular $\mathcal{R} = \emptyset$, but $X \neq \emptyset$, then by Definition 6.1 we have $\mathcal{R}^\wedge = \emptyset$, and thus also $\mathcal{R}^\vee = \mathcal{R}^{\wedge -1} = \emptyset$ which cannot be topologically simple.

17. A Few Basic Facts on Symmetric Relators

In contrast to the reflexivity property of a relator $\mathcal{R}$ on X, we may naturally introduce a great abundance of important symmetry and transitivity properties of the relator $\mathcal{R}$ [148, 149].

For instance, the relator $\mathcal{R}$ may be naturally called *strongly symmetric* if each member of $\mathcal{R}$ is symmetric. Moreover, the relator $\mathcal{R}$ may be naturally called *weakly symmetric* if the relation $R = \bigcap \mathcal{R}$ is symmetric.

However, it is now more important to note that we may also naturally have

Definition 17.1. A relator R on X is called

(1) *properly symmetric* if $R \in \mathcal{R}$ implies $R^{-1} \in \mathcal{R}$;
(2) *uniformly symmetric* if for each $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S \subseteq R^{-1}$;
(3) *proximally symmetric* if for each $A \subseteq X$ and $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S[A] \subseteq R^{-1}[A]$;
(4) *topologically symmetric* if for each $x \in X$ and $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S(x) \subseteq R^{-1}(x)$.

Remark 17.2. By using the operations -1, $*$, $\#$ and $\wedge$, the above properties can be reformulated in more concise forms.

For instance, by using some basic properties of $\#$, we can easily prove

Theorem 17.3. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximally symmetric; *(2) $\mathcal{R}^{-1} \subseteq \mathcal{R}^\#$;*
(3) $\mathcal{R}^{-1\,\#} = \mathcal{R}^\#$; *(4) $\mathcal{R}^{\#-1} \subseteq \mathcal{R}^\#$;* *(5) $\mathcal{R}^{\#-1} = \mathcal{R}^\#$.*

Thus, in particular we can also state the following

Corollary 17.4. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}^\#$ is properly symmetric;

(2) $\mathcal{R}$ is proximally symmetric;

(3) $\mathcal{R}^{-1}$ is proximally symmetric;

(4) $\mathcal{R}$ and $\mathcal{R}^{-1}$ are proximally equivalent.

Remark 17.5. Concerning topological symmetry, we can only prove that $\mathcal{R}$ is topologically symmetric if and only if $\mathcal{R}^{-1} \subseteq \mathcal{R}^\wedge$, or equivalently $\mathcal{R}^{-1\,\wedge} \subseteq \mathcal{R}^\wedge$.

Hence, it is clear that, topological symmetry is already is not a genuine symmetry property. Therefore, in addition to Definition 17.1, we must also have

Definition 17.6. A relator $\mathcal{R}$ on X is called

(1) *properly topologically symmetric* if $\mathcal{R}^\wedge$ is properly symmetric;
(2) *topologically bisymmetric* if both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are topologically symmetric.

Remark 17.7. Thus, by Remark 17.5, we can see that $\mathcal{R}$ is topologically bisymmetric if and only if $\mathcal{R}^\wedge = \mathcal{R}^{-1\,\wedge}$. That is, $\mathcal{R}$ and $\mathcal{R}^{-1}$ are topologically equivalent.

Moreover, by using Definitions 17.6 and 6.10 and Theorem 16.7, we can also easily prove the following two theorems.

Theorem 17.8. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly topologically symmetric; *(2) $\mathcal{R}^{\vee} \subseteq \mathcal{R}^{\wedge}$;*

(3) $\mathcal{R}^{\vee} = \mathcal{R}^{\wedge}$; *(4) $\mathcal{R}^{\vee\wedge} = \mathcal{R}^{\wedge}$;* *(5) $\mathcal{R}^{\vee\vee} = \mathcal{R}^{\vee}$.*

Theorem 17.9. *If $\mathcal{R}$ is a nonvoid relator on X, then under the notation $R = \bigcap \mathcal{R}$ the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly topologically symmetric;

(2) $R^{-1} \in \mathcal{R}^{\wedge}$; *(3) $\mathcal{R}^{\wedge} = \{R^{-1}\}^{\wedge}$.*

Proof. If (1) holds, then by Theorems 17.8 and 16.7 we have $\mathcal{R}^{\wedge} = \mathcal{R}^{\vee\wedge} = \{R^{-1}\}^{\wedge}$, and thus (3) holds.

While, if (3) holds, then

$$\mathcal{R}^{\wedge -1} = \{R^{-1}\}^{\wedge -1} = \{R^{-1}\}^{* -1} = \{R^{-1}\}^{-1 *} = \{R\}^{*} = \{R\}^{\wedge},$$

and thus (1) also holds. The equivalence of (2) and (3) is even more obvious. □

From this theorem, we can easily derive the following

Corollary 17.10. *For a nonvoid relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly topologically symmetric;

(2) $\mathcal{R}$ is topologically simple and weakly symmetric.

Proof. To derive the symmetry of $R = \bigcap \mathcal{R}$ from (1), note that if (1) holds, then by Corollary 4.10, Definition 4.1 and Theorems 17.9 and 6.4, we have

$$R^{-1} = \rho_{\mathcal{R}} = \rho_{\mathcal{R}^{\wedge}} = \rho_{\{R^{-1}\}^{\wedge}} = \rho_{R^{-1}} = R.$$

□

Remark 17.11. According to Remark 17.5, a relator $\mathcal{R}$ on X may be naturally called *paratopologically symmetric* if $\mathcal{R}^{-1} \subseteq \mathcal{R}^{\triangle}$, or equivalently $\mathcal{R}^{-1\triangle} \subseteq \mathcal{R}^{\triangle}$.

Moreover, by Definition 17.6, the relator $\mathcal{R}$ may, for instance, be naturally called *properly paratopologically symmetric* if the relator $\mathcal{R}^{\triangle}$ is properly symmetric.

On the other hand, by Remark 17.5, the relator $\mathcal{R}$ may also be naturally called *quasi-topologically symmetric* if $\mathcal{R}^{-1\wedge\infty} \subseteq \mathcal{R}^{\wedge}$, or equivalently $\mathcal{R}^{-1\wedge\infty} \subseteq \mathcal{R}^{\wedge\infty}$.

18. Some Further Theorems on Symmetric Relators

By Theorems 6.4 and 8.12, we can at once state the following two theorems.

Theorem 18.1. *For a relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically symmetric;

(2) $\mathrm{cl}_{\mathcal{R}} \subseteq \mathrm{cl}_{\mathcal{R}^{-1}}$; *(3) $\mathrm{int}_{\mathcal{R}^{-1}} \subseteq \mathrm{int}_{\mathcal{R}}$.*

Theorem 18.2. *For a nonvoid relator $\mathcal{R}$ on X, the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-topologically symmetric;

(2) $\mathcal{T}_{\mathcal{R}^{-1}} \subseteq \mathcal{T}_{\mathcal{R}}$; *(3) $\mathcal{F}_{\mathcal{R}^{-1}} \subseteq \mathcal{F}_{\mathcal{R}}$.*

Remark 18.3. Thus, in particular we can also state $\mathcal{R}$ is quasi-topologically bisymmetric if and only if $\mathcal{T}_{\mathcal{R}^{-1}} = \mathcal{T}_{\mathcal{R}}$, or equivalently $\mathcal{F}_{\mathcal{R}^{-1}} = \mathcal{F}_{\mathcal{R}}$.

However, it is now more important to note that, by using our former results, we can also prove the following three theorems.

Theorem 18.4. *If $\mathcal{R}$ is a nonvoid relator on X, then under the notation $R = \bigcap \mathcal{R}$ the following assertions are equivalent :*

(1) $\mathcal{R}$ *is properly topologically symmetric ;*

(2) $\operatorname{cl}_{\mathcal{R}}(A) = R[A]$ *for all* $A \subseteq X$*;*

(3) $\operatorname{int}_{\mathcal{R}}(A) = R[A^c]^c$ *for all* $A \subseteq X$.

Proof. To prove the implication (1) $\Longrightarrow$ (2), note that if (1) holds, then by Theorem 16.7 we have $\mathcal{R}^{\wedge} = \{R^{-1}\}^{\wedge}$. Hence, by using Theorems 6.4 and 4.9 we can infer that $\operatorname{cl}_{\mathcal{R}}(A) = \operatorname{cl}_{R^{-1}}(A) = R[A]$ for all $A \subseteq X$. Therefore, (2) also holds. □

Theorem 18.5. *For a nonvoid relator $\mathcal{R}$ on X, the following assertions are equivalent :*

(1) $\mathcal{R}$ *is properly topologically symmetric ;*

(2) $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$ *implies* $B^c \subseteq \operatorname{int}_{\mathcal{R}}(A^c)$ *for all* $A, B \subseteq X$*;*

(3) $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ *implies* $B \cap \operatorname{cl}_{\mathcal{R}}(A) \neq \emptyset$ *for all* $A, B \subseteq X$.

Proof. To prove the implication (1) $\Longrightarrow$ (3), note that if (1) holds and $R = \bigcap \mathcal{R}$, then by Theorem 18.4 and Corollary 17.10

$$A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset \implies A \cap R[B] \neq \emptyset \implies R^{-1}[A] \cap B \neq \emptyset \implies R[A] \cap B \neq \emptyset \implies \operatorname{cl}_{\mathcal{R}}(A) \cap B \neq \emptyset$$

for all $A, B \subseteq X$. Therefore, (3) also holds. □

Theorem 18.6. *For a nonvoid relator $\mathcal{R}$ on X, the following assertions are equivalent :*

(1) $\mathcal{R}$ *is properly topologically symmetric ;*

(2) $A \subseteq \operatorname{int}_{\mathcal{R}}(\operatorname{cl}_{\mathcal{R}}(A))$ *for all* $A \subseteq X$*;*

(3) $\operatorname{cl}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(A)) \subseteq A$ *for all* $A \subseteq X$.

Proof. To prove the implication (1) $\Longrightarrow$ (2), note that if $A \subseteq X$, then by Theorem 4.5 we have $\operatorname{cl}_{\mathcal{R}}(A)^c = \operatorname{int}_{\mathcal{R}}(A^c)$, and thus also $\operatorname{cl}_{\mathcal{R}}(A)^c \subseteq \operatorname{int}_{\mathcal{R}}(A^c)$. Therefore, if (1) holds, then by Theorem 18.5 we also have $A \subseteq \operatorname{int}_{\mathcal{R}}(\operatorname{cl}_{\mathcal{R}}(A))$. Thus, (2) also holds. □

Remark 18.7. The latter three theorems can be naturally generalized to topologically simple relators.

For instance, it can be shown that a nonvoid relator on X to Y is topologically simple if and only if under the notation $\mathcal{S} = \mathcal{R}^{-1}$ or $\mathcal{R}^{\vee}$ we have $A \subseteq \operatorname{int}_{\mathcal{R}}(\operatorname{cl}_{\mathcal{S}}(A))$ for all $A \subseteq X$.

19. Some Generalized Topologically Open Sets

Notation 19.1. *In the sequel, we shall always assume that X is a set and $\mathcal{R}$ is a relator on X.*

Moreover, to shorten the subsequent proofs, we shall use the notations

$$A^- = \operatorname{cl}_{\mathcal{R}}(A), \qquad A^\circ = \operatorname{int}_{\mathcal{R}}(A) \qquad \text{and} \qquad A^\dagger = \operatorname{res}_{\mathcal{R}}(A).$$

Parts (2) and (3) of the following definition has been mainly suggested by [94, Theorem 1 and Definition 1] of Levine.

While, for the motivations of parts (1) and (4), see Corson and Michael [26], Mashhour et al. [102], and a survey by Dontchev [50].

Definition 19.2. A subset A of the relator space $X(\mathcal{R})$ will be called *topologically*

(1) *preopen* if $A \subseteq \operatorname{int}_{\mathcal{R}}(\operatorname{cl}_{\mathcal{R}}(A))$;

(2) *semi-open* if $A \subseteq \operatorname{cl}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(A))$;

(3) *quasi-open* if there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}}(V)$;

(4) *pseudo-open* if there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq \mathrm{cl}_{\mathcal{R}}(A)$.

And, the families of all such subsets A of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{\kappa}$ with $\kappa = p$, s , q and ps , respectively.

Remark 19.3. The inclusions $A \subseteq A^{-\circ}$ and $A \subseteq A^{\circ -}$ mean only that the set A is open with respect to the composite operations $- \circ$ and $\circ -$, respectively.

While, the inclusions $V \subseteq A \subseteq V^{-}$ and $A \subseteq V \subseteq A^{-}$ mean that A is near to V from above and below, or can be approximated by V from below and above.

The next simple example shows that, even in a very particular case, the families $\mathcal{T}_{\mathcal{R}}^{q}$ and $\mathcal{T}_{\mathcal{R}}^{ps}$ may be strictly smaller than the families $\mathcal{T}_{\mathcal{R}}^{s}$ and $\mathcal{T}_{\mathcal{R}}^{p}$, respectively.

Example 19.4. If $X = \{1, 2\}$ and R is a relation on X such that

$$R(1) = \{2\} \quad \text{and} \quad R(2) = \{1\}$$

then R is an injective function of X onto itself, with $R^{-1} = R$, such that

(1) $\mathcal{T}_{R} = \mathcal{T}_{R}^{q} = \mathcal{T}_{R}^{ps} = \{\emptyset , X\}$; (2) $\mathcal{T}_{R}^{s} = \mathcal{T}_{R}^{p} = \mathcal{P}(X)$.

To check this, note that now, for any $A \subseteq X$, we have $R[A^{c}] = R[A]^{c}$. Therefore, by Theorems 4.9 and 4.5, we can state that

$$A^{-} = R^{-1}[A] = R[A] \quad \text{and} \quad A^{\circ} = A^{c-c} = R[A^{c}]^{c} = R[A]^{cc} = R[A] .$$

Thus, by the corresponding definitions, we have $\mathcal{T}_{R} = \{\emptyset, X\}$ and $\mathcal{T}_{R}^{q} = \mathcal{T}_{R}^{ps} = \{\emptyset, X\}$. Moreover, for any $A \subseteq X$, we also have

$$A^{\circ -} = R[R[A]] = (R \circ R)[A] = (R \circ R^{-1})[A] = \Delta_{X}[A] = A ,$$

and quite similarly $A^{-\circ} = A$. Therefore, $\mathcal{T}_{R}^{s} = \mathcal{T}_{R}^{p} = \mathcal{P}(X)$ also holds.

However, the appropriateness of Definition 19.2 can only be completely clear from the subsequent generalizations of the corresponding topological results.

Theorem 19.5. *We have*

(1) $\mathcal{T}_{\mathcal{R}}^{q} \subseteq \mathcal{T}_{\mathcal{R}}^{s}$*;* *(2)* $\mathcal{T}_{\mathcal{R}}^{ps} \subseteq \mathcal{T}_{\mathcal{R}}^{p}$.

Proof. If $A \in \mathcal{T}_{\mathcal{R}}^{q}$, then by Definition 19.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that

$$V \subseteq A \subseteq V^{-}.$$

Hence, by using the definition of $\mathcal{T}_{\mathcal{R}}$ and the increasingness of $\circ$, we can infer that $V \subseteq V^{\circ} \subseteq A^{\circ}$. Now, by using the increasingness of $-$, we can also see that

$$A \subseteq V^{-} \subseteq A^{\circ -}.$$

Therefore, by Definition 19.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^{s}$.

While, if $A \in \mathcal{T}_{\mathcal{R}}^{ps}$, then by Definition 19.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that

$$A \subseteq V \subseteq A^{-}.$$

Hence, by using the definition of $\mathcal{T}_{\mathcal{R}}$ and the increasingness of $\circ$, we can infer that

$$A \subseteq V \subseteq V^{\circ} \subseteq A^{-\circ}.$$

Therefore, by Definition 19.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^{p}$. □

Theorem 19.6. *If* $\mathcal{R}$ *is reflexive relator on* X*, then*

$$\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{q} \cap \mathcal{T}_{\mathcal{R}}^{ps} .$$

Proof. If $A \in \mathcal{T}_{\mathcal{R}}$, then by taking $V = A$ we have $V \in \mathcal{T}_{\mathcal{R}}$. And, by using Theorem 9.3, we can see that

$$V = A \subseteq A^- = V^- \quad \text{and} \quad A = V \subseteq V^- = A^- .$$

Therefore, by Definition 19.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^{q}$ and $A \in \mathcal{T}_{\mathcal{R}}^{ps}$, and thus also $A \in \mathcal{T}_{\mathcal{R}}^{q} \cap \mathcal{T}_{\mathcal{R}}^{ps}$. □

From this theorem, by using Theorem 19.5, we can immediately derive

Corollary 19.7. *If $\mathcal{R}$ is reflexive relator on X, then $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}$.*

20. Further Basic Properties of Generalized Topologically Open Sets

In addition to Theorem 19.5, we can also prove the following

Theorem 20.1. *If $\mathcal{R}$ is a topological relator on X, then*

(1) $\mathcal{T}_{\mathcal{R}}^{q} = \mathcal{T}_{\mathcal{R}}^{s}$; *(2)* $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{T}_{\mathcal{R}}^{p}$.

Proof. If $A \in \mathcal{T}_{\mathcal{R}}^{s}$, then by Definition 19.2, we have $A \subseteq A^{\circ -}$. Hence, by taking $V = A^{\circ}$, we get $A \subseteq V^-$. Moreover, by using Theorems 10.3, 9.3 and 10.2, we can see that

$$V = A^{\circ} \subseteq A \quad \text{and} \quad V = A^{\circ} \in \mathcal{T}_{\mathcal{R}} .$$

Thus, by Definition 19.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^{q}$. This proves that $\mathcal{T}_{\mathcal{R}}^{s} \subseteq \mathcal{T}_{\mathcal{R}}^{q}$.

While, if $A \in \mathcal{T}_{\mathcal{R}}^{p}$, then by Definition 19.2 we have $A \subseteq A^{-\circ}$. Hence, by taking $V = A^{-\circ}$, we get $A \subseteq V$. Moreover, by using Theorems 10.3, 9.3 and 10.2, we can see that

$$V = A^{-\circ} \subseteq A^- \quad \text{and} \quad V = A^{-\circ} \in \mathcal{T}_{\mathcal{R}} .$$

Thus, by Definition 19.2, we also have $A \in \mathcal{T}_{\mathcal{R}}^{ps}$. This proves that $\mathcal{T}_{\mathcal{R}}^{p} \subseteq \mathcal{T}_{\mathcal{R}}^{ps}$.

Now, by Theorem 19.5, we can see that assertions (1) and (2) are also true. □

By using our former results on topological relators, we can also prove

Theorem 20.2. *If $\mathcal{R}$ is a topological relator on X, then for any $A \subseteq X$, the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{s}$;

(2) $\mathrm{cl}_{\mathcal{R}}(A) = \mathrm{cl}_{\mathcal{R}}(\mathrm{int}_{\mathcal{R}}(A))$;

(3) $\mathrm{cl}_{\mathcal{R}}(A) = \mathrm{cl}_{\mathcal{R}}(V)$ *for some* $V \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$.

Proof. If (1) holds, then by Definition 19.2 we have $A \subseteq A^{\circ -}$. Hence, by using the increasingness of $-$, we can infer that $A^- \subseteq A^{\circ --}$. Moreover, from Theorems 10.3 and 10.2, we can see that $A^{\circ -} \in \mathcal{F}_{\mathcal{R}}$. Thus, by the definition of $\mathcal{F}_{\mathcal{R}}$, we also have $A^{\circ --} \subseteq A^{\circ -}$. Therefore, $A^- \subseteq A^{\circ -}$.

On the other hand, by Theorems 10.3 and 9.3, we have $A^{\circ} \subseteq A$. Hence, by using the increasingness of $-$, we can infer that $A^{\circ -} \subseteq A^-$. Therefore, we actually have $A^- = A^{\circ -}$, and thus assertion (2) also holds.

While, if (2) holds, i. e., $A^- = A^{\circ -}$, then by taking $V = A^{\circ}$ we get $A^- = V^-$. Moreover, by Theorems 10.3, 9.3 and 10.2, we can see that $V \subseteq A$ and $V \in \mathcal{T}_{\mathcal{R}}$, and thus $V \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$. Therefore, (3) also holds.

Finally, if (3) holds, then there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A$ and $A^- = V^-$. Hence, by using Theorems 10.3 and 9.3, we can see that $A \subseteq A^- = V^-$. Therefore, by Definition 19.2, we have $A \in \mathcal{T}_{\mathcal{R}}^{q}$. Thus, by Theorem 19.5, assertion (1) also holds. □

Remark 20.3. Note that if $\mathcal{R}$ is a topological relator on X, then the family $\mathcal{T}_{\mathcal{R}}$ need not be closed under finite intersections.

Therefore, Theorem 20.2 is a generalization of the corresponding observations of Njastad [110, p.961], Isomichi [75, Theorem 2], Noiri [111, Lemma 2] and Pipitone and Russo [123, Lemma 2.2].

Njåstad and Isomichi, being not aware of Levine's paper [94], investigated semi-open sets under the names "β–sets" and "subcondensed sets", respectively. While, Noiri, Pipitone and Russo already used Levine's terminology.

Theorem 20.2 is also a certain generalization of [6, Proposition 3.2] of Al-shami since, for every supra topology (generalized topology) $\mathcal{T}$ on X, there exists a nonvoid preorder relator $\mathcal{R}$ on X such that $\mathcal{T} = \mathcal{T}_{\mathcal{R}}$ [159].

Now, as an immediate consequence of Theorem 20.2, we can also state

Corollary 20.4. *If $\mathcal{R}$ is a topological relator on X, then for any $A \subseteq X$, the following assertions are equivalent:*

(1) $A \in \mathcal{F}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{R}}^{s}$;

(2) $A = \operatorname{cl}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(A))$;

(3) $A = \operatorname{cl}_{\mathcal{R}}(V)$ *for some* $V \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$.

21. Another Important Property of Topologically Semi-Open and Quasi-Open Sets

By using similar arguments as in the proof of Theorem 20.2, we can also prove the following two generalizations of [94, Theorem 3] of Levine.

Theorem 21.1. *If $\mathcal{R}$ is a quasi-topological relator on X,*

$$A \in \mathcal{T}_{\mathcal{R}}^{s} \quad \text{and} \quad A \subseteq B \subseteq \operatorname{cl}_{\mathcal{R}}(A),$$

then $B \in \mathcal{T}_{\mathcal{R}}^{s}$ also holds.

Proof. By Definition 19.2, we have $A \subseteq A^{\circ -}$. Hence, by using the increasingness of $-$, we can infer that $A^{-} \subseteq A^{\circ --}$. Moreover, by Theorem 10.2, we now also have $A^{\circ -} \in \mathcal{F}_{\mathcal{R}}$. Hence, by using the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that $A^{\circ --} \subseteq A^{\circ -}$. Therefore, we also have $A^{-} \subseteq A^{\circ -}$.

On the other hand, because of the inclusion $A \subseteq B$ and the increasingness properties of $\circ$ and $-$, we also have $A^{\circ -} \subseteq B^{\circ -}$. Therefore, $A^{-} \subseteq B^{\circ -}$ also holds. Hence, because of $B \subseteq A^{-}$, we can already see that $B \subseteq B^{\circ -}$. Thus, by Definition 19.2, the required assertion is also true. □

Theorem 21.2. *If $\mathcal{R}$ is a quasi-topological relator on X,*

$$A \in \mathcal{T}_{\mathcal{R}}^{q} \quad \text{and} \quad A \subseteq B \subseteq \operatorname{cl}_{\mathcal{R}}(A),$$

then $B \in \mathcal{T}_{\mathcal{R}}^{q}$ also holds.

Proof. By Definition 19.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A \subseteq V^{-}$. Hence, since $A \subseteq B$, it is clear that $V \subseteq B$. Moreover, by using the increasingness of $-$, we can also see that $A^{-} \subseteq V^{--}$.

On the other hand, by Theorem 10.2, we now also have $V^{-} \in \mathcal{F}_{\mathcal{R}}$. Hence, by the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that $V^{--} \subseteq V^{-}$. Therefore, $A^{-} \subseteq V^{-}$ also holds. Hence, because of $B \subseteq A^{-}$, we can already see that $B \subseteq V^{-}$ also holds. Thus, by Definition 19.2, the required assertion is also true. □

Remark 21.3. Note that, in the above proof, we have only used an apparently weaker property, than the quasi-topologicalness of $\mathcal{R}$.

Moreover, we have proved a little more than that was stated. Namely, that the "approximating set" V used for the set B does not depend on B.

Now, as an immediate consequence of Theorem 21.1, we can also state

Theorem 21.4. *If $\mathcal{R}$ is a topological relator on X, then for every $A \in \mathcal{T}_{\mathcal{R}}^{s}$ we have* $\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}^{s}$.

Proof. To derive this from Theorem 21.1, note that now by Theorems 10.3 the relator $\mathcal{R}$ is reflexive and quasi-topological. Moreover, by Theorem 9.3, we have $A \subseteq A^{-}$, and thus also $A \subseteq A^{-} \subseteq A^{-}$. Therefore, by Theorem 21.1, we also have $A^{-} \in \mathcal{T}_{\mathcal{R}}^{s}$. □

From this theorem, by using Corollary 19.7, we can immediately derive

Corollary 21.5. *If $\mathcal{R}$ is a topological relator on X, then for every $V \in \mathcal{T}_{\mathcal{R}}$ we have* $\operatorname{cl}_{\mathcal{R}}(V) \in \mathcal{T}_{\mathcal{R}}^{s}$.

Hence, by using Theorems 10.2 and 10.3, we can immediately derive

Corollary 21.6. *If $\mathcal{R}$ is a topological relator on X, then for every $A \subseteq X$ we have* $\mathrm{cl}_{\mathcal{R}}(\,\mathrm{int}_{\mathcal{R}}(A)) \in \mathcal{T}_{\mathcal{R}}^{\,s}$.

Now, by using Theorems 19.6 and 21.1, we can also prove the following generalization [94, Theorem 5] of Levine.

Theorem 21.7. *If $\mathcal{R}$ is a topological relator on X, then $\mathcal{A} = \mathcal{T}_{\mathcal{R}}^{\,s}$ is the smallest family of subsets of X such that*

(1) $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{A}$; *(2) $A \in \mathcal{A}$ and $A \subseteq B \subseteq \mathrm{cl}_{\mathcal{R}}(A)$ imply $B \in \mathcal{A}$.*

Proof. To prove the stated minimality property of $\mathcal{T}_{\mathcal{R}}^{\,s}$, note that if $A \in \mathcal{T}_{\mathcal{R}}^{\,s}$, then by Theorem 20.1 we also have $A \in \mathcal{T}_{\mathcal{R}}^{\,q}$. Thus, Definition 19.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $V \subseteq A \subseteq V^{-}$. Therefore, if $\mathcal{A}$ is a family of subsets of X such that (1) holds, then we have $V \in \mathcal{A}$. Moreover, if (2) also holds, then we also have $A \in \mathcal{A}$. Therefore, $\mathcal{T}_{\mathcal{R}}^{\,s} \subseteq \mathcal{A}$. □

Moreover, we can also prove the following generalization of [94, Lemma 2] of Levine.

Theorem 21.8. *If $\mathcal{R}$ is a topological relator on X, then*

$$\mathcal{T}_{\mathcal{R}} = \{\, \mathrm{int}_{\mathcal{R}}(A) : \quad A \in \mathcal{T}_{\mathcal{R}}^{\,s} \,\}.$$

Proof. Now, by Theorem 10.3, $\mathcal{R}$ is reflexive and quasi-topological. Thus, if in particular $V \in \mathcal{T}_{\mathcal{R}}$, then by Corollary 19.7 we have also have $V \in \mathcal{T}_{\mathcal{R}}^{\,s}$.

Moreover, by the definition of $\mathcal{T}_{\mathcal{R}}$ and Theorem 9.3, we also have $V \subseteq V^{\circ}$ and $V^{\circ} \subseteq V$, and thus also $V = V^{\circ}$. Therefore, $V \in (\mathcal{T}_{\mathcal{R}}^{\,s})^{\circ}$, and thus $\mathcal{T}_{\mathcal{R}} \subseteq (\mathcal{T}_{\mathcal{R}}^{\,s})^{\circ}$ also holds.

On the other hand, by Theorem 10.2, we have $A^{\circ} \in \mathcal{T}_{\mathcal{R}}$ for all $A \subseteq X$. Thus, $\mathcal{P}(X)^{\circ} \subseteq \mathcal{T}_{\mathcal{R}}$, and thus in particular $(\mathcal{T}_{\mathcal{R}}^{\,s})^{\circ} \subseteq \mathcal{T}_{\mathcal{R}}$ also holds. Therefore, the required equality $\mathcal{T}_{\mathcal{R}} = (\mathcal{T}_{\mathcal{R}}^{\,s})^{\circ}$ is also true. □

Remark 21.9. Note that, because of Theorem 20.1, here we may also write $\mathcal{T}_{\mathcal{R}}^{\,q}$ in place of $\mathcal{T}_{\mathcal{R}}^{\,s}$.

22. Another Important Property of Topologically Preopen and Pseudo-Open Sets

Analogously to Theorems 21.1 and 21.2, we can also prove the following two theorems.

Theorem 22.1. *If $\mathcal{R}$ is a quasi-topological relator on X,*

$$A \in \mathcal{T}_{\mathcal{R}}^{\,p} \quad and \quad B \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}}(B),$$

then $B \in \mathcal{T}_{\mathcal{R}}^{\,p}$ also holds.

Proof. By Definition 19.2, we have $A \subseteq A^{-\circ}$. Hence, by using that $B \subseteq A$, we can see that $B \subseteq A^{-\circ}$.

Moreover, from the inclusion $A \subseteq B^{-}$, by using the increasingness of $-$ and Theorem 10.2, we can infer that

$$A^{-} \subseteq B^{--} \subseteq B^{-}.$$

Hence, by using the increasingness of $\circ$, we can infer that $A^{-\circ} \subseteq B^{-\circ}$. Therefore, because of $B \subseteq A^{-\circ}$, we also have $B \subseteq B^{-\circ}$. Hence, by Definition 19.2, we can see that $B \in \mathcal{T}_{\mathcal{R}}^{\,p}$ also holds. □

Theorem 22.2. *If $\mathcal{R}$ is a quasi-topological relator on X,*

$$A \in \mathcal{T}_{\mathcal{R}}^{\,sp} \quad and \quad B \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}}(B),$$

then $B \in \mathcal{T}_{\mathcal{R}}^{\,sp}$ also holds.

Proof. By Definition 19.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq A^{-}$. Hence, by using that $B \subseteq A$, we can see that $B \subseteq V$.

Moreover, from the inclusion $A \subseteq B^{-}$, by using the increasingness of $-$ and Theorem 10.2, we can infer that

$$A^{-} \subseteq B^{--} \subseteq B^{-}.$$

Hence, by using the inclusion $V \subseteq A^{-}$, we can see that $V \subseteq B^{-}$. Therefore, we actually have $B \subseteq V \subseteq B^{-}$. Hence, by Definition 19.2, we can see that $B \in \mathcal{T}_{\mathcal{R}}^{\,sp}$ also holds. □

Remark 22.3. Again, we have proved a little more than that was stated. Namely, that the "approximating set" V used for the set B does not depend on B.

Instead of some counterparts of Theorem 21.4 and its corollaries, we shall only prove here the following two theorems.

Theorem 22.4. *If $\mathcal{R}$ is a nonvoid relator on X, then $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\,p}$.*

Proof. If $A \in \mathcal{D}_{\mathcal{R}}$, then by the definition of $\mathcal{D}_{\mathcal{R}}$ we have $A^- = X$. Moreover, since $\mathcal{R} \neq \emptyset$, by the definition of $\circ$, we also have $X^\circ = X$. Therefore, we actually have $A^{-\circ} = X^\circ = X$. Thus, the required inclusion $A \subseteq A^{-\circ}$ trivially holds. □

Theorem 22.5. *If $\mathcal{R}$ is a topologically filtered relator on X and $A = V \cap B$ for some $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{D}_{\mathcal{R}}$, then $A \in \mathcal{T}_{\mathcal{R}}^{\,ps}$.*

Proof. In this case, we have $A \subseteq V$. Moreover, by the definition of $\mathcal{D}_{\mathcal{R}}$ and Corollary 14.4, we also have

$$V = X \cap V = B^- \cap V \subseteq (B \cap V)^- = A^- .$$

Therefore, we actually have $A \subseteq V \subseteq A^-$. Thus, by Definition 19.2, we have $A \in \mathcal{T}_{\mathcal{R}}^{\,ps}$. □

From this theorem, by taking $V = X$ whenever $\mathcal{R} \neq \emptyset$, we can get

Corollary 22.6. *If $\mathcal{R}$ is a nonvoid, topologically filtered relator on X, then $\mathcal{D}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\,ps}$.*

Now, as some close analogues of Theorems 21.7 and 21.8, we can also prove the following two theorems.

Theorem 22.7. *If $\mathcal{R}$ is a topological relator on X, then $\mathcal{A} = \mathcal{T}_{\mathcal{R}}^{\,p}$ is the smallest family of subsets of X such that*

(1) $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{A}$; *(2) $A \in \mathcal{A}$ and $B \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}}(B)$ imply $B \in \mathcal{A}$.*

Proof. To prove the stated minimality property of $\mathcal{T}_{\mathcal{R}}^{\,p}$, note that if $A \in \mathcal{T}_{\mathcal{R}}^{\,p}$, then by Theorem 20.1 we also have $A \in \mathcal{T}_{\mathcal{R}}^{\,ps}$. Thus, Definition 19.2, there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq A^-$. Therefore, if $\mathcal{A}$ is a family of subsets of X such that (1) holds, then we have $V \in \mathcal{A}$. Moreover, if (2) also holds, then we also have $A \in \mathcal{A}$. Therefore, $\mathcal{T}_{\mathcal{R}}^{\,p} \subseteq \mathcal{A}$. □

Theorem 22.8. *If $\mathcal{R}$ is a topological relator on X, then*

$$\mathcal{T}_{\mathcal{R}} = \{\, \mathrm{int}_{\mathcal{R}}(A) : \quad A \in \mathcal{T}_{\mathcal{R}}^{\,p} \,\}.$$

Proof. Now, by Theorem 10.3, $\mathcal{R}$ is reflexive and quasi-topological. Thus, if in particular $V \in \mathcal{T}_{\mathcal{R}}$, then by Corollary 19.7 we have also have $V \in \mathcal{T}_{\mathcal{R}}^{\,p}$.

Moreover, by the definition of $\mathcal{T}_{\mathcal{R}}$ and Theorem 9.3, we also have $V \subseteq V^\circ$ and $V^\circ \subseteq V$, and thus also $V = V^\circ$. Therefore, $V \in (\mathcal{T}_{\mathcal{R}}^{\,p})^\circ$, and thus $\mathcal{T}_{\mathcal{R}} \subseteq (\mathcal{T}_{\mathcal{R}}^{\,p})^\circ$ also holds.

On the other hand, by Theorem 10.2, we have $A^\circ \in \mathcal{T}_{\mathcal{R}}$ for all $A \subseteq X$. Thus, $\mathcal{P}(X)^\circ \subseteq \mathcal{T}_{\mathcal{R}}$, and thus in particular $(\mathcal{T}_{\mathcal{R}}^{\,p})^\circ \subseteq \mathcal{T}_{\mathcal{R}}$ also holds. Therefore, the required equality $\mathcal{T}_{\mathcal{R}} = (\mathcal{T}_{\mathcal{R}}^{\,p})^\circ$ is also true. □

Remark 22.9. Note that, because of Theorem 20.1, here we may also write $\mathcal{T}_{\mathcal{R}}^{\,ps}$ in place of $\mathcal{T}_{\mathcal{R}}^{\,p}$.

23. Topologically Regular Open Sets

By Kuratowski [87], in addition to Definition 19.2, we may also naturally introduce the following

Definition 23.1. A subset A of the relator space $X(\mathcal{R})$ will be called *topologically regular open* if

$$A = \mathrm{int}_{\mathcal{R}}(\,\mathrm{cl}_{\mathcal{R}}(A)).$$

And, the family of all such subsets of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{\,r}$.

Remark 23.2. Note that thus $\mathcal{T}_{\mathcal{R}}^{\,r}$ is just the family of all fixed points of the composite operation $-\circ = -c - c$.

However, $\mathcal{T}_{\mathcal{R}}^{\,r}$ need not be a subset of $\mathcal{T}_{\mathcal{R}}$. Namely, if X and R are as in Example 19.4, then $\mathcal{T}_R = \{\emptyset, X\}$, but $\mathcal{T}_R^{\,r} = \mathcal{P}(X)$.

Of course, as an immediate consequence of Theorem 10.2, we can state

Theorem 23.3. *If $\mathcal{R}$ is a quasi-topological relator on X, then $\mathcal{T}_{\mathcal{R}}^{\,r} \subseteq \mathcal{T}_{\mathcal{R}}$.*

Moreover, by using the plausible notation $\mathcal{F}_{\mathcal{R}}^{\,s} = \{A^c : \ A \in \mathcal{T}_{\mathcal{R}}^{\,s}\}$, we can prove the following generalization of a statement of Dontchev [50, p. 4]. (See also Ekici [54, Theorem 8] and Jamunarani et al. [77, Theorem 2.2] for some more general results.)

Theorem 23.4. *We have*

$$\mathcal{T}_{\mathcal{R}}^{\,r} = \mathcal{T}_{\mathcal{R}}^{\,p} \cap \mathcal{F}_{\mathcal{R}}^{\,s}.$$

Proof. By Definitions 23.1 and 19.2, it is clear that, for any $A \subseteq X$, we have

$$A \in \mathcal{T}_{\mathcal{R}}^{\,r} \iff A = A^{-\circ} \iff A \subseteq A^{-\circ}, \quad A^{-\circ} \subseteq A \iff A \in \mathcal{T}_{\mathcal{R}}^{\,p}, \quad A^c \subseteq A^{-\circ c}.$$

Moreover, by using the equalities $\circ c = c-$ and $-c = c\circ$, we can see that

$$A^{-\circ c} = A^{-c-} = A^{c\circ-}.$$

Therefore,

$$A^c \subseteq A^{-\circ c} \iff A^c \subseteq A^{c\circ-} \iff A^c \in \mathcal{T}_{\mathcal{R}}^{\,s} \iff A \in \mathcal{F}_{\mathcal{R}}^{\,s}.$$

Consequently,

$$A \in \mathcal{T}_{\mathcal{R}}^{\,r} \iff A \in \mathcal{T}_{\mathcal{R}}^{\,p}, \quad A \in \mathcal{F}_{\mathcal{R}}^{\,s} \iff A \in \mathcal{T}_{\mathcal{R}}^{\,p} \cap \mathcal{F}_{\mathcal{R}}^{\,s},$$

and thus the required equality is true. □

From this theorem, by using Theorem 20.1, we can immediately derive

Corollary 23.5. *If $\mathcal{R}$ is a topological relator on X, then $\mathcal{T}_{\mathcal{R}}^{\,r} = \mathcal{T}_{\mathcal{R}}^{\,ps} \cap \mathcal{F}_{\mathcal{R}}^{\,q}$.*

Proof. To check this, note that by Theorem 20.1, for any $A \subseteq X$, we have

$$A \in \mathcal{F}_{\mathcal{R}}^{\,s} \iff A^c \in \mathcal{T}_{\mathcal{R}}^{\,s} \iff A^c \in \mathcal{T}_{\mathcal{R}}^{\,q} \iff A \in \mathcal{F}_{\mathcal{R}}^{\,q}.$$

□

Moreover, from Theorem 23.4, by using Corollary 19.7 and Theorem 23.3, we can easily derive the following partial dual of Corollary 20.4.

Theorem 23.6. *If $\mathcal{R}$ is a topological relator on X, then*

$$\mathcal{T}_{\mathcal{R}}^{\,r} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^{\,s}.$$

Proof. By Theorem 10.3, $\mathcal{R}$ is reflexive and quasi-topological. Thus, by Corollary 19.7 and Theorem 23.3, we have $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\,p}$ and $\mathcal{T}_{\mathcal{R}}^{\,r} \subseteq \mathcal{T}_{\mathcal{R}}$. Hence, by using Theorem 23.4, we can already infer that

$$\mathcal{T}_{\mathcal{R}}^{\,r} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{R}}^{\,r} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{R}}^{\,p} \cap \mathcal{F}_{\mathcal{R}}^{\,s} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^{\,s}.$$

□

From this theorem, by using Theorem 20.1, we can immediately derive

Corollary 23.7. *If $\mathcal{R}$ is a topological relator on X, then $\mathcal{T}_{\mathcal{R}}^{\,r} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^{\,q}$.*

By using Theorem 23.6, we can also easily prove the following generalization of a statement of Kuratowski [87].

Theorem 23.8. *If $\mathcal{R}$ is a topological relator on X, then for any $A \in \mathcal{T}_{\mathcal{R}}^{s}$ we have*

$$\mathrm{cl}_{\mathcal{R}}(A)^{c} \in \mathcal{T}_{\mathcal{R}}^{r}.$$

Proof. By Theorem 23.6, we have $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^{s}$. Moreover, by Theorems 10.3 and 10.2, we have $A^{-} \in \mathcal{F}_{\mathcal{R}}$, and thus $A^{-c} \in \mathcal{T}_{\mathcal{R}}$. Therefore, to obtain the required assertion $A^{-c} \in \mathcal{T}_{\mathcal{R}}^{r}$, we need only show that $A^{-c} \in \mathcal{F}_{\mathcal{R}}^{s}$, i.e., $A^{-} \in \mathcal{T}_{\mathcal{R}}^{s}$ also holds. However, this is true by Theorem 21.4. □

Remark 23.9. Note that thus, under the conditions of the above theorem, we also have $A^{-\circ} = A^{-c-c} \in \mathcal{T}_{\mathcal{R}}^{r}$.

However, the latter fact seems to be of no importance, since by using Theorem 23.6 we can also prove the following

Theorem 23.10. *If $\mathcal{R}$ is a topological relator on X, then for any $A \subseteq X$ we have*

$$\mathrm{int}_{\mathcal{R}}(\,\mathrm{cl}_{\mathcal{R}}(A)) \in \mathcal{T}_{\mathcal{R}}^{r}.$$

Proof. By Theorem 23.6, we have $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^{s}$. Moreover, by Theorems 10.3 and 10.2, we have $A^{-\circ} \in \mathcal{T}_{\mathcal{R}}$. Therefore, to obtain the required assertion $A^{-\circ} \in \mathcal{T}_{\mathcal{R}}^{r}$, we need only show that $A^{-\circ} \in \mathcal{F}_{\mathcal{R}}^{s}$, i.e., $A^{-\circ c} \in \mathcal{T}_{\mathcal{R}}^{s}$ also holds.

For this, note that, because of the equalities $\circ\, c = c\, -$ and $-c = c\,\circ$ and Corollary 21.6, we have

$$A^{-\circ c} = A^{-c-} = A^{c\circ -} \in \mathcal{T}_{\mathcal{R}}^{s}.$$

□

Remark 23.11. The above two theorems can also be proved directly, by using only Theorems 10.3, 9.3 and 10.2 and the increasingness of the operations $-$ and $\circ$.

By using Theorems 12.5, 23.8 and 23.10, it can be shown that if $\mathcal{R}$ is a topologically filtered, topological relator on X, then the family $\mathcal{T}_{\mathcal{R}}^{r}$ forms a Boolean algebra under the operations defined by

$$A' = A^{-c}, \quad A \wedge B = A \cap B \quad \text{and} \quad A \vee B = (\,A \cup B)''$$

for all $A,\, B \in \mathcal{T}_{\mathcal{R}}^{r}$. (See Stone [144] and Givant and Halmos [64].)

However, the above statement is of no particular importance for us now, since unions, intersections and complements of generalized topologically open sets will only be investigated in a subsequent paper.

24. Characterizations of topologically semi-open and quasi-open sets

Now, as a generalization of [51, Lemma 1] of Duszyński and Noiri, we can also prove the following

Theorem 24.1. *If $\mathcal{R}$ is a reflexive relator on X, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{s}$;

(2) there exists $B \subseteq X$ *such that*

$$A = \mathrm{int}_{\mathcal{R}}(A) \cup B \quad \textit{and} \quad B \subseteq \mathrm{res}_{\mathcal{R}}(\mathrm{int}_{\mathcal{R}}(A)).$$

Proof. By Theorem 9.3, we have $A^{\circ} \subseteq A$. Moreover, if (1) holds, then by Definition 19.2 we have $A \subseteq A^{\circ -}$. Hence, by defining $B = A \setminus A^{\circ}$, we can already see that

$$A = A^{\circ} \cup (\,A \setminus A^{\circ}) = A^{\circ} \cup\, B \quad \text{and} \quad B = A \setminus A^{\circ} \subseteq A^{\circ -} \setminus\, A^{\circ} = A^{\circ \dagger}.$$

Therefore, (2) also holds.

By Theorem 9.3, we also have $A^{\circ} \subseteq A^{\circ -}$. Therefore, if (2) holds, then we have

$$A = A^{\circ} \cup\, B \subseteq A^{\circ} \cup A^{\circ \dagger} = A^{\circ} \cup (\,A^{\circ -} \setminus\, A^{\circ}) = A^{\circ -}.$$

Thus, by Definition 19.2, assertion (1) also holds. □

Remark 24.2. Note that if B is as in (2), then $A = A^\circ \cup B$. And, because of $B \subseteq A^{\circ\dagger} = A^{\circ -} \setminus A^\circ$, we have $A^\circ \cap B = \emptyset$.

Moreover, by Theorem 9.3, we have $A^{\circ\circ} \subseteq A^\circ$. Therefore, by using the notation $A^\ddagger = \operatorname{bnd}_{\mathcal{R}}(A)$, we can also state that $B \subseteq A^{\circ\dagger} = A^{\circ -} \setminus A^\circ \subseteq A^{\circ -} \setminus A^{\circ\circ} = A^{\circ\ddagger}$.

Now, by using our former results, we can also easily prove the following generalization of an observation of Dlaska, Ergun and Ganster [47, p. 1163].

Theorem 24.3. *If $\mathcal{R}$ is a topological relator on X, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{\,s}$;

(2) there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \subseteq X$ such that

$$A = V \cup B \quad \text{and} \quad B \subseteq \operatorname{res}_{\mathcal{R}}(V).$$

Proof. If (1) holds, then by Theorem 24.1, there exists $B \subseteq X$ such that

$$A = A^\circ \cup B \quad \text{and} \quad B \subseteq A^{\circ\dagger}.$$

Moreover, by Theorems 10.3 and 10.2, we have $A^\circ \in \mathcal{T}_{\mathcal{R}}$. Hence, by taking $V = A^\circ$, we can see that (2) also holds.

On the other hand, by Theorems 10.3 and 9.3, we have $V \subseteq V^-$ for any $V \subseteq X$. Therefore, if (2) holds, then we have not only $V \subseteq A$ but also

$$A = V \cup B \subseteq V \cup V^\dagger = V \cup (V^- \setminus V) = V^-.$$

Hence, by Definition 19.2, we can see that $A \in \mathcal{T}_{\mathcal{R}}^{\,q}$. Thus, by Theorem 20.1, assertion (1) also holds. □

Remark 24.4. Note that if in particular $\mathcal{R}$ is a topologically filtered, topological relator on X, then by Corollary 15.6 we have $V^\dagger \in \mathcal{N}_{\mathcal{R}}$. Hence, since $B \subseteq V^\dagger$, it is clear that $B \in \mathcal{N}_{\mathcal{R}}$ also holds.

Therefore, analogously to [94, Theorem 7] of Levine, we can also state

Theorem 24.5. *If $\mathcal{R}$ is a topologically filtered, topological relator on X and $A \in \mathcal{T}_{\mathcal{R}}^{\,s}$, then there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that*

$$A = V \cup B \quad \text{and} \quad V \cap B = \emptyset.$$

The following obvious reformulation of [94, Example 2] of Levine shows that the converse of this theorem is false.

Example 24.6. Define $X = \mathbb{R}$ and

$$R_n = \{(x, y) \in X^2 : \quad d(x, y) < n^{-1}\}$$

for all $n \in \mathbb{N}$.

Then, $\mathcal{R} = \{R_n : \ n \in \mathbb{N}\}$ is a properly filtered, strongly topological relator on X such that, under the notations

$$V =]\,0,\, 1\,[\quad \text{and} \quad B = \{2\},$$

we have $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that $A = V \cup B \notin \mathcal{T}_{\mathcal{R}}^{\,s}$.

To check the required properties, note that

(1) $R_n = R_n \cap R_m$ if $n, m \in \mathbb{N}$ such that $m \leq n$;

(2) $R_n(x) =]\,x - n^{-1},\ x + n^{-1}\,[$ for all $n \in \mathbb{N}$ and $x \in X$;

(3) for each $n \in \mathbb{N}$, $x \in X$ and $y \in R_n(x)$, we have $R_m(y) \subseteq R_n(x)$ if $m \in \mathbb{N}$ such that $m^{-1} < n^{-1} - d(x, y)$.

By using (2), we can easily see that $A^\circ = V$ and $A^{\circ -} = V^- = [\,0,\, 1\,]$, and thus $A \not\subseteq A^{\circ -}$. Therefore, by Definition 19.2, $A \notin \mathcal{T}_{\mathcal{R}}^{\,s}$. Although, $A = V \cup B$, $V \cap B = \emptyset$, $V \in \mathcal{T}_{\mathcal{R}}$ and $B^{-\circ} = B^\circ = \emptyset$.

While, by using assertion (3) and the property $\sup(\mathbb{N}) = +\infty$, we can easily see that $R_n(x) \in \mathcal{T}_{\mathcal{R}}$ for all $n \in \mathbb{N}$ and $x \in X$. Therefore, $\mathcal{R}$ is strongly quasi-topological.

Remark 24.7. Assertion (3) can also be easily derived from (2). However, it seems not to be a consequence of the very strong reflexivity, symmetry and transitivity properties of $\mathcal{R}$.

Now, in addition to $\Delta_X \subseteq R_n$ and $R_n = R_n^{-1}$, we also have $R_{2n} \circ R_{2n} \subseteq R_n$ for all $n \in \mathbb{N}$. Thus, $\mathcal{R}$ is a strictly uniformly transitive tolerance relator on X.

25. Characterizations of preopen and pseudo-open sets

As a conterpart of [59, Proposition 2.1] of Ganster, we can also prove

Theorem 25.1. *If $\mathcal{R}$ is a topologically filtered, topological relator on X, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{\,p}$ *;*

(2) there exist $V \in \mathcal{T}_{\mathcal{R}}$ *and* $B \in \mathcal{D}_{\mathcal{R}}$ *such that* $A = V \cap B$ *;*

(3) there exist $V \in \mathcal{T}_{\mathcal{R}}^{\,r}$ *and* $B \in \mathcal{D}_{\mathcal{R}}$ *such that* $A = V \cap B$.

Proof. If (1) holds, then by Definition 19.2 we have $A \subseteq A^{-\circ}$. Hence, we can infer that

$$A = A^{-\circ} \setminus (A^{-\circ} \setminus A) = A^{-\circ} \cap (A^{-\circ} \cap A^c)^c = A^{-\circ} \cap (A^{-\circ c} \cup A).$$

Now, by defining

$$V = A^{-\circ} \quad \text{and} \quad B = A^{-\circ c} \cup A,$$

we can state that $A = V \cap B$. And, by Theorem 23.10, we can state that $V = A^{-\circ} \in \mathcal{T}_{\mathcal{R}}^{\,r}$.

Moreover, by using Theorem 12.5, we can see that

$$B^{-} = (A^{-\circ c} \cup A)^{-} = A^{-\circ c -} \cup A^{-}.$$

And, by using the equality $\circ c = c -$ and Theorems 10.3 and 9.3, we can also see that

$$A^{-\circ c -} = A^{-c--} \supseteq A^{-c-} \supseteq A^{-c}.$$

Therefore,

$$B^{-} = A^{-\circ c -} \cup A^{-} \supseteq A^{-c} \cup A^{-} = X,$$

and thus $B^{-} = X$. Hence, by the definition of $\mathcal{D}_{\mathcal{R}}$, we can see that $B \in \mathcal{D}_{\mathcal{R}}$, and thus (3) also holds.

From Theorem 23.6, we know that $\mathcal{T}_{\mathcal{R}}^{\,r} \subseteq \mathcal{T}_{\mathcal{R}}$. Therefore, assertion (2) is an immediate consequence of (3). Moreover, from Theorems 22.5 and 19.5, we can see that (2) implies (1) even if the relator $\mathcal{R}$ is assumed to be only topologically filtered. □

In addition to the above theorem, we can also prove the following

Theorem 25.2. *If $\mathcal{R}$ is a topologically filtered, topological relator on X, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{\,p}$ *;*

(2) there exists $V \in \mathcal{T}_{\mathcal{R}}$ *such that* $A \subseteq V$ *and* $\operatorname{cl}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(V)$ *;*

(3) there exists $V \in \mathcal{T}_{\mathcal{R}}^{\,r}$ *such that* $A \subseteq V$ *and* $\operatorname{cl}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(V)$.

Proof. If (1) holds, then by Theorem 25.1 there exist $V \in \mathcal{T}_{\mathcal{R}}^{\,r}$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B$. Thus, in particular $A \subseteq V$. Hence, by using the increasingness $-$, we can infer that $A^{-} \subseteq V^{-}$.

Moreover, by Theorem 23.6, we have $\mathcal{T}_{\mathcal{R}}^{\,r} \subseteq \mathcal{T}_{\mathcal{R}}$, and thus $V \in \mathcal{T}_{\mathcal{R}}$. Therefore, by the definition of $\mathcal{D}_{\mathcal{R}}$ and Corollary 14.4, we can also state that

$$V = V \cap X = V \cap B^{-} \subseteq (V \cap B)^{-} = A^{-}.$$

Hence, by using the increasingness of $-$, Theorems 10.3 and 10.2 and the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that $V^- \subseteq A^{--} \subseteq A^-$. Therefore, we actually have $A^- = V^-$, and thus (3) also holds.

Again, by the inclusion $\mathcal{T}_{\mathcal{R}}^{\,r} \subseteq \mathcal{T}_{\mathcal{R}}$, it is clear that (2) is an immediate consequence of (3). Therefore, we need only show that (2) also implies (1).

For this, note that if (2) holds, then by Theorems 10.3 and 9.3, in addition to $A \subseteq V$, we also have $V \subseteq V^- = A^-$. Therefore, by Definition 19.2, we have $A \in \mathcal{T}_{\mathcal{R}}^{\,ps}$. Hence, by Theorem 19.5, we can see that (1) also holds. □

Remark 25.3. Note that, by Theorem 20.1, in the above two theorems we may again write $\mathcal{T}_{\mathcal{R}}^{\,ps}$ in place of $\mathcal{T}_{\mathcal{R}}^{\,p}$.

26. A further important property of topologically semi-open and preopen sets

By using Corollary 14.4, we can also prove the following improvement of [112, Lemma 2.5] of Noiri.

Theorem 26.1. *If $\mathcal{R}$ is a topologically filtered, topological relator on X, and*

$$A \in \mathcal{T}_{\mathcal{R}}^{\,s}, \qquad B \in \mathcal{T}_{\mathcal{R}}^{\,p}, \qquad \operatorname{int}_{\mathcal{R}}(A) \subseteq C \subseteq \operatorname{cl}_{\mathcal{R}}(A), \qquad B \subseteq D \subseteq X,$$

then

$$\operatorname{cl}_{\mathcal{R}}(A) \cap B = \operatorname{cl}_{\mathcal{R}}(C \cap D) \cap B.$$

Proof. By Definition 19.2, we have

$$A \subseteq A^{\circ -} \quad \text{and} \quad B \subseteq B^{- \circ}.$$

Hence, by using the increasingness of the operation $-$, Theorems 10.3 and 10.2, and the definition $\mathcal{F}_{\mathcal{R}}$, we can infer that

$$A^- \subseteq A^{\circ - -} \subseteq A^{\circ -}.$$

Moreover, from Theorems 10.3, 10.2 and 9.3, we can see that A°, $B^{-\circ} \in \mathcal{T}_{\mathcal{R}}$ and $B^{-\circ} \subseteq B^-$. Hence, by using Corollary 14.4, the corresponding properties of $-$, and the assumptions $A^\circ \subseteq C$ and $B \subseteq D$, we can infer that

$$A^- \cap B \subseteq A^{\circ -} \cap B^{-\circ} \subseteq (A^\circ \cap B^{-\circ})^- \subseteq (A^\circ \cap B^-)^- \subseteq (A^\circ \cap B)^{--} \subseteq (A^\circ \cap B)^- \subseteq (C \cap D)^-,$$

and thus $A^- \cap B \subseteq (C \cap D)^- \cap B$.

Moreover, by using the corresponding properties of $-$ and the assumption $C \subseteq A^-$, we can also see that

$$(C \cap D)^- \subseteq C^- \subseteq A^{--} \subseteq A^-,$$

and thus $(C \cap D)^- \cap B \subseteq A^- \cap B$. Therefore, the required equality is also true. □

From this theorem, by choosing C and D appropriately, we can immediately derive a great number equalities for the set $A^- \cap B$. For instance, we can at once state the following

Corollary 26.2. *If $\mathcal{R}$ is a topologically filtered, topological relator on X, and $A \in \mathcal{T}_{\mathcal{R}}^{\,s}$ and $B \in \mathcal{T}_{\mathcal{R}}^{\,p}$, then*

$$\operatorname{cl}_{\mathcal{R}}(A) \cap B = \operatorname{cl}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(A)) \cap \operatorname{int}_{\mathcal{R}}(\operatorname{cl}_{\mathcal{R}}(B))\big) \cap B.$$

Theorem 26.1 strongly suggests that some of the equalities stated by Dontchev [50, p. 4], without proofs and references, cannot be true.

To see that this is really the case, we can use the following

Example 26.3. If X and $\mathcal{R}$ are as in Example 24.6, and

$$A = [\,0,\ 1\,] \quad \text{and} \quad B = \,]\,0,\ 1\,[\, \cap \mathbb{Q},$$

then $\mathcal{R}$ is a properly filtered, strongly topological relator on X, and $A \in \mathcal{T}_{\mathcal{R}}^{\,s}$ and $B \in \mathcal{T}_{\mathcal{R}}^{\,p}$ such that

$$\operatorname{cl}_{\mathcal{R}}(A) \cap B = \,]\,0,\ 1\,[\, \cap \mathbb{Q} \quad \text{and} \quad \operatorname{cl}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(A) \cap B) = [\,0,\ 1\,],$$

and thus even $\operatorname{cl}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(A) \cap B) \not\subseteq \operatorname{cl}_{\mathcal{R}}(A) \cap B$.

Note that the inclusions $A \in \mathcal{T}_{\mathcal{R}}^{\,s}$ and $B \in \mathcal{T}_{\mathcal{R}}^{\,p}$ are immediate consequences of Corollary 21.5 and Theorems 22.5 and 19.5.

27. Some further generalized topologically open sets

Parts (1) and (2) of the following definition has been suggested by Njåstad [110] and Abd El-Monsef et al. [1]. While, for some motivations of parts (3) and (4), see Theorems 21.1 and 22.1 and [8, 2.1 Definition] of Andrijević.

Definition 27.1. A subset A of the relator space $X(\mathcal{R})$ will be called *topologically*

(1) *α–open* if $A \subseteq \operatorname{int}_{\mathcal{R}}(\operatorname{cl}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(A)))$;

(2) *β–open* if $A \subseteq \operatorname{cl}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(\operatorname{cl}_{\mathcal{R}}(A)))$;

(3) *γ–open* if there exists $V \in \mathcal{T}_{\mathcal{R}}^{s}$ such that $A \subseteq V \subseteq \operatorname{cl}_{\mathcal{R}}(A)$;

(4) *δ–open* if there exists $V \in \mathcal{T}_{\mathcal{R}}^{p}$ such that $V \subseteq A \subseteq \operatorname{cl}_{\mathcal{R}}(V)$.

And, the family of all such subsets of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{\kappa}$ with $\kappa = \alpha,\ \beta,\ \gamma$ and δ, respectively.

Remark 27.2. Note that if the relator $\mathcal{R}$ is not topological, then by using the families $\mathcal{T}_{\mathcal{R}}^{q}$ and $\mathcal{T}_{\mathcal{R}}^{ps}$ instead of $\mathcal{T}_{\mathcal{R}}^{s}$ and $\mathcal{T}_{\mathcal{R}}^{p}$, respectively, we can get some stronger forms of generalized topologically open sets.

Now, by using Definition 27.1 and the increasingness of $\circ$ and $-$, we can easily prove the following

Theorem 27.3. *We have*

$$\mathcal{T}_{\mathcal{R}}^{\gamma} \cup \mathcal{T}_{\mathcal{R}}^{\delta} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}.$$

Proof. If $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$, then by Definition 27.1 there exists $V \in \mathcal{T}_{\mathcal{R}}^{s}$ such that

$$A \subseteq V \subseteq A^{-}.$$

Hence, by using the definition of $\mathcal{T}_{\mathcal{R}}^{s}$ and the increasingness of $\circ -$, we can see that

$$A \subseteq V \subseteq V^{\circ -} \subseteq A^{-\circ -}.$$

Therefore, by Definition 27.1, we also have $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$.

While, if $A \in \mathcal{T}_{\mathcal{R}}^{\delta}$, then by Definition 27.1 there exists $V \in \mathcal{T}_{\mathcal{R}}^{p}$ such that

$$V \subseteq A \subseteq V^{-}.$$

Hence, by using the definition of $\mathcal{T}_{\mathcal{R}}^{p}$ and the increasingness of $-$ and $- \circ -$, we can see that

$$A \subseteq V^{-} \subseteq V^{-\circ -} \subseteq A^{-\circ -}.$$

Therefore, by Definition 27.1, we also have $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$. □

Moreover, by Definition 27.1, Theorem 9.3 and Corollary 19.7, it is clear that we also have the following

Theorem 27.4. *If $\mathcal{R}$ is a reflexive relator on X, then*

(1) $\mathcal{T}_{\mathcal{R}}^{s} \cup \mathcal{T}_{\mathcal{R}}^{ps} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma}$; *(2)* $\mathcal{T}_{\mathcal{R}}^{p} \cup \mathcal{T}_{\mathcal{R}}^{q} \subseteq \mathcal{T}_{\mathcal{R}}^{\delta}$.

Proof. To prove (1), note that if $A \in \mathcal{T}_{\mathcal{R}}^{s}$, then by taking $V = A$ we evidently have $V \in \mathcal{T}_{\mathcal{R}}^{s}$ such that $A \subseteq V$. Moreover, by Theorem 9.3, we can also see that $V \subseteq V^{-} = A^{-}$. Thus, by Definition 27.1, $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$ also holds.

While, if $A \in \mathcal{T}_{\mathcal{R}}^{ps}$, then by Definition 19.2 there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $A \subseteq V \subseteq A^{-}$. Moreover, by Corollary 19.7, we have $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{s}$, and thus in particular $V \in \mathcal{T}_{\mathcal{R}}^{s}$. Therefore, by Definition 27.1, $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$ also holds. □

Now, in addition to Theorem 19.6 and Corollary 19.7, we can also prove

Theorem 27.5. *If $\mathcal{R}$ is a reflexive relator on X, then $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\kappa}$ also holds with $\kappa = \alpha,\ \beta,\ \gamma$ and δ.*

Proof. By Corollary 19.7 and Theorems 27.4 and 27.3, we have

$$\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma} \cap \mathcal{T}_{\mathcal{R}}^{\delta} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma} \cup \mathcal{T}_{\mathcal{R}}^{\delta} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}.$$

Therefore, we need only show that $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\alpha}$ also holds.

For this, note that if $A \in \mathcal{T}_{\mathcal{R}}$, then by Theorem 9.3 and the definition of $\mathcal{T}_{\mathcal{R}}$, we have

$$A \subseteq A^{-} \quad \text{and} \quad A \subseteq A^{\circ}.$$

Hence, by using the increasingness of $\circ$ and $-\circ$, we can infer that

$$A^{\circ} \subseteq A^{-\circ} \quad \text{and} \quad A^{-\circ} \subseteq A^{\circ-\circ}.$$

Therefore, $A \subseteq A^{\circ-\circ}$. Hence, by Definition 27.1, we can see that $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$. □

Concerning reflexive relators, we can also easily prove the following

Theorem 27.6. *If $\mathcal{R}$ is a reflexive relator on X, then*

(1) $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}$; *(2)* $\mathcal{T}_{\mathcal{R}}^{s} \cup \mathcal{T}_{\mathcal{R}}^{p} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$.

Proof. From Theorems 27.4 and 27.4, we can see that

$$\mathcal{T}_{\mathcal{R}}^{s} \cup \mathcal{T}_{\mathcal{R}}^{p} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma} \cup \mathcal{T}_{\mathcal{R}}^{\delta} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta},$$

and thus in particular (2) also holds. Therefore, we need only prove (1).

For this, note that if $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$, then by Definition 27.1 we have $A \subseteq A^{\circ-\circ}$. Moreover, from Theorem 9.3, we can see $A^{\circ-\circ} \subseteq A^{\circ-}$. Therefore, we also have $A \subseteq A^{\circ-}$. Hence, by Definition 19.2, we can see that $A \in \mathcal{T}_{\mathcal{R}}^{s}$.

Moreover, by Theorem 9.3, we also have $A^{\circ} \subseteq A$. Hence, by using the increasingness of the operation $-\circ$, we can infer that $A^{\circ-\circ} \subseteq A^{-\circ}$. Therefore, we also have $A \subseteq A^{-\circ}$. Hence, by Definition 19.2, we can see that $A \in \mathcal{T}_{\mathcal{R}}^{p}$ also holds. Therefore, we actually have $A \in \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}$, and thus (1) is true. □

Remark 27.7. Assertions (1) and (2) actually depend on the fact that if $\mathcal{R}$ is a reflexive relator on X, then for any $A \subseteq X$ we have

$$A^{\circ-\circ} \subseteq A^{\circ-} \cap A^{-\circ} \quad \text{and} \quad A^{\circ-} \cup A^{-\circ} \subseteq A^{-\circ-}.$$

28. Characterizations of topologically α–open sets

Now, as a straightforward generalization of [112, Lemma 2.1] of Noiri and [128, Theorem 3] of Reilly and Vamanamurthy, we can also prove the following

Theorem 28.1. *If $\mathcal{R}$ is a topological relator on X, then*

$$\mathcal{T}_{\mathcal{R}}^{\alpha} = \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}.$$

Proof. From Theorem 27.6, we know that $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}$ even if the relator $\mathcal{R}$ is assumed to be only reflexive.

Moreover, if $A \in \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}$, i.e., $A \in \mathcal{T}_{\mathcal{R}}^{s}$ and $A \in \mathcal{T}_{\mathcal{R}}^{p}$, then by Definition 19.2, we have

$$A \subseteq A^{\circ-} \quad \text{and} \quad A \subseteq A^{-\circ}.$$

Hence, by using the increasingness of the operations $-$ and $\circ$, Theorems 10.3 and 10.2, and the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that

$$A^{-} \subseteq A^{\circ--} \subseteq A^{\circ-} \quad \text{and thus} \quad A^{-\circ} \subseteq A^{\circ-\circ}.$$

Therefore, we also have $A \subseteq A^{\circ-\circ}$. Hence, by Definition 27.1, we can see that $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$ also holds even if $\mathcal{R}$ is assumed to be only quasi-topological. □

Moreover, as an improvement of [110, Proposition 4] of Njåstad, we can prove

Theorem 28.2. *If $\mathcal{R}$ is a topologically filtered, topological relator on X, then for any $A \subseteq X$ the following assertions are equivalent :*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$ *;*

(2) there exist $V \in \mathcal{T}_{\mathcal{R}}$ *and* $B \in \mathcal{N}_{\mathcal{R}}$ *such that* $A = V \setminus B$ *;*

(3) there exist $V \in \mathcal{T}_{\mathcal{R}}$ *and* $B \subseteq \operatorname{res}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(A))$ *such that* $A = V \setminus B$.

Proof. If (1) holds, then by Definition 27.1 we have $A \subseteq A^{\circ - \circ}$, and thus

$$A = A^{\circ - \circ} \setminus (A^{\circ - \circ} \setminus A) .$$

Hence, by defining

$$V = A^{\circ - \circ} \quad \text{and} \quad B = A^{\circ - \circ} \setminus A ,$$

we can obtain $A = V \setminus B$. Moreover, by Theorems 10.3, 10.2 and 9.3, we can state that

$$V \in \mathcal{T}_{\mathcal{R}} \quad \text{and} \quad B \subseteq A^{\circ -} \setminus A^{\circ} = A^{\circ \dagger}.$$

Therefore, (3) also holds.

To prove the implication (3) $\Longrightarrow$ (2), it is enough to note only that, by Theorems 10.3 and 10.2, we have $A^{\circ} \in \mathcal{T}_{\mathcal{R}}$. Thus, by Corollary 15.6, we have $A^{\circ \dagger} \in \mathcal{N}_{\mathcal{R}}$. Therefore, if $B \subseteq A^{\circ \dagger}$, then we also have $B \in \mathcal{N}_{\mathcal{R}}$.

Finally, if (2) holds, then by Theorems 12.5, 10.3, 10.2 and 9.3 we can see that

$$A^{\circ} = (V \setminus B)^{\circ} = (V \cap B^{c})^{\circ} = V^{\circ} \cap B^{c\,\circ} = V \cap B^{c\,\circ} .$$

Hence, by using Corollary 14.4, we can infer that

$$V \cap B^{c\,\circ\,-} \subseteq (V \cap B^{c\,\circ})^{-} = A^{\circ -}.$$

Moreover, by using that $c \circ = - c$ and $c - = \circ c$, we can see that

$$B^{c\,\circ\,-} = B^{-\,c\,-} = B^{-\,\circ\,c} = \emptyset^{c} = X.$$

Therefore, we actually have $V \subseteq A^{\circ -}$, and hence also

$$A \subseteq V = V^{\circ} \subseteq A^{\circ - \circ}.$$

Thus, by Definition 27.1, assertion (1) also holds. □

Remark 28.3. If $\mathcal{R}$ is a topological relator on X, then by Theorems 10.3, 10.2 and 9.3, for any $A \subseteq X$, we have $A^{\circ\circ} = A^{\circ}$.

Therefore, by using the notation $A^{\ddagger} = \operatorname{bnd}_{\mathcal{R}}(A)$, we can also state that $A^{\circ \dagger} = A^{\circ -} \setminus A^{\circ} = A^{\circ -} \setminus A^{\circ\circ} = A^{\circ \ddagger}$.

Now, by using the above theorem, we can also prove the following improvement of [110, Corollary] of Njåstad.

Corollary 28.4. *For a nonvoid, topologically filtered, topological relator $\mathcal{R}$ on X, the following assertions are equivalent :*

(1) $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}$ *;* *(2)* $\mathcal{T}_{\mathcal{R}}^{\alpha} = \mathcal{T}_{\mathcal{R}}$ *;*

(3) $\mathcal{N}_{\mathcal{R}} \subseteq \mathcal{F}_{\mathcal{R}}$ *;* *(4)* $\mathcal{N}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}} \setminus \mathcal{E}_{\mathcal{R}}$.

Proof. From Theorem 27.5, we can see that $\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\alpha}$. Therefore, (1) and (2) are equivalent even if $\mathcal{R}$ is only reflexive.

Moreover, if $A \in \mathcal{N}_{\mathcal{R}}$, then by using Theorem 9.3, the increasingness of $\circ$, and the definitions of $\mathcal{D}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$, we can see that $A^{\circ} \subseteq A^{- \circ} = \emptyset$, and thus $A^{\circ} = \emptyset$, i. e., $A \notin \mathcal{E}_{\mathcal{R}}$. Therefore, (3) and (4) are also equivalent even if $\mathcal{R}$ is only reflexive.

From Theorem 10.3, we know that a topological relator is reflexive. Therefore, to complete the proof, we need only show that now (1) and (3) are also equivalent. For this, note that now $X \in \mathcal{T}_{\mathcal{R}}$ since $\mathcal{R} \neq \emptyset$. Therefore, if $B \in \mathcal{N}_{\mathcal{R}}$, then by Theorem 28.2 we also have $B^c = X \setminus B \in \mathcal{T}_{\mathcal{R}}^{\alpha}$. Hence, if (1) holds, we can infer that $B^c \in \mathcal{T}_{\mathcal{R}}$, and thus $B \in \mathcal{F}_{\mathcal{R}}$. Therefore, (1) implies (3).

On the other hand, if $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$, then by Theorem 28.2 there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that $A = V \setminus B$. Moreover, if (3) holds, we can also state that $B \in \mathcal{F}_{\mathcal{R}}$, and thus $B^c \in \mathcal{T}_{\mathcal{R}}$. Hence, by using Corollary 12.6, we can infer that $A = V \setminus B = V \cap B^c \in \mathcal{T}_{\mathcal{R}}$. Therefore, (1) also holds. $\square$

29. Characterizations of topologically β–open sets

By using the plausible notation $\mathcal{F}_{\mathcal{R}}^{r} = \{A^c : \ A \in \mathcal{T}_{\mathcal{R}}^{r}\}$, as a partial counterpart of [54, Theorem 26] of Ekici and [77, Theorem 3.7] of Jamunarani et al., we can also prove the following

Theorem 29.1. *If $\mathcal{R}$ is a topological relator on X, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$; *(2)* $\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}^{s}$;

(3) $\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}^{q}$; *(4)* $\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}^{r}$.

Proof. From Theorem 20.1, we know that (2) and (3) are equivalent. Therefore, we need only prove the equivalence of (1), (2) and (4).

For this, note that if (1) holds, then by Definition 27.1 we have $A \subseteq A^{-\circ-}$. Hence, by using the increasingness of $-$, Theorems 10.3 and 10.2, and the definition of $\mathcal{F}_{\mathcal{R}}$, we can infer that

$$A^{-} \subseteq A^{-\circ--} \subseteq A^{-\circ-}.$$

Therefore, by Definition 19.2, assertion (2) also holds.

While, if (2) holds, then by using Theorems 10.3, 9.3, 10.2 and 20.2, we can see that

$$A^{-} = A^{--} = A^{-\circ-}.$$

Hence, by using that $-c = c\circ$ and $\circ c = c-$, we can infer that

$$A^{-c} = A^{-\circ-c} = A^{-\circ c\circ} = A^{-c-\circ}.$$

Thus, by Definition 23.1, we have $A^c \in \mathcal{T}_{\mathcal{R}}^{r}$, and thus (4) also holds.

Finally, if (4) holds, then $A^c \in \mathcal{T}_{\mathcal{R}}^{r}$. Therefore, by Definition 23.1, we have $A^{-c} = A^{-c-\circ}$. Hence, by using a similar argument as above, we can infer that $A^{-} = A^{-\circ-}$. Moreover, by Theorems 10.3 and 9.3, we also have $A \subseteq A^{-}$. Therefore, $A \subseteq A^{-\circ-}$, and thus by Definition 27.1 assertion (1) also holds. $\square$

Remark 29.2. From this theorem, by using our former results on the families $\mathcal{T}_{\mathcal{R}}^{s}$ and $\mathcal{T}_{\mathcal{R}}^{r}$, we can derive several properties of the family $\mathcal{T}_{\mathcal{R}}^{\beta}$.

For instance, from Theorem 29.1, by using Theorems 20.2 and 24.3, we can immediately derive the following

Theorem 29.3. *If $\mathcal{R}$ is a topological relator on X, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$;

(2) there exists $V \in \mathcal{T}_{\mathcal{R}}$ *such that* $\operatorname{cl}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(V)$;

(3) there exist $V \in \mathcal{T}_{\mathcal{R}}$ *and* $B \subseteq X$ *such that*

$$\operatorname{cl}_{\mathcal{R}}(A) = V \cup B \quad \textit{and} \quad B \subseteq \operatorname{res}_{\mathcal{R}}(V).$$

Hence, we can easily derive the following counterpart of [54, Theorem 27] of Ekici and [77, Theorem 3.8] of Jamunarani at al.

Corollary 29.4. *If $\mathcal{R}$ is a topological relator on X, $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$ and*

$$A \subseteq B \subseteq \mathrm{cl}_{\mathcal{R}}(A),$$

then $B \in \mathcal{T}_{\mathcal{R}}^{\beta}$ also holds.

Proof. By using the increasingness of $-$, Theorems 10.3 and 10.2, and the definition of $\mathcal{F}_{\mathcal{R}}$, we can see that

$$A^{-} \subseteq B^{-} \subseteq A^{--} \subseteq A^{-},$$

and thus $A^{-} = B^{-}$. Hence, by Theorem 29.3, it is clear that the required assertion is also true. □

Moreover, from Theorem 29.1, by using Theorem 24.5, we can also easily derive

Theorem 29.5. *If $\mathcal{R}$ is a topologically filtered, topological relator on X and $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then there exist $V \in \mathcal{T}_{\mathcal{R}}$ and $B \in \mathcal{N}_{\mathcal{R}}$ such that*

$$\mathrm{cl}_{\mathcal{R}}(A) = V \cup B \quad \textit{and} \quad V \cap B = \emptyset .$$

Now, analogously to [54, Theorem 23] of Ekici and [77, Theorem 3.5] of Jamunarani et al., we can also prove the following

Theorem 29.6. *If $\mathcal{R}$ is a topological relator on X and $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then there exist $V \in \mathcal{T}_{\mathcal{R}}^{s}$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B$.*

Proof. Define

$$V = A^{-} \quad \text{and} \quad B = A \cup A^{-c}.$$

Then, by Theorem 29.1, we have $V \in \mathcal{T}_{\mathcal{R}}^{s}$. Moreover, by using the increasingness of $-$ and Theorems 10.3 and 9.3, we can see that

$$B^{-} = (A \cup A^{-c})^{-} \subseteq A^{-} \cup A^{-c-} \subseteq A^{-} \cup A^{-c} = X.$$

Therefore, $B^{-} = X$, and thus $B \in \mathcal{D}_{\mathcal{R}}$.

Moreover, by using Theorems 10.3 and 9.3, we can also see that

$$V \cap B = A^{-} \cap (A \cup A^{-c}) = (A^{-} \cap A) \cup (A^{-} \cap A^{-c}) = A \cup \emptyset = A.$$

Therefore, the required assertion is also true. □

The following example shows that the converse of this theorem is also false.

Example 29.7. If $X = \{1, 2, 3, 4\}$ and R_i is a relations on X, for every $i = 1, 2, 3$, such that

$R_1(1) = \{1\}$, $\quad R_1(2) = R_1(3) = R_1(4) = X$;

$R_2(1) = R_2(2) = \{1, 2\}$, $\quad R_2(3) = R_2(4) = X$;

$R_3(1) = X$, $\quad R_3(2) = R_3(3) = \{2, 3\}$, $\quad R_3(4) = X$;

then $\mathcal{R} = \{R_1, R_2, R_3\}$ is a preorder relator on X such that, under the notations

$$V = \{2, 3\} \quad \text{and} \quad B = \{1, 3, 4\},$$

we have $V \in \mathcal{T}_{\mathcal{R}}^{s}$ and $B \in \mathcal{D}_{\mathcal{R}}$ such that $A = V \cap B \notin \mathcal{T}_{\mathcal{R}}^{\beta}$.

To check this, note that R_1, R_2 and R_3 are just the Pervin preorders generated by the sets $\{1\}$, $\{1, 2\}$ and $\{2, 3\}$, respectively. Moreover, we have

$$A^{-\circ-} = \{3\}^{-\circ-} = \{3, 4\}^{\circ-} = \emptyset^{-} = \emptyset,$$

$$V^{\circ-} = \{2, 3\}^{\circ-} = \{2, 3\}^{-} = \{2, 3, 4\} \quad \text{and} \quad B^{-} = \{1, 3, 4\}^{-} = X.$$

30. Some further results on topologically α–open and β–open sets

By using our former results and the plausible notation $\mathcal{F}_{\mathcal{R}}^{\kappa} = \{ A^c : \ A \in \mathcal{T}_{\mathcal{R}}^{\kappa} \}$ with $\kappa = s$ and $\kappa = \beta$, we can also easily prove the following counterpart of [54, Theorem 7] of Ekici and [77, Theorem 2.1] of Jamunarani et al.

Theorem 30.1. *If $\mathcal{R}$ is a topological relator on X, then*

$$\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}}^{\alpha} \cap \mathcal{F}_{\mathcal{R}}^{\beta} .$$

Proof. By Theorem 23.6, we have $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}^{s}$. Moreover, by Theorems 27.5 and 27.6, we have

$$\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}^{\alpha} \quad \text{and} \quad \mathcal{T}_{\mathcal{R}}^{s} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta} ,$$

and thus also $\mathcal{F}_{\mathcal{R}}^{s} \subseteq \mathcal{F}_{\mathcal{R}}^{\beta}$. Therefore, $\mathcal{T}_{\mathcal{R}}^{r} \subseteq \mathcal{T}_{\mathcal{R}}^{\alpha} \cap \mathcal{F}_{\mathcal{R}}^{\beta}$.

On the other hand, if $A \in \mathcal{T}_{\mathcal{R}}^{\alpha} \cap \mathcal{F}_{\mathcal{R}}^{\beta}$, then we have

$$A \in \mathcal{T}_{\mathcal{R}}^{\alpha} \quad \text{and} \quad A \in \mathcal{F}_{\mathcal{R}}^{\beta} ,$$

and thus also $A^c \in \mathcal{T}_{\mathcal{R}}^{\beta}$. Hence, by using Definition 27.1, we can infer that

$$A \subseteq A^{\circ - \circ} \quad \text{and} \quad A^c \subseteq A^{c - \circ -} ,$$

and thus also $A^{c - \circ - c} \subseteq A$. Moreover, by using the equalities $c\, - = \circ\, c$, $-\, c = c\, \circ$ and $- = c \circ c$, we can see that

$$A^{c - \circ - c} = A^{\circ c \circ - c} = A^{\circ c \circ c \circ} = A^{\circ - \circ} .$$

Therefore, $A^{\circ - \circ} \subseteq A$, and thus $A = A^{\circ - \circ}$ also holds. Hence, by using Theorem 23.10, we can already infer that $A \in \mathcal{T}_{\mathcal{R}}^{r}$. Therefore, $\mathcal{T}_{\mathcal{R}}^{\alpha} \cap \mathcal{F}_{\mathcal{R}}^{\beta} \subseteq \mathcal{T}_{\mathcal{R}}^{r}$, and thus the required equality is also true. □

Now, by using Theorem 27.3 and 29.1, we can also easily prove the following two counterparts of [8, Theorem 2.4] of Andrijević.

Theorem 30.2. *If $\mathcal{R}$ is a topological relator on X, then $\mathcal{T}_{\mathcal{R}}^{\gamma} = \mathcal{T}_{\mathcal{R}}^{\beta}$.*

Proof. By Theorem 27.3, we always have $\mathcal{T}_{\mathcal{R}}^{\gamma} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$. Therefore, we need only prove that now $\mathcal{T}_{\mathcal{R}}^{\beta} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma}$ also holds.

For this, note that if $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then by Theorem 29.1 we have $A^{-} \in \mathcal{T}_{\mathcal{R}}^{s}$. Hence, by defining $V = A^{-}$, we can note that $V \in \mathcal{T}_{\mathcal{R}}^{s}$ such that $V \subseteq A^{-}$. Moreover, by Theorems 10.3 and 9.3, it is clear that $A \subseteq V$ is also true. Therefore, by Definition 27.1 we also have $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$. □

Theorem 30.3. *If $\mathcal{R}$ is a topologically filtered, topological relator on X, then $\mathcal{T}_{\mathcal{R}}^{\delta} = \mathcal{T}_{\mathcal{R}}^{\beta}$.*

Proof. By Theorem 27.3, we always have $\mathcal{T}_{\mathcal{R}}^{\delta} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$. Therefore, we need only prove that now $\mathcal{T}_{\mathcal{R}}^{\beta} \subseteq \mathcal{T}_{\mathcal{R}}^{\delta}$ also holds.

For this, note that if $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$, then by Theorem 29.1 we have $A^{-} \in \mathcal{T}_{\mathcal{R}}^{s}$. Hence, by using Theorems 10.3, 9.3, 10.2 and 20.2, we can again infer that

$$A^{-} = A^{--} = A^{- \circ -} .$$

Now, by defining $V = A \cap A^{- \circ}$, we can note that $V \subseteq A$. Moreover, by using Corollary 14.4 and Theorems 10.3, 10.2 and 9.3, we can see that

$$V^{-} = \left(A \cap A^{- \circ} \right)^{-} \supseteq A^{-} \cap A^{- \circ} = A^{- \circ} .$$

Hence, we can infer that

$$V^{- \circ} \supseteq A^{- \circ \circ} = A^{- \circ} \supseteq A \cap A^{- \circ} = V.$$

Thus, by Definition 19.2, we also have $V \in \mathcal{T}_{\mathcal{R}}^{p}$.

Moreover, we can now also note that

$$A \subseteq A^{-} = A^{- \circ -} \subseteq V^{--} = V^{-} .$$

Therefore, by Definition 27.1, we also have $A \in \mathcal{T}_{\mathcal{R}}^{\delta}$. □

Unfortunately, concerning the relationship of the families $\mathcal{T}_{\mathcal{R}}^{\alpha}$ and $\mathcal{T}_{\mathcal{R}}^{\gamma}$, we can only prove the following

Theorem 30.4. *If $\mathcal{R}$ is a reflexive relator on X, then $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}^{\gamma} \cap \mathcal{T}_{\mathcal{R}}^{\delta}$.*

Proof. If $A \in \mathcal{T}_{\mathcal{R}}^{\alpha}$, then by Theorem 27.6 we also have $A \in \mathcal{T}_{\mathcal{R}}^{s}$ and $A \in \mathcal{T}_{\mathcal{R}}^{p}$. Hence, by Theorem 27.4, we can see that $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$ and $A \in \mathcal{T}_{\mathcal{R}}^{\delta}$ also hold. Therefore, the required inclusion is also true. □

Remark 30.5. Later, we shall see that the corresponding equality need not be true. Moreover, $\mathcal{T}_{\mathcal{R}}^{\delta}$ may also be a proper subset of $\mathcal{T}_{\mathcal{R}}^{\gamma}$.

31. Topologically *a*–open and *b*–open sets

In topological spaces, β–open sets were actually called semi-preopen by Andrijević [8]. Later, this terminology was also used by Ganster and Andrijević [60] and Dontchev [48].

In a subsequent paper [12], Andrijević also introduced the notion of a b–open subset of a topological space. Motivated by his definition, we may also naturally introduce the following

Definition 31.1. A subset A of a relator space $X(\mathcal{R})$ will be called *topologically*

(1) *a–open* if $A \subseteq \mathrm{cl}_{\mathcal{R}}(\,\mathrm{int}_{\mathcal{R}}(A)) \cap \mathrm{int}_{\mathcal{R}}(\,\mathrm{cl}_{\mathcal{R}}(A))$;

(2) *b–open* if $A \subseteq \mathrm{cl}_{\mathcal{R}}(\,\mathrm{int}_{\mathcal{R}}(A)) \cup \mathrm{int}_{\mathcal{R}}(\,\mathrm{cl}_{\mathcal{R}}(A))$.

And, the family of all such subsets of $X(\mathcal{R})$ will be denoted by $\mathcal{T}_{\mathcal{R}}^{a}$ and $\mathcal{T}_{\mathcal{R}}^{b}$, respectively.

Thus, analogously to Theorem 27.6, we evidently have the following

Theorem 31.2. *We have*

(1) $\mathcal{T}_{\mathcal{R}}^{a} = \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p}$; *(2)* $\mathcal{T}_{\mathcal{R}}^{s} \cup \mathcal{T}_{\mathcal{R}}^{p} \subseteq \mathcal{T}_{\mathcal{R}}^{b}$.

Moreover, by using Theorems 27.6 and 9.3 and Definition 27.1, we can also prove

Theorem 31.3. *If $\mathcal{R}$ is reflexive relator on X, then*

(1) $\mathcal{T}_{\mathcal{R}}^{\alpha} \subseteq \mathcal{T}_{\mathcal{R}}^{a}$ *(2)* $\mathcal{T}_{\mathcal{R}}^{b} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$.

Proof. Since (1) follows immediately from Theorems 27.6 and 31.2, we need only prove (2).

For this, note that if $A \in \mathcal{T}_{\mathcal{R}}^{b}$, then by Definition 31.1 and Remark 27.7, we have $A \subseteq A^{\circ -} \cup A^{- \circ} \subseteq A^{- \circ -}$. Thus, by Definition 27.1, $A \in \mathcal{T}_{\mathcal{R}}^{\beta}$ also holds. □

Remark 31.4. Note that, by Theorem 31.2, we always have $\mathcal{T}_{\mathcal{R}}^{a} \subseteq \mathcal{T}_{\mathcal{R}}^{b}$. Thus, if $\mathcal{R}$ is reflexive relator on X, then by Theorem 31.3 we also have $\mathcal{T}_{\mathcal{R}}^{a} \subseteq \mathcal{T}_{\mathcal{R}}^{\beta}$.

Now, in accordance with [12, Remark 1] of Adrijević, we can also prove

Theorem 31.5. *If $\mathcal{R}$ is a topologically filtered, topological relator on X, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{b}$;

(2) there exist $B \in \mathcal{T}_{\mathcal{R}}^{s}$ *and* $C \in \mathcal{T}_{\mathcal{R}}^{p}$ *such that* $A = B \cup C$.

Proof. If (2) holds then by Definition 19.2 we have

$$B \subseteq B^{\circ -} \quad \text{and} \quad C \subseteq C^{- \circ}.$$

Moreover, we can see that $B \subseteq A$ and $C \subseteq A$. Hence, by using the increasingness of $\circ -$ and $- \circ$, we can infer that

$$B^{\circ -} \subseteq A^{\circ -} \quad \text{and} \quad C^{- \circ} \subseteq A^{- \circ}.$$

Therefore, we have

$$A = B \cup C \subseteq A^{\circ -} \cup A^{- \circ}.$$

Thus, by Definition 31.1, assertion (1) also holds even if $\mathcal{R}$ is not assumed to have any particular properties.

Conversely, if (1) holds, then by Definition 31.1, we have $A \subseteq A^{\circ -} \cup A^{- \circ}$. Hence, we can infer that

$$A = A \cap (A^{\circ -} \cup A^{- \circ}) = (A \cap A^{\circ -}) \cup (A \cap A^{- \circ}).$$

Thus, by defining

$$B = A \cap A^{\circ -} \quad \text{and} \quad C = A \cap A^{- \circ},$$

we can at once state that $A = B \cup C$.

Now, by using Theorems 12.5, 10.3, 10.2 and 9.3, we can also see that

$$B^{\circ} = (A \cap A^{\circ -})^{\circ} = A^{\circ} \cap A^{\circ - \circ} \supseteq A^{\circ} \cap A^{\circ \circ} = A^{\circ} \cap A^{\circ} = A^{\circ},$$

and thus

$$B^{\circ -} \supseteq A^{\circ -} \supseteq A \cap A^{\circ -} = B.$$

Therefore, by Definition 19.2, we have $B \in \mathcal{T}_{\mathcal{R}}^{s}$.

Moreover, by using Corollary 14.4, Theorem 12.5, 10.3, 10.2 and 9.3, we can also see that

$$C^{-} = (A \cap A^{- \circ})^{-} \supseteq A^{-} \cap A^{- \circ} = A^{- \circ},$$

and thus

$$C^{- \circ} \supseteq A^{- \circ \circ} = A^{- \circ} \supseteq A \cap A^{- \circ} = C.$$

Therefore, by Definition 19.2, we also have $C \in \mathcal{T}_{\mathcal{R}}^{p}$. □

Remark 31.6. The above proof shows that if $\mathcal{R}$ is a topologically filtered, topological relator on X, then for any $A \subseteq X$ we have

(1) $A \cap A^{\circ -} \in \mathcal{T}_{\mathcal{R}}^{s}$; (2) $A \cap A^{- \circ} \in \mathcal{T}_{\mathcal{R}}^{p}$.

32. Generalized topologically open sets derived from topologically simple relators

By Definition 19.2 and Theorem 18.6, we can at once state the following

Theorem 32.1. *For a nonvoid relator $\mathcal{R}$ on X, the following assertions are equivalent:*
(1) $\mathcal{T}_{\mathcal{R}}^{p} = \mathcal{P}(X)$; (2) $\mathcal{R}$ is a properly topologically symmetric.

Hence, by using Theorems 23.4 and 31.2, we can immediately derive

Corollary 32.2. *If $\mathcal{R}$ is a nonvoid, properly topologically symmetric relator on X, then*
(1) $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{F}_{\mathcal{R}}^{s}$; (2) $\mathcal{T}_{\mathcal{R}}^{a} = \mathcal{T}_{\mathcal{R}}^{s}$; (3) $\mathcal{T}_{\mathcal{R}}^{b} = \mathcal{P}(X)$.

Proof. Namely, by Theorems 23.4 and 31.2 and 32.1, we have

$$\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}}^{p} \cap \mathcal{F}_{\mathcal{R}}^{s} = \mathcal{P}(X) \cap \mathcal{F}_{\mathcal{R}}^{s} = \mathcal{F}_{\mathcal{R}}^{s}$$

and

$$\mathcal{T}_{\mathcal{R}}^{a} = \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{T}_{\mathcal{R}}^{p} = \mathcal{T}_{\mathcal{R}}^{s} \cap \mathcal{P}(X) = \mathcal{T}_{\mathcal{R}}^{s}.$$

Moreover, by Theorems 31.2 and 32.1, we also have

$$\mathcal{P}(X) = \mathcal{T}_{\mathcal{R}}^{s} \cup \mathcal{P}(X) = \mathcal{T}_{\mathcal{R}}^{s} \cup \mathcal{T}_{\mathcal{R}}^{p} \subseteq \mathcal{T}_{\mathcal{R}}^{b},$$

and thus $\mathcal{T}_{\mathcal{R}}^{b} = \mathcal{P}(X)$ also holds. □

Remark 32.3. By using equality (1), we can also easily see that

$$A \in \mathcal{F}_{\mathcal{R}}^{r} \iff A^{c} \in \mathcal{T}_{\mathcal{R}}^{r} \iff A^{c} \in \mathcal{F}_{\mathcal{R}}^{s} \iff A \in \mathcal{T}_{\mathcal{R}}^{s}.$$

Therefore, in addition to Corollary 32.2, we can also state that $\mathcal{F}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}}^{s}$.

Now, in addition to Corollary 20.4, we can also state the following

Corollary 32.4. *If $\mathcal{R}$ is a nonvoid, properly topologically symmetric relator on X, then for any $A \subseteq X$, the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{s}$; *(2)* $A = \operatorname{cl}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(A))$.

Proof. By Remark 32.3 and the equalities $c\,- = \circ\, c$ and $c \circ c = -$, we have

$$A \in \mathcal{T}_{\mathcal{R}}^{s} \iff A \in \mathcal{F}_{\mathcal{R}}^{r} \iff A^{c} \in \mathcal{T}_{\mathcal{R}}^{r} \iff A^{c} = A^{c\,-\,\circ} \iff A = A^{c\,-\,\circ\,c} \iff A = A^{\circ\,c\,\circ\,c} \iff A = A^{\circ\,-}.$$

□

Moreover, from Theorem 32.1, by Theorems 20.1 and 28.1, we can also derive

Corollary 32.5. *If $\mathcal{R}$ is a nonvoid, properly topologically symmetric, topological relator on X, then*

(1) $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{P}(X)$; *(2)* $\mathcal{T}_{\mathcal{R}}^{\alpha} = \mathcal{T}_{\mathcal{R}}^{s}$.

From Corollary 17.10, we know that a relator $\mathcal{R}$ is properly topologically symmetric if and only if it is topologically simple and weakly symmetric.

Therefore, as a straightforward generalization of properly symmetric relators, we may also naturally consider here topologically simple relators.

Theorem 32.6. *If $\mathcal{R}$ is a topologically simple relator on X and $R = \bigcap \mathcal{R}$, then for any $A \subseteq X$ we have*

(1) $A \in \mathcal{T}_{\mathcal{R}}^{s}$ *if and only if* $A \subseteq R^{-1}[\,R^{-1}[\,A^{c}\,]^{c}\,]$;

(2) $A \in \mathcal{T}_{\mathcal{R}}^{p}$ *if and only if* $A \subseteq R^{-1}[\,R^{-1}[\,A\,]^{c}\,]^{c}$;

(3) $A \in \mathcal{T}_{\mathcal{R}}^{q}$ *if and only if there exists* $V \subseteq X$ *such that*

$$R[\,V\,] \subseteq V \subseteq A \subseteq R^{-1}[\,V\,].$$

(4) $A \in \mathcal{T}_{\mathcal{R}}^{ps}$ *if and only if there exists* $V \subseteq X$ *such that*

$$A \cup R[\,V\,] \subseteq V \subseteq R^{-1}[\,A\,].$$

Proof. By Theorem 16.5, we have $\mathcal{R}^{\wedge} = \{R\}^{\wedge}$. Hence, by using Theorems 6.4 and 4.9, we can see that

$$\operatorname{cl}_{\mathcal{R}}(A) = \operatorname{cl}_{R}(A) = R^{-1}[\,A\,] \quad \text{and} \quad \operatorname{int}_{\mathcal{R}}(A) = \operatorname{int}_{R}(A) = R^{-1}[\,A^{c}\,]^{c}$$

for all $A \subseteq X$. Moreover, by the corresponding definitions and Theorem 6.4, we can also see that

$$A \in \mathcal{T}_{\mathcal{R}} \iff A \subseteq \operatorname{int}_{\mathcal{R}}(A) \iff A \subseteq \operatorname{int}_{R}(A) \iff R[\,A\,] \subseteq A.$$

Hence, by Definition 19.2, it is clear that required assertions are true. □

Remark 32.7. By Definitions 23.1 and 27.1, in addition to assertions (1)–(4), we can, for instance, also state that

(a) $A \in \mathcal{T}_{\mathcal{R}}^{r}$ if and only if $A = R^{-1}[\,R^{-1}[\,A\,]^{c}\,]^{c}$;

(b) $A \in \mathcal{T}_{\mathcal{R}}^{\gamma}$ if and only if there exists $V \subseteq X$ such that

$$A \subseteq V \subseteq R^{-1}[\,A\,] \cap R^{-1}[\,R^{-1}[\,A^{c}\,]^{c}\,].$$

By using our former results on the various refinements of relators, we can also easily prove the following

Theorem 32.8. *For any $\square = *$, $\#$ or $\wedge$ and $\kappa = s, p, q, ps, r, \alpha, \beta, \gamma, \delta, a$ or b, we have*

$$\mathcal{T}_{\mathcal{R}}^{\kappa} = \mathcal{T}_{\mathcal{R}^{\square}}^{\kappa}.$$

Proof. By Theorems 6.7 and 6.4, we have $\mathcal{R}^{\wedge} = \mathcal{R}^{\Box\wedge}$. Thus, by Theorem 6.4 and the definition of $\mathcal{T}_{\mathcal{R}}$, we have

$$\mathrm{cl}_{\mathcal{R}} = \mathrm{cl}_{\mathcal{R}^{\Box}}, \quad \mathrm{int}_{\mathcal{R}} = \mathrm{int}_{\mathcal{R}^{\Box}} \quad \text{and} \quad \mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\Box}}.$$

Hence, by Definitions 19.2, 23.1, 27.1 and 31.1, it is clear that the required equality is also true. □

Remark 32.9. By [110, 29, 31, 27, 13, 10, 11], for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, the possible consequences and equivalents of the inclusion $\mathcal{T}_{\mathcal{R}}^{\kappa} \subseteq \mathcal{T}_{\mathcal{S}}^{\kappa}$ should also be investigated.

33. Topologically semi-open and preopen sets derived from the paratopological refinements of relators

In addition to Theorem 32.8, we can also prove the following

Theorem 33.1. *If $\mathcal{R}$ is a non-degenerated relator on X, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}_{\mathcal{R}^{\vartriangle}}^{s} \setminus \{\emptyset\}$; *(2) $A \in \mathcal{E}_{\mathcal{R}}$ and $\mathcal{R}$ is non-partial.*

Proof. By Definition 19.2, assertion (1) is equivalent to the statement that

(a) $A \subseteq \mathrm{cl}_{\mathcal{R}^{\vartriangle}}(\mathrm{int}_{\mathcal{R}^{\vartriangle}}(A))$ and $A \neq \emptyset$.

Therefore, if (1) holds, then in particular we have

(b) $\mathrm{cl}_{\mathcal{R}^{\vartriangle}}(\mathrm{int}_{\mathcal{R}^{\vartriangle}}(A)) \neq \emptyset$.

Next, we show that (b) already implies (2). For this, note that if $A \notin \mathcal{E}_{\mathcal{R}}$, then by Corollary 7.8 we have $\mathrm{int}_{\mathcal{R}^{\vartriangle}}(A) = \emptyset$. Therefore, if (b) holds, then

$$\mathrm{cl}_{\mathcal{R}^{\vartriangle}}(\emptyset) = \mathrm{cl}_{\mathcal{R}^{\vartriangle}}(\mathrm{int}_{\mathcal{R}^{\vartriangle}}(A)) \neq \emptyset.$$

Hence, by using Corollary 7.8, we can infer that $\emptyset \in \mathcal{D}_{\mathcal{R}}$. Thus, by Theorem 9.10, the relator $\mathcal{R}$ cannot be non-degenerated. This contradiction proves that (b) implies $A \in \mathcal{E}_{\mathcal{R}}$.

However, if $A \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 7.8 we have $\mathrm{int}_{\mathcal{R}^{\vartriangle}}(A) = X$. Therefore, if (b) holds, then

$$\mathrm{cl}_{\mathcal{R}^{\vartriangle}}(X) = \mathrm{cl}_{\mathcal{R}^{\vartriangle}}(\mathrm{int}_{\mathcal{R}^{\vartriangle}}(A)) \neq \emptyset.$$

Hence, by using Corollary 7.8, we can infer that $X \in \mathcal{D}_{\mathcal{R}}$. Thus, by Theorem 9.8, the relator $\mathcal{R}$ is non-partial. Consequently, (b) implies (2), and thus (1) also implies (2).

In the sequel, we shall show that (2) implies that

(c) $\mathrm{cl}_{\mathcal{R}^{\vartriangle}}(\mathrm{int}_{\mathcal{R}^{\vartriangle}}(A)) = X$ and $A \neq \emptyset$.

For this, note that if $A \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 7.8, we have $\mathrm{int}_{\mathcal{R}^{\vartriangle}}(A) = X$. Moreover, if $\mathcal{R}$ is non-partial, then by Theorem 9.8 we have $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ and $X \in \mathcal{D}_{\mathcal{R}}$. Thus, in particular $A \neq \emptyset$. Moreover, by Corollary 7.8, we have $\mathrm{cl}_{\mathcal{R}^{\vartriangle}}(X) = X$. Therefore, if (2) holds, then in addition to $A \neq \emptyset$ we also have

$$\mathrm{cl}_{\mathcal{R}^{\vartriangle}}(\mathrm{int}_{\mathcal{R}^{\vartriangle}}(A)) = \mathrm{cl}_{\mathcal{R}^{\vartriangle}}(X) = X,$$

and thus (c) also holds.

Hence, since (2) $\Longrightarrow$ (c) $\Longrightarrow$ (a) $\Longrightarrow$ (1), we can see that (2) also implies (1). □

Now, by using the above theorem, we can also easily establish

Corollary 33.2. *If $\mathcal{R}$ is a non-partial, non-degenerated relator on X, then*

$$\mathcal{T}_{\mathcal{R}^{\vartriangle}}^{s} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\} \quad \textit{with} \quad \emptyset \notin \mathcal{E}_{\mathcal{R}}.$$

Proof. To prove this, note that by Theorem 33.1 we have $\mathcal{T}_{\mathcal{R}^{\vartriangle}}^{s} \setminus \{\emptyset\} = \mathcal{E}_{\mathcal{R}}$. Moreover, by Definition 19.2, we have $\emptyset \in \mathcal{T}_{\mathcal{R}}^{s}$ for any relator $\mathcal{R}$ on X. And, by and Theorem 9.8, we have $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ for any non-partial relator $\mathcal{R}$ on X. □

Concerning the family $\mathcal{T}^{\,p}_{\mathcal{R}^{\vartriangle}}$, instead of an analogue of Theorem 33.1, we can only prove the following counterpart of Corollary 33.2.

Theorem 33.3. *If $\mathcal{R}$ is a non-partial, non-degenerated relator on X, then*

$$\mathcal{T}^{\,p}_{\mathcal{R}^{\vartriangle}} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \quad \textit{with} \quad \emptyset \notin \mathcal{D}_{\mathcal{R}} \,.$$

Proof. If $A \in \mathcal{T}^{\,p}_{\mathcal{R}^{\vartriangle}} \setminus \{\emptyset\}$, then by Definition 19.2 we have

(a) $A \subseteq \operatorname{int}_{\mathcal{R}^{\vartriangle}}\big(\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(A)\big)$ and $A \neq \emptyset$.

Thus, in particular we have

(b) $\operatorname{int}_{\mathcal{R}^{\vartriangle}}\big(\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(A)\big) \neq \emptyset$.

Next, we show that (b) already implies $A \in \mathcal{D}_{\mathcal{R}}$. For this, note that if $A \notin \mathcal{D}_{\mathcal{R}}$, then by Corollary 7.8 we have $\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(A) = \emptyset$. Therefore, if (b) holds, then

$$\operatorname{int}_{\mathcal{R}^{\vartriangle}}(\emptyset) = \operatorname{int}_{\mathcal{R}^{\vartriangle}}\big(\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(A)\big) \neq \emptyset \,.$$

Hence, by using Corollary 7.8, we can infer that $\emptyset \in \mathcal{E}_{\mathcal{R}}$. Thus, by Theorem 9.8, the relator $\mathcal{R}$ cannot be non-partial. This contradiction proves that (b) implies $A \in \mathcal{D}_{\mathcal{R}}$, and thus $A \in \mathcal{T}^{\,p}_{\mathcal{R}^{\vartriangle}} \setminus \{\emptyset\}$ also implies $A \in \mathcal{D}_{\mathcal{R}}$.

In the sequel, we shall show that if $A \in \mathcal{D}_{\mathcal{R}}$, then

(c) $\operatorname{int}_{\mathcal{R}^{\vartriangle}}\big(\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(A)\big) = X$ and $A \neq \emptyset$.

For this, note that if $A \in \mathcal{D}_{\mathcal{R}}$, then by Corollary 7.8 we have $\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(A) = X$. Moreover, by Theorems 9.10 and 9.8, we have $\emptyset \notin \mathcal{D}_{\mathcal{R}}$ and $X \in \mathcal{D}_{\mathcal{R}}$. Thus, in particular $A \neq \emptyset$. Moreover, by Corollary 7.8, we have $\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(X) = X$. Therefore, in addition to $A \neq \emptyset$, we also have

$$\operatorname{int}_{\mathcal{R}^{\vartriangle}}\big(\operatorname{cl}_{\mathcal{R}^{\vartriangle}}(A)\big) = \operatorname{int}_{\mathcal{R}^{\vartriangle}}(X) = X,$$

and thus (c) also holds. Hence, since $A \in \mathcal{D}_{\mathcal{R}} \implies$ (c) $\implies$ (a) $\implies$ $A \in \mathcal{T}^{\,p}_{\mathcal{R}^{\vartriangle}} \setminus \{\emptyset\}$, we can see that $A \in \mathcal{D}_{\mathcal{R}}$ also implies $A \in \mathcal{T}^{\,p}_{\mathcal{R}^{\vartriangle}} \setminus \{\emptyset\}$.

Thus, we have proved that $\mathcal{T}^{\,p}_{\mathcal{R}^{\vartriangle}} \setminus \{\emptyset\} = \mathcal{D}_{\mathcal{R}}$. Therefore, to obtain the required assertion, we need only note that, by Definition 19.2, we have $\emptyset \in \mathcal{T}^{\,p}_{\mathcal{R}}$ for any relator $\mathcal{R}$ on X. Moreover, by Theorem 9.10 we have $\emptyset \notin \mathcal{D}_{\mathcal{R}}$ for any non-degenerated relator $\mathcal{R}$ on X. $\square$

From Theorem 33.3 and Corollary 33.2, by using Theorem 23.4, we can derive

Theorem 33.4. *If $\mathcal{R}$ is a non-partial, non-degenerated relator on X, then $\mathcal{T}^{\,r}_{\mathcal{R}^{\vartriangle}} = \{\emptyset, X\}$.*

Proof. If $A \in \mathcal{T}^{\,r}_{\mathcal{R}^{\vartriangle}}$, then by Theorem 23.4 we have

$$A \in \mathcal{T}^{\,p}_{\mathcal{R}^{\vartriangle}} \quad \text{and} \quad A \in \mathcal{F}^{\,s}_{\mathcal{R}^{\vartriangle}} \,,$$

and thus $A^{c} \in \mathcal{T}^{\,s}_{\mathcal{R}^{\vartriangle}}$. Hence, by using Theorem 33.3 and Corollary 33.2, we can infer that

$$\big(\, A \in \mathcal{D}_{\mathcal{R}} \quad \text{or} \quad A = \emptyset \,\big) \quad \text{and} \quad \big(\, A^{c} \in \mathcal{E}_{\mathcal{R}} \quad \text{or} \quad A^{c} = \emptyset \,\big),$$

and thus $A \notin \mathcal{D}_{\mathcal{R}}$ or $A = X$. Therefore, if $A \in \mathcal{D}_{\mathcal{R}}$, then only $A = X$ can hold. Hence, we can see that if $A \in \mathcal{T}^{\,r}_{\mathcal{R}^{\vartriangle}}$, then either $A = X$ or $A = \emptyset$ holds. Thus, $\mathcal{T}^{\,r}_{\mathcal{R}^{\vartriangle}} \subseteq \{\emptyset, X\}$.

On the other hand, from Theorems 33.3 and 9.8, we can see that $\emptyset,\, X \in \mathcal{T}^{\,p}_{\mathcal{R}^{\vartriangle}}$. Moreover, from Corollary 33.2 and Theorem 9.10, we can see that $\emptyset,\, X \in \mathcal{T}^{\,s}_{\mathcal{R}^{\vartriangle}}$, and thus also $\emptyset,\, X \in \mathcal{F}^{\,s}_{\mathcal{R}^{\vartriangle}}$. Therefore, by Theorem 23.4, we also have $\emptyset,\, X \in \mathcal{T}^{\,r}_{\mathcal{R}^{\vartriangle}}$, and thus also $\{\emptyset, X\} \subseteq \mathcal{T}^{\,r}_{\mathcal{R}^{\vartriangle}}$. $\square$

34. Topologically quasi-open and pseudo-open sets derived from the paratopological refinements of relators

In addition to Theorem 33.1, we can also prove the following

Theorem 34.1. *If $\mathcal{R}$ is a non-degenerated relator on X, then for any $A \subseteq X$ the following assertions are equivalent :*

(1) $A \in \mathcal{T}^{\,q}_{\mathcal{R}^{\vartriangle}} \setminus \{\emptyset\}$ *;*

(2) there exists $V \in \mathcal{E}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}}$ *such that* $V \subseteq A$ *.*

Proof. If (1) holds, then $A \neq \emptyset$. Moreover, by Definition 19.2, there exists $V \subseteq X$ such that

$$V \in \mathcal{T}_{\mathcal{R}^{\vartriangle}} \quad \text{and} \quad V \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}^{\vartriangle}}(V)\,.$$

Thus, in particular $\mathrm{cl}_{\mathcal{R}^{\vartriangle}}(V) \neq \emptyset$. Hence, by using Corollary 7.8, we can infer that $V \in \mathcal{D}_{\mathcal{R}}$. Moreover, from Theorem 9.10, we can see that $\emptyset \notin \mathcal{D}_{\mathcal{R}}$, and thus $V \neq \emptyset$. Furthermore, from Corollary 7.9, we can see that $\mathcal{T}_{\mathcal{R}^{\vartriangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$. Therefore, we necessarily have $V \in \mathcal{E}_{\mathcal{R}}$, and thus (2) also holds.

On the other hand, if (2) holds, then we have both $V \in \mathcal{E}_{\mathcal{R}}$ and $V \in \mathcal{D}_{\mathcal{R}}$. Hence, by Corollary 7.9, we can see that $V \in \mathcal{T}_{\mathcal{R}^{\vartriangle}}$. Moreover, from Theorem 9.10, we can see that $V \neq \emptyset$, and thus $A \neq \emptyset$. Furthermore, from Corollary 7.8, we can see that $\mathrm{cl}_{\mathcal{R}^{\vartriangle}}(V) = X$, and thus $A \subseteq \mathrm{cl}_{\mathcal{R}^{\vartriangle}}(V)$ trivially holds. Therefore, by Definition 19.2, assertion (1) also holds. □

Now, by using the above theorem, we can also easily establish

Corollary 34.2. *If $\mathcal{R}$ is a non-degenerated relator on X, then*

$$\mathcal{T}^{\,q}_{\mathcal{R}^{\vartriangle}} = \mathcal{E}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \quad \textit{with} \quad \emptyset \notin \mathcal{D}_{\mathcal{R}}\,.$$

Proof. To derive this from Theorem 34.1, note that $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$ are ascending families of subsetes of X. Therefore, if assertion (2) in Theorem 34.1 holds, then we necessarily have $A \in \mathcal{E}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}}$. Moreover, by Definition 19.2, we have $\emptyset \in \mathcal{T}^{\,q}_{\mathcal{R}}$. And, by Theorem 9.10, we have $\emptyset \notin \mathcal{D}_{\mathcal{R}}$. □

Analogously, to Theorem 34.1, we can also prove

Theorem 34.3. *If $\mathcal{R}$ is a non-degenerated relator on X, then for any $A \subseteq X$ the following assertions are equivalent :*

(1) $A \in \mathcal{T}^{\,ps}_{\mathcal{R}^{\vartriangle}} \setminus \{\emptyset\}$ *;*

(2) $A \in \mathcal{D}_{\mathcal{R}}$ *and there exists* $V \in \mathcal{E}_{\mathcal{R}}$ *such that* $A \subseteq V$ *.*

Proof. If (1) holds, then $A \neq \emptyset$. Moreover, by Definition 19.2, there exists $V \subseteq X$ such that

$$V \in \mathcal{T}_{\mathcal{R}^{\vartriangle}} \quad \text{and} \quad A \subseteq V \subseteq \mathrm{cl}_{\mathcal{R}^{\vartriangle}}(A)\,.$$

Thus, in particular $V \neq \emptyset$ and $\mathrm{cl}_{\mathcal{R}^{\vartriangle}}(A) \neq \emptyset$. Hence, by using Corollary 7.8, we can infer that $A \in \mathcal{D}_{\mathcal{R}}$. Moreover, from Corollary 7.9, we can see that $\mathcal{T}_{\mathcal{R}^{\vartriangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$. Therefore, $V \in \mathcal{E}_{\mathcal{R}}$, and thus (2) also holds.

On the other hand, if (2) holds, then by Theorem 9.10 and Corollary 7.8 we have $A \neq \emptyset$ and $\mathrm{cl}_{\mathcal{R}^{\vartriangle}}(A) = X$. Thus, we trivially have $V \subseteq \mathrm{cl}_{\mathcal{R}^{\vartriangle}}(A)$. Moreover, from Corollary 7.9, we can see that $V \in \mathcal{T}_{\mathcal{R}^{\vartriangle}}$. Therefore, by Definition 19.2, assertion (1) also holds. □

Now, by using the above theorem, we can also easily establish

Corollary 34.4. *If $\mathcal{R}$ is a non-degenerated relator on X, then*

$$\mathcal{T}^{\,ps}_{\mathcal{R}^{\vartriangle}} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \quad \textit{with} \quad \emptyset \notin \mathcal{D}_{\mathcal{R}}\,.$$

Proof. To derive this from Theorem 34.3, note that, by Definition 19.2, for any relator $\mathcal{R}$ on X we have $\emptyset \in \mathcal{T}^{\,ps}_{\mathcal{R}}$. Moreover, by Theorem 9.10, we have $X \in \mathcal{E}_{\mathcal{R}}$. Therefore, in assertion (2) of Theorem 34.3, we can take $V = X$. Furthermore, by Theorem 9.10, we have $\emptyset \notin \mathcal{D}_{\mathcal{R}}$. □

Remark 34.5. If $\mathcal{R}$ is a non-partial, non-degenerated relator on X, then by Theorem 33.3 and Corollary 34.4 we have $\mathcal{T}^{p}_{\mathcal{R}^{\triangle}} = \mathcal{T}^{ps}_{\mathcal{R}^{\triangle}}$.

However, the following theorem shows that, for a non-degenerated relator $\mathcal{R}$ on X, the above equality need not be true.

Theorem 34.6. *If $\mathcal{R}$ is a partial relator on X, then $\mathcal{T}^{ps}_{\mathcal{R}^{\triangle}} = \{\emptyset\}$ and $\mathcal{T}^{p}_{\mathcal{R}^{\triangle}} = \mathcal{P}(X)$.*

Proof. By Theorem 9.8, we have $\emptyset \in \mathcal{E}_{\mathcal{R}}$ and $X \notin \mathcal{D}_{\mathcal{R}}$. Hence, since $\mathcal{D}_{\mathcal{R}}$ is an ascending family of subsets of X, we can see that, for any $A \subseteq X$, we also have $A \notin \mathcal{D}_{\mathcal{R}}$. Now, by using Corollary 7.8, we can see that

$$\operatorname{int}_{\mathcal{R}^{\triangle}}(\operatorname{cl}_{\mathcal{R}^{\triangle}}(A)) = \operatorname{int}_{\mathcal{R}^{\triangle}}(\emptyset) = X.$$

Thus, by Definition 19.2, we have $\mathcal{T}^{p}_{\mathcal{R}^{\triangle}} = \mathcal{P}(X)$. Moreover, by Corollary 34.4, we can also see that

$$\mathcal{T}^{ps}_{\mathcal{R}^{\triangle}} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} = \emptyset \cup \{\emptyset\} = \{\emptyset\}.$$

Namely, since $\mathcal{R}$ is a partial relator on X, there exist $x \in X$ and $R \in \mathcal{R}$ such that $R(x) = \emptyset$. Thus, in particular $X \neq \emptyset$ and $\mathcal{R} \neq \emptyset$. Therefore, $\mathcal{R}$ is a non-degenerated relator on X. □

35. Topologically α–open, β–open, γ–open and δ–open sets derived from the paratopological refinements of relators

By using Theorems 9.8 and 9.10 and Corollary 7.8, we can also prove

Theorem 35.1. *If $\mathcal{R}$ is a non-partial, non-degenerated relator on X, then*

$$\mathcal{T}^{\alpha}_{\mathcal{R}^{\triangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\} \quad \textit{with} \quad \emptyset \notin \mathcal{E}_{\mathcal{R}}.$$

Proof. If $A \in \mathcal{T}^{\alpha}_{\mathcal{R}^{\triangle}}$, then by Definition 27.1 we have

$$A \subseteq \operatorname{int}_{\mathcal{R}^{\triangle}}(\operatorname{cl}_{\mathcal{R}^{\triangle}}(\operatorname{int}_{\mathcal{R}^{\triangle}}(A))).$$

Hence, if $A \neq \emptyset$, we can infer that

$$\operatorname{int}_{\mathcal{R}^{\triangle}}(\operatorname{cl}_{\mathcal{R}^{\triangle}}(\operatorname{int}_{\mathcal{R}^{\triangle}}(A))) \neq \emptyset.$$

Therefore, by Corollary 7.8, we necessarily have

$$\operatorname{cl}_{\mathcal{R}^{\triangle}}(\operatorname{int}_{\mathcal{R}^{\triangle}}(A)) \in \mathcal{E}_{\mathcal{R}}.$$

Moreover, from Theorem 9.8, we can see that $\emptyset \notin \mathcal{E}_{\mathcal{R}}$, and thus

$$\operatorname{cl}_{\mathcal{R}^{\triangle}}(\operatorname{int}_{\mathcal{R}^{\triangle}}(A)) \neq \emptyset.$$

Hence, by using Corollary 7.8, we can infer that $\operatorname{int}_{\mathcal{R}^{\triangle}}(A) \in \mathcal{D}_{\mathcal{R}}$. Moreover, from Theorem 9.10, we can see that $\emptyset \notin \mathcal{D}_{\mathcal{R}}$, and thus $\operatorname{int}_{\mathcal{R}^{\triangle}}(A) \neq \emptyset$. Hence, by using Corollary 7.8, we can already infer that $A \in \mathcal{E}_{\mathcal{R}}$. Thus, we have proved that $\mathcal{T}^{\alpha}_{\mathcal{R}^{\triangle}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$, and hence $\mathcal{T}^{\alpha}_{\mathcal{R}^{\triangle}} \subseteq \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$.

On the other hand, if $A \in \mathcal{E}_{\mathcal{R}}$, then by Corollary 7.8 we have $\operatorname{int}_{\mathcal{R}^{\triangle}}(A) = X$. Moreover, from Theorem 9.8, we can see that $X \in \mathcal{D}_{\mathcal{R}}$. Hence, by using Corollary 7.8, we can infer that

$$\operatorname{cl}_{\mathcal{R}^{\triangle}}(\operatorname{int}_{\mathcal{R}^{\triangle}}(A)) = \operatorname{cl}_{\mathcal{R}^{\triangle}}(X) = X.$$

Moreover, from Theorem 9.10, we can see that $X \in \mathcal{E}_{\mathcal{R}}$. Hence, by using Corollary 7.8, we can infer that

$$\operatorname{int}_{\mathcal{R}^{\triangle}}(\operatorname{cl}_{\mathcal{R}^{\triangle}}(\operatorname{int}_{\mathcal{R}^{\triangle}}(A))) = \operatorname{int}_{\mathcal{R}^{\triangle}}(X) = X.$$

Therefore,

$$A \subseteq \operatorname{int}_{\mathcal{R}^{\triangle}}(\operatorname{cl}_{\mathcal{R}^{\triangle}}(\operatorname{int}_{\mathcal{R}^{\triangle}}(A))),$$

and thus $A \in \mathcal{T}^{\alpha}_{\mathcal{R}^{\triangle}}$ trivially holds. Thus, we have proved that $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{T}^{\alpha}_{\mathcal{R}^{\triangle}}$. Hence, since $\emptyset \in \mathcal{T}^{\alpha}_{\mathcal{R}^{\triangle}}$ trivially holds, we can see that $\mathcal{E}_{\mathcal{R}} \cup \{\emptyset\} \subseteq \mathcal{T}^{\alpha}_{\mathcal{R}^{\triangle}}$ is also true. □

Now, quite similarly, we can also prove the following

Theorem 35.2. *If $\mathcal{R}$ is a non-partial, non-degenerated relator on X, then*

$$\mathcal{T}^{\beta}_{\mathcal{R}^{\triangle}} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \quad \textit{with} \quad \emptyset \notin \mathcal{D}_{\mathcal{R}} .$$

Remark 35.3. Thus, in contrast to Theorem 27.6, we usually have $\mathcal{T}^{\alpha}_{\mathcal{R}^{\triangle}} \not\subseteq \mathcal{T}^{\beta}_{\mathcal{R}^{\triangle}}$. This, in accordance with Remark 11.11, also shows that the relator $\mathcal{R}^{\triangle}$ is not, in general, reflexive.

By [132, Definition 31.1] relator $\mathcal{R}$ on X to Y may be naturally called *hyperconnected* if $\mathcal{E}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{R}}$. That is, the identity function Δ_Y of Y is *fatness reversing* in the sense of [168, Definition 12.3] .

Such relators have several remarkable properties. For instance, it can be easily shown that a relator $\mathcal{R}$ on X to Y is hyperconnected if and only if $R(x) \cap S(y) \neq \emptyset$ for all $x, y \in X$ and $R, S \in \mathcal{R}$.

Thus, in particular, a hyperconnected relator is non-partial. Therefore, if $\mathcal{R}$ is a non-degenerated, hyperconnected relator on X, then by Theorems 35.1 and 35.2 we already have $\mathcal{T}^{\alpha}_{\mathcal{R}^{\triangle}} \subseteq \mathcal{T}^{\beta}_{\mathcal{R}^{\triangle}}$.

Now, by using Corollaries 7.8 and 33.2, we can also prove the following

Theorem 35.4. *If $\mathcal{R}$ is a non-partial, non-degenerated relator on X, then for any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \mathcal{T}^{\gamma}_{\mathcal{R}^{\triangle}} \setminus \{\emptyset\}$ *;*

(2) $A \in \mathcal{D}_{\mathcal{R}}$ *and there exists* $V \in \mathcal{E}_{\mathcal{R}}$ *such that* $A \subseteq V$.

Proof. If (1) holds, then $A \neq \emptyset$. Moreover, by Definition 27.1, there exists $V \subseteq X$ such that

$$V \in \mathcal{T}^{s}_{\mathcal{R}^{\triangle}} \quad \text{and} \quad A \subseteq V \subseteq \mathrm{cl}_{\mathcal{R}^{\triangle}}(A) .$$

Thus, in particular, $V \neq \emptyset$ and $\mathrm{cl}_{\mathcal{R}^{\triangle}}(A) \neq \emptyset$ also hold. Hence, by using Corollaries 33.2 and 7.8, we can already infer that $V \in \mathcal{E}_{\mathcal{R}}$ and $A \in \mathcal{D}_{\mathcal{R}}$. Therefore, (2) also holds.

Conversely, if (2) holds, then by using Corollaries 33.2 and 7.8 we can see that

$$V \in \mathcal{T}^{s}_{\mathcal{R}^{\triangle}} \quad \text{and} \quad \mathrm{cl}_{\mathcal{R}^{\triangle}}(A) = X.$$

Thus, in particular $V \subseteq \mathrm{cl}_{\mathcal{R}^{\triangle}}(A)$ trivially holds. Hence, by Definition 27.1, we can see that $\mathcal{T}^{\gamma}_{\mathcal{R}^{\triangle}}$. Moreover, by Theorem 9.10, we can also note that $A \neq \emptyset$. Thus, (1) also holds. □

Hence, analogously to Corollary 34.4, we can also derive

Corollary 35.5. *If $\mathcal{R}$ is a non-partial, non-degenerated relator on X, then*

$$\mathcal{T}^{\gamma}_{\mathcal{R}^{\triangle}} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \quad \textit{with} \quad \emptyset \notin \mathcal{D}_{\mathcal{R}} .$$

Concerning the family $\mathcal{T}^{\delta}_{\mathcal{R}}$, instead of an analogue of Theorem 35.4, we can only prove the following counterpart of Corollary 35.5.

Theorem 35.6. *If $\mathcal{R}$ is a non-partial, non-degenerated relator on X, then*

$$\mathcal{T}^{\delta}_{\mathcal{R}^{\triangle}} = \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \quad \textit{with} \quad \emptyset \notin \mathcal{D}_{\mathcal{R}} .$$

Proof. If $A \in \mathcal{T}^{\delta}_{\mathcal{R}^{\triangle}}$, then by Definition 27.1 there exists $V \subseteq X$ such that

$$V \in \mathcal{T}^{p}_{\mathcal{R}^{\triangle}} \qquad \text{and} \qquad V \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}^{\triangle}}(V) .$$

Hence, if $A \neq \emptyset$, then we can infer that $\mathrm{cl}_{\mathcal{R}^{\triangle}}(V) \neq \emptyset$. Thus, by Corollary 7.8, we necessarily have $V \in \mathcal{D}_{\mathcal{R}}$. Hence, since $\mathcal{D}_{\mathcal{R}}$ is ascending, it is clear that $A \in \mathcal{D}_{\mathcal{R}}$ also holds. This already shows that $\mathcal{T}^{\delta}_{\mathcal{R}^{\triangle}} \subseteq \mathcal{D}_{\mathcal{R}} \cup \{\emptyset\}$.

On the other hand, if $A \in \mathcal{D}_{\mathcal{R}}$, then by using Theorem 33.3 and Corollary 7.8, we can see that

$$A \in \mathcal{T}^{p}_{\mathcal{R}^{\triangle}} \quad \text{and} \quad \mathrm{cl}_{\mathcal{R}^{\triangle}}(V) = X.$$

Hence, by taking $V = A$, we can see that

$$V \in \mathcal{T}^{p}_{\mathcal{R}^{\triangle}} \quad \text{and} \quad V \subseteq A \subseteq \mathrm{cl}_{\mathcal{R}^{\triangle}}(V) .$$

Thus, by Definition 27.1, we also have $A \in \mathcal{T}^{\delta}_{\mathcal{R}^{\triangle}}$. Hence, since $\emptyset \in \mathcal{T}_{\mathcal{R}}$ is always true, we can already infer that $\mathcal{D}_{\mathcal{R}} \cup \{\emptyset\} \subseteq \mathcal{T}^{\delta}_{\mathcal{R}^{\triangle}}$. □

36. A Further Illustrating Example and a Diagram

The following example will show that, for a non-reflexive relator $\mathcal{R}$, the six inclusions derivable from Theorems 19.6 and 27.5 need not be true.

Example 36.1. If $X = \{1, 2, 3\}$, and R_1 and R_2 are relations on X such that

$R_1(1) = \{1\}$, $R_1(2) = \{1\}$, $R_1(3) = \{2, 3\}$;

$R_2(1) = \{1\}$, $R_2(2) = \{2, 3\}$, $R_2(3) = \{1, 2\}$;

then $\mathcal{R} = \{R_1, R_2\}$ is a non-partial relator on X such that:

(1) $\mathcal{T}_{\mathcal{R}}^{r} = \{\emptyset, X\}$;

(2) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}}^{q} \cup \{\{2, 3\}\}$;

(3) $\mathcal{T}_{\mathcal{R}}^{q} = \mathcal{T}_{\mathcal{R}}^{ps} = \{\emptyset, \{1\}, \{1, 2\}, X\}$;

(4) $\mathcal{T}_{\mathcal{R}}^{s} = \mathcal{T}_{\mathcal{R}}^{p} = \mathcal{T}_{\mathcal{R}}^{a} = \mathcal{T}_{\mathcal{R}}^{b} = \mathcal{T}_{\mathcal{R}}^{\alpha} = \mathcal{T}_{\mathcal{R}}^{\beta} = \mathcal{T}_{\mathcal{R}}^{\gamma} = \mathcal{T}_{\mathcal{R}}^{\delta} = \mathcal{T}_{\mathcal{R}}^{q} \cup \{\{1, 3\}\}$.

To check this, recall that for any $x \in X$ and $A \subseteq X$ we have

$$x \in A^{\circ} \iff \exists\, R \in \mathcal{R}: \ R(x) \subseteq A \quad (x \in A^{-} \iff \forall\, R \in \mathcal{R}: \ R(x) \cap A \neq \emptyset).$$

Therefore, concerning the relevant operations on subsets of X, we can state:

A	A°	A^{-}	$A^{\circ-}$	$A^{-\circ}$	$A^{\circ-\circ}$	$A^{-\circ-}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1\}$	$\{1,2\}$	$\{1\}$	X	$\{1,2\}$	X	X
$\{2\}$	$\emptyset$	$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1,2\}$	X	X	X	X	X	X
$\{1,3\}$	$\{1,2\}$	X	X	X	X	X
$\{2,3\}$	$\{2,3\}$	$\{3\}$	$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$
X	X	X	X	X	X	X

By using Theorems 19.5, 19.6, 23.4, 27.3, 27.4, 27.5, 31.2 and 31.3, we can easily establish the following

Diagram 36.2. For a reflexive relator $\mathcal{R}$ on X, the following implications hold:

$$
\begin{tikzcd}
 & & A \in \mathcal{T}_{\mathcal{R}}^{\beta} & & \\
A \in \mathcal{T}_{\mathcal{R}}^{\delta} \arrow[urr, Rightarrow] & & A \in \mathcal{T}_{\mathcal{R}}^{b} \arrow[u, Rightarrow] & & A \in \mathcal{T}_{\mathcal{R}}^{\gamma} \arrow[ull, Rightarrow] \\
 & A \in \mathcal{T}_{\mathcal{R}}^{s} \arrow[ur, Rightarrow] \arrow[urrr, Rightarrow] & A \in \mathcal{T}_{\mathcal{R}}^{r} \arrow[r, Rightarrow] & A \in \mathcal{T}_{\mathcal{R}}^{p} \arrow[ul, Rightarrow] \arrow[ulll, Rightarrow] & \\
 & & A \in \mathcal{T}_{\mathcal{R}}^{a} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & & \\
 & A \in \mathcal{T}_{\mathcal{R}}^{q} \arrow[u, Rightarrow] \arrow[uuul, Rightarrow] & A \in \mathcal{T}_{\mathcal{R}}^{\alpha} \arrow[u, Rightarrow] & A \in \mathcal{T}_{\mathcal{R}}^{ps} \arrow[u, Rightarrow] \arrow[uuur, Rightarrow] & \\
 & & A \in \mathcal{T}_{\mathcal{R}} \arrow[ul, Rightarrow] \arrow[u, Rightarrow] \arrow[ur, Rightarrow] & &
\end{tikzcd}
$$

Remark 36.3. Note that, by Theorems 19.5, 23.4 and 31.2, nine from the above implications do not require the relator $\mathcal{R}$ to be reflexive.

37. Two Clarifying Examples to the Above Diagram

The following simple example will show only that eight implications in Diagram 1 are not reversible.

Example 37.1. If $X = \{1, 2, 3\}$ and R is a relation on X such that

$$R(1) = \{1, 2\} \quad \text{and} \quad R(2) = R(3) = X,$$

then $\mathcal{R} = \{R\}$ is a reflexive relator on X such that:

(1) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}}^{q} = \{\emptyset, X\}$;

(2) $\mathcal{T}_{\mathcal{R}}^{s} = \mathcal{T}_{\mathcal{R}}^{a} = \mathcal{T}_{\mathcal{R}}^{\alpha} = \{\emptyset, \{1, 2\}, X\}$;

(3) $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{T}_{\mathcal{R}}^{p} = \mathcal{T}_{\mathcal{R}}^{b} = \mathcal{T}_{\mathcal{R}}^{\beta} = \mathcal{T}_{\mathcal{R}}^{\gamma} = \mathcal{T}_{\mathcal{R}}^{\delta} = \mathcal{P}(X) \setminus \{\{3\}\}$.

To check this, note that now we have:

A	A°	A^{-}	$A^{\circ-}$	$A^{-\circ}$	$A^{\circ-\circ}$	$A^{-\circ-}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{2\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{3\}$	$\emptyset$	$\{2,3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1,2\}$	$\{1\}$	X	X	X	X	X
$\{1,3\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{2,3\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
X	X	X	X	X	X	X

The following, more difficult example will already show that sixteen implications in Diagram 1 are not reversible.

Example 37.2. If $X = \{1, 2, 3, 4\}$ and R_1 and R_2 are relations on X such that

$$R_1(1) = R_1(2) = \{1, 2, 3\}, \qquad R_1(3) = R_1(4) = \{1, 3, 4\};$$

$$R_2(1) = \{1, 2, 3\}, \qquad R_2(2) = \{1, 2\}, \qquad R_2(3) = R_2(4) = \{3, 4\};$$

then $\mathcal{R} = \{R_1, R_2\}$ is a reflexive relator on X such that:

(1) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}}^{\alpha} = \{\emptyset, \{3, 4\}, X\}$;

(2) $\mathcal{T}_{\mathcal{R}}^{r} = \mathcal{T}_{\mathcal{R}} \cup \{\{2\}\}$;

(3) $\mathcal{T}_{\mathcal{R}}^{q} = \mathcal{T}_{\mathcal{R}}^{a} = \mathcal{T}_{\mathcal{R}} \cup \{\{1, 3, 4\}\}$;

(4) $\mathcal{T}_{\mathcal{R}}^{s} = \mathcal{T}_{\mathcal{R}}^{q} \cup \{\{1, 2\}\}$;

(5) $\mathcal{T}_{\mathcal{R}}^{p} = \mathcal{P}(X) \setminus \{\{1\}, \{1, 2\}\}$;

(6) $\mathcal{T}_{\mathcal{R}}^{ps} = \mathcal{T}_{\mathcal{R}}^{p} \setminus \{\{2\}\}$;

(7) $\mathcal{T}_{\mathcal{R}}^{b} = \mathcal{T}_{\mathcal{R}}^{\delta} = \mathcal{P}(X) \setminus \{\{1\}\}$;

(8) $\mathcal{T}_{\mathcal{R}}^{\beta} = \mathcal{T}_{\mathcal{R}}^{\gamma} = \mathcal{P}(X)$.

To check this, note that now we have:

A	A°	A^-	$A^{\circ-}$	$A^{-\circ}$	$A^{\circ-\circ}$	$A^{-\circ-}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1\}$	$\emptyset$	$\{1,2\}$	$\emptyset$	$\{2\}$	$\emptyset$	$\{1,2\}$
$\{2\}$	$\emptyset$	$\{1,2\}$	$\emptyset$	$\{2\}$	$\emptyset$	$\{1,2\}$
$\{3\}$	$\emptyset$	$\{1,3,4\}$	$\emptyset$	$\{3,4\}$	$\emptyset$	$\{1,3,4\}$
$\{4\}$	$\emptyset$	$\{3,4\}$	$\emptyset$	$\{3,4\}$	$\emptyset$	$\{1,3,4\}$
$\{1,2\}$	$\{2\}$	$\{1,2\}$	$\{1,2\}$	$\{2\}$	$\{2\}$	$\{1,2\}$
$\{1,3\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{1,4\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{2,3\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{2,4\}$	$\emptyset$	X	$\emptyset$	X	$\emptyset$	X
$\{3,4\}$	$\{3,4\}$	$\{1,3,4\}$	$\{1,3,4\}$	$\{3,4\}$	$\{3,4\}$	$\{1,3,4\}$
$\{1,2,3\}$	$\{1,2\}$	X	$\{1,2\}$	X	$\{2\}$	X
$\{1,2,4\}$	$\{2\}$	X	$\{1,2\}$	X	$\{2\}$	X
$\{1,3,4\}$	$\{3,4\}$	X	$\{1,3,4\}$	X	$\{3,4\}$	X
$\{2,3,4\}$	$\{3,4\}$	X	$\{1,3,4\}$	X	$\{3,4\}$	X
X	X	X	X	X	X	X

Remark 37.3. Examples 37.1 and 37.2, together, already show that seventeen implications in Diagram 36.2 are not reversible.

However, unfortunately, they cannot be used to show that the remaining implication $A \in \mathcal{T}_{\mathcal{R}}^{\gamma} \implies A \in \mathcal{T}_{\mathcal{R}}^{\beta}$ is also not reversible.

Note that, by Theorem 27.3, the above implication does not require the relator $\mathcal{R}$ to be reflexive. Moreover, if $\mathcal{R}$ is topological, then by Theorem 30.2 the reverse implication is also true.

Acknowledgements

This paper is dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th Birthday.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ. **12**, 77-90, 1983.
[2] T. Aho and T. Nieminen, *Spaces in which preopen subsets are semiopen*, Richerche Mat. **43**, 45-49, 1994.
[3] P. Alexandroff, *Zur Begründung der n-dimensionalen mengentheorischen topologie*, Math. Ann. **94**, 296-308, 1925.
[4] T. Al-Hawary, *Generalized preopen sets*, Quest. Answ. Gen. Topology **29**, 73-80, 2011.
[5] T. Al-Hawary, *On generalized preopen sets*, Proyecciones J. Math. **32**, 296-308, 2013.
[6] T. M. Al-shami, *On supra semi open sets and some applications on topological spaces*, J. Adv. Stud. Topol. **8**, 144-153, 2017.
[7] D. Andrijević, *Some properties of the topology of α–open sets*, Mat. Vesnik **36**, 1-10, 1984.
[8] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik **38**, 24-32, 1986.
[9] D. Andrijević, *On the topology generated by preopen sets*, Mat. Vesnik **39**, 367-376, 1987.
[10] D. Andrijević, *On SPO–equivalent topologies*, Suppl. Rend. Circ. Mat. Palermo **29**, 317-328, 1992.
[11] D. Andrijević, *A note on α–equivalent topologies*, Mat. Vesnik **45**, 65-69, 1993.
[12] D. Andrijević, *On b–open sets*, Mat. Vesnik **48**, 59-64, 1996.
[13] D. Andrijević and M. Ganster, *On PO–equivalent topologies*, Suppl. Rend. Circ. Mat. Palermo **24**, 251-256, 1990.
[14] H. Arianpoor, *Preorder relators and generalized topologies*, J. Lin. Top. Algebra **5**, 271-277, 2016.
[15] J. Ávila and F. Molina, *Generalized weak structures*, Int. Math. Forum **7**, 2589-2595, 2012.
[16] M. Babahmed, *Semi-open sets and semi-separability in topological spaces*, Available on the Internet, 18 pp., 2017.

[17] C. Boonpok and J. Khampakdee, *Between closed sets and generalized closed sets in closure spaces*, Acta Math. Univ. Ostrav. **16**, 3-14, 2008.
[18] N. Bourbaki, *General Topology, Chapters 1–4*, Springer-Verlag, Berlin, 1989.
[19] M. Caldas, *A note on some applications of α–open sets*, IJMMS **2**, 125-130, 2003.
[20] M. Caldas, S. Jafari and R. K. Saraf, *Semi–Θ–open sets and new classes of maps*, Bull. Iran. Math. Soc. **31**, 37-52, 2005.
[21] C. Carpintero, N. Rajesh and E. Rosas, *On $[\gamma,\gamma]$–semiopen sets*, Bol. Mat. **17**, 125-136, 2010.
[22] E. Cech, *Topological Spaces*, Academia, Prague, 1966.
[23] T. A. Chapman, *A further note on closure and interior operators*, Amer. Math. Monthly **69**, 524-529, 1962.
[24] M. P. Chaudhary and V. Kumar, *On α–open sets in a topological space*, IJMA **2**, 703-706, 2011.
[25] M. Chicourrat and C. D. Horváth, *Pretopologies, preuniformities and probabilistic metric spaces*, Acta Math. Hungar. **110**, 91-116, 2006.
[26] H. H. Corson and E. Michael, *Metrizability of countable unions*, Illinois J. Math. **8**, 351-360, 1964.
[27] S. G. Crossley, *A note on semitopological classes*, Proc. Amer. Math. Soc. **43**, 416-420, 1974.
[28] S. G. Crossley, *A note on semitopological properties*, Proc. Amer. Math. Soc. **72**, 409-412, 1978.
[29] S. G. Crossley and S. K. Hildebrandt, *Semi-closure*, Texas J. Sci. **22**, 99-112, 1971.
[30] S. G. Crossley and S. K. Hildebrandt, *Semi-closed sets and semi-continuity in topological spaces*, Texas J. Sci. **22**, 123-126, 1971.
[31] S. G. Crossley and S. K. Hildebrandt, *Semi-topological properties*, Fund. Math. **74**, 233-254, 1972.
[32] Á. Császár, *Foundations of General Topology*, Pergamon Press, London, 1963.
[33] Á. Császár, *Generalized open sets*, Acta Math. Hungar. **75**, 65-87, 1997.
[34] Á. Császár, *On the γ–interior and γ–closure of a set*, Acta Math. Hungar. **80**, 89-93, 1998.
[35] Á. Császár, *γ–quasi-open sets*, Studia Sci. Math. Hungar. **38**, 171-176, 2001.
[36] Á. Császár, *Remarks on γ-quasi-open sets*, Studia Sci. Math. Hungar. **39**, 137-141, 2002.
[37] Á. Császár, *Generalized open sets in generalized topologies*, Acta Math. Hungar. **106**, 53-66, 2005.
[38] Á. Császár, *Further remarks on the formula for the γ–interior*, Acta Math. Hungar. **113**, 325-332, 2006.
[39] Á. Császár, *Remarks on quasi-topologies*, Acta Math. Hungar. **119**, 197-200, 2008.
[40] Á. Császár, *Weak structures*, Acta Math. Hungar. **131**, 193-195, 2011.
[41] H. M. Darwesh and S. F. Namiq, *Some properties of preopen set in closure spaces*, J. Garmian Univ. **5**, 229-246, 2018.
[42] P. Das, *Note on some applications of semi-open sets*, Progress Math. **7**, 33-44, 1973.
[43] A. K. Das, *A note on weak structures due to Császár*, Bul. Acad. Stiinte Repub. Mold. Mat. **2**, 114-116, 2015.
[44] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 2002.
[45] A. S. Davis, *Indexed systems of neighbordoods for general topological spaces*, Amer. Math. Monthly **68**, 886-893, 1961.
[46] R. Devi, S. Sampathkumar and M. Caldas, *On α–open sets and $s\alpha$–continuous maps*, Gen. Math. **16**, 77-84, 2008.
[47] K. Dlaska, N. Ergun, M. Ganster, *On the topology generated by semi-regular sets*, Indian J. Pure Appl. Math. **25**, 1163-1170, 1994.
[48] J. Dontchev, *The characterization of spaces and maps via semi-preopen sets*, Indian J. Pure Appl. Math. **25**, 939-947, 1994.
[49] J. Dontchev, *On generalizing semi-preopen sets*, Mem. Fac. Sci. Kochi Univ. **16**, 35-48, 1995.
[50] J. Dontchev, *Survey on preopen sets*, Meetings on Topological Spaces, Theory and Applications, Yatsushiro College of Technology, Kumamoto, Japan, 18 pp., 1998.
[51] Z. Duszyński and T. Noiri, *Semi-open, semi-closed sets and semi-continuity of functions*, Math. Pannon. **23**, 195-200, 2012.
[52] V. A. Efremovič, *The geometry of proximity*, Mat. Sb. **31**, 189-200, 1952, (Russian).
[53] V. A. Efremović and A. S. Švarc, *A new definition of uniform spaces. Metrization of proximity spaces*, Dokl. Acad. Nauk. SSSR **89**, 393-396, 1953, (Russian).
[54] E. Ekici, *On weak structures due to Császár*, Acta Math. Hungar. **134**, 565-570, 2012.
[55] E. Ekici and B. Roy, *New generalized topologies on generalized topological spaces due to Császár*, Acta Math. Hungar. **132**, 117-124, 2011.
[56] N. Elez and O. Papaz, *The new operators in topological spaces*, Math. Moravica **17**, 63-68, 2013.
[57] M. E. El-Shafei, M. Abo-elhamayel and T. M. Al-shami, *On supra R-open sets and some applications on topological spaces*, J. Progr. Res. Math. **8**, 1237-1248, 2016.
[58] P. Fletcher and W. F. Lindgren, *Quasi-Uniform Spaces*, Marcel Dekker, New York, 1982.
[59] M. Ganster, *Preopen sets and resolvable spaces*, Kyungpook Math. J. **27**, 135-143, 1987.
[60] M. Ganster and D. Andrijević, *On some questions concerning semi-preopen sets*, J. Inst. Math. Com. Sci. **1**, 65-72, 1988.
[61] M. Ganster and M. Steiner, *On some questions about b–open sets*, Quest. Answers in Gen. Topology **25**, 45-52, 2007.
[62] B. Ganter and R. Wille, *Formal Concept Analysis*, Springer-Verlag, Berlin, 1999.
[63] R. Gargouri and A. Rezgui, *A unification of weakening of open and closed subsets in a topological spaces*, Bull. Malays. Math. Sci. Soc. **40**, 1219-1230, 2017.
[64] S. Givant and P. Halmos, *Introduction to Boolean Algebras*, Springer-Verlag, Berlin, 2009.
[65] T. Glavosits, *Generated preorders and equivalences*, Acta Acad. Paed. Agrienses, Sect. Math. **29**, 95-103, 2002.
[66] A. Gupta and R. D. Sarma, *A note on some generalized closure and interior operators in a topological space*, Math. Appl. **6**, 11-20, 2017.
[67] A. Güldürdek and O. B. Özbakir, *On γ–semi-open sets*, Acta Math. Hungar. **109**, 347-355, 2005.
[68] T. R. Hamlett, *Correction to the paper "Semi-open sets and semi-continuity in topological spaces" by Norman Levine*, Proc. Amer. Math. Soc. **70**, 36-41, 1963.
[69] P. C. Hammer, *Kuratowski's closure theorem*, Nieuw Arch. Wisk. **8**, 74-80, 1960.
[70] F. Hausdorff, *Grundzüge der Mengenlehre*, De Gruyter, Berlin, 1914.
[71] W. Hunsaker and W. Lindgren, *Construction of quasi-uniformities*, Math. Ann. **188**, 39-42, 1970.
[72] S. Hussain, *On generalized open sets*, Hacet. J. Math. Stat. **47**, 1438-1446, 2018.
[73] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A **27**, 222-224, 1941.
[74] B. M. Ittanagi and S. S. Benchalli, *On paraopen sets and maps in topological spaces*, Kyungpook Math. J. **56**, 301-310, 2016.
[75] Y. Isomichi, *New concept in the theory of topological spaces–Supercondensed set, subcondensed set, and condensed set*, Pacific J. Math. **38**, 657-668, 1971.

[76] S. Jafari and S. Tahiliani, *Supra β–open sets and supra β–continuity on topological spaces*, Ann. Univ. Sci. Budapest Sect. Math. **56**, 1-9, 2013.
[77] R. Jamunarani, P. Jeyanthi and T. Noiri, *On generalized weak structures*, **47**, 21-26, 2016.
[78] D. S. Janković and I. L. Reilly, *On semi separation properties*, Indian J. Pure Apppl. Math. **16**, 957-964, 1985.
[79] Y. B. Jun, S. W. Jeong, H. j. Lee and J. W. Lee, *Applications of pre-open sets*, Appl. Gen. Top. **9**, 213-228, 2008.
[80] S.-M. Jung, *Interiors and cosures of sets and applications*, Int. J. Pure Math. **3**, 41-45, 2016.
[81] S.-M. Jung, D. Nam, *Some properties of interior and closure in general topology*, Mathematics **7 (7)**, 624, 1-10, 2019.
[82] J. L. Kelley, *General Topology*, Van Nostrand Reinhold Company, New York, 1955.
[83] J. Khampakdee, *Semi-Open Sets in Closure Spaces*, Ph.D. Thesis under the supervision of J. Šlapal. 55 pp., Department of Mathematics, Brno University of Technology, Brno, 2009.
[84] Y. K. Kim, W. K. Min, *σ–structures and quasi-enlarging operations*, Int. J. Pure Appl. Math. **86**, 791-798, 2013.
[85] V. S. Kumar, *On semi-open sets and semi-separability*, Global J Sci. Front. Res. Math. Dec. Sci. **13**, 17-20, 2013.
[86] V. Kumari, T. C. K. Raman, *On characterization of b-open sets in topological spaces*, Int. J. Appl. Innov. Eng. Manag. **2**, 229-235, 2013.
[87] K. Kuratowski, *Sur l'opération $\overline{A}$ de l'analysis situs*, Fund. Math. **3**, 182-199, 1922. (An English translation: On the operation $\overline{A}$ in analysis situs, prepared by M. Bowron in 2010).
[88] K. Kuratowski, *Topology I*, Academic Press, New York, 1966.
[89] J. Kurdics, *A note on connection properties*, Acta Math. Acad. Paedagog. Nyházi. **12**, 57-59, 1990.
[90] J. Kurdics and Á. Száz, *Well-chainedness characterizations of connected relators*, Math. Pannon. **4**, 37-45, 1993.
[91] R. M. Latif, *Semi-convergence of filters and nets*, Math. J. Okayama Univ. **41**, 103-109, 1999.
[92] Y. Lei and J. Zhang, *Generalizing topological set operators*, Electon. Notes Theor. Comput. Sci. **345**, 63-76, 2019.
[93] N. Levine, *On the commutivity of the closure and interior operators in topological spaces*, Amer. Math. Montly **68**, 474-477, 1961.
[94] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70**, 36-41, 1963.
[95] N. Levine, *Some remarks on the closure operator in topological spaces*, Amer. Math. Monthly **70**, p. 553, 1963.
[96] N. Levine, *On uniformities generated by equivalence relations*, Rend. Circ. Mat. Palermo **18**, 62-70, 1969.
[97] N. Levine, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo **19**, 89-96, 1970.
[98] N. Levine, *On Pervin's quasi uniformity*, Math. J. Okayama Univ. **14**, 97-102, 1970.
[99] J. Mala, *Relators generating the same generalized topology*, Acta Math. Hungar. **60**, 291-297, 1992.
[100] J. Mala and Á. Száz, *Properly topologically conjugated relators*, Pure Math. Appl., Ser. B **3**, 119-136, 1992.
[101] J. Mala and Á. Száz, *Modifications of relators*, Acta Math. Hungar. **77**, 69-81, 1997.
[102] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt **53**, 47-53, 1982.
[103] M. Mirmiran, *A note on extremally disconnected spaces*, Res. Open J. Inf. Sci. **1**, 1-3, 2013.
[104] S. Modak, *Remarks on dense sets*, Int. Math. Forum **6**, 2153-2158, 2011.
[105] A. Mukharjee and R. M. Roy, *On generalized preopen sets*, Mat. Stud. **51**, 195-199, 2019.
[106] S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge University Press, Cambridge, 1970.
[107] H. Nakano and K. Nakano, *Connector theory*, Pacific J. Math. **56**, 195-213, 1975.
[108] A. A. Nasef and R. Mareay, *More on simply open sets and its applications*, South Asian J. Math. **5**, 100-108, 2015.
[109] M. Navaneethakrishnan and S. Thamaraiselvi, *On weak structures of Császár*, Acta Math. Hungar. **137**, 224-229, 2012.
[110] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math. **15**, 195-213, 1965.
[111] T. Noiri, *On semi-continuous mappings*, Lincei-Rend. Sci. Fis. Mat. Nat. **54**, 210-214, 1973.
[112] T. Noiri, *Hyperconnectedness and preopen sets*, Rev. Roum. Math. Pures Appl. **29**, 329-334, 1984.
[113] T. Noiri, *On α–continuous functions*, Časopis Pěst. Mat. **109**, 118-126, 1984.
[114] T. Noiri, *Remarks on δ–semi-open sets and δ–preopen sets*, Demonstration Math. **36**, 1007-1020, 2003.
[115] T. M. Nour, *A note on some applications of semi-open sets*, Int. J. Math. Math. Sci. **21**, 205-207, 1998.
[116] G. Pataki, *Supplementary notes to the theory of simple relators*, Radovi Mat. **9**, 101-118, 1999.
[117] G. Pataki, *On the extensions, refinements and modifications of relators*, Math. Balk. **15**, 155-186, 2001.
[118] G. Pataki and Á. Száz, *A unified treatment of well-chainedness and connectedness properties*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **19**, 101-165, 2003.
[119] D. Peleg, *A generalized closure and complement phenomenon*, Discrete Math. **50**, 285-293, 1984.
[120] W. J. Pervin, *Uniformizations of neighborhood axioms*, Math. Ann. **147**, 313-315, 1962.
[121] W. J. Pervin, *Quasi-uniformization of topological spaces*, Math. Ann. **147**, 316-317, 1962.
[122] W. J. Pervin, *Quasi-proximities for topological spaces*, Math. Ann. **150**, 325-326, 1963.
[123] V. Pipitone and G. Russo, *Spazi semiconnessi e spazi semiaperty*, Rend. Circ. Mat. Palermo **24**, 273-285, 1975.
[124] I. Ramabhadrasarma and V. Srinivasakumar, *On semi-open sets and semi-continuity*, J. Adv. Stud. Topol. **3**, 6-10, 2012.
[125] V. V. S. Ramachandram, *A note on pre-open sets*, Res. J. Pure Algebra **1**, 132-134, 2011.
[126] Th. M. Rassias, M. Salih and Á. Száz, *Set-theoretic properties of generalized topologically open sets in relator spaces*, In: I. N. Parasidis, E. Providas and Th. M. Rassias (Eds.), Mathematical Analysis in Interdisciplinary Research, Springer, to appear.
[127] Th. M. Rassias, M. Salih and Á. Száz, *Characterizations and set-theoretic properties of generalized proximally open sets in relator spaces*, in preparation.
[128] I. L. Reilly and M. K. Vamanamurthy, *On α-continuity in topological spaces*, Acta Math. Hungar. **45**, 27-32, 1985.
[129] I. L. Reilly and M. K. Vamanamurthy, *On some questions concerning preopen sets*, Kyungpook Math. J. **30**, 87-93, 1990.
[130] F. Riesz, *Die Genesis der Raumbegriffs*, Math. Naturwiss. Ber. Ungarn **24**, 309-353, 1906.
[131] F. Riesz, *Stetigkeitsbegriff und abstrakte Mengenlehre*, Atti IV Congr. Intern.Mat., Roma **II**, 18-24, 1908.
[132] M. Salih and Á. Száz, *Generalizations of some ordinary and extreme connectedness properties of topological spaces to relator spaces*, Elec. Res. Arch. **28**, 471-548, 2020.

[133] R. D. Sarma, *On extremally disconnected generalized topologies*, Acta Math. Hungar. **134**, 583-588, 2012.
[134] M. S. Sarsak, *On b–open sets and associated generalized open sets*, Questions Answers Gen. Topology **27**, 157-173, 2009.
[135] M. S. Sarsak, *On some properties of generalized open sets in generalized topological spaces*, Demonstr. Math. **46**, 415-427, 2013.
[136] O. R. Sayed, *Supra pre-open sets and supra pre-continuity on topological spaces*, Sci. Stud. Res. Ser. Math. Inform. **20**, 79-88, 2010.
[137] O. R. Sayed and T. Noiri, *On supra b-open sets and supra b-continuity on topological spaces*, Eur. J. Pure Appl. Math. **3**, 295-302, 2010.
[138] W. Sierpinski, *General Topology*, (Mathematical Expositions No. 7), University of Toronto Press, Toronto, 1956.
[139] D. Sivaraj, *A note on semitopological properties*, Math. Chronicle **13**, 73-78, 1984.
[140] P. Sivagami, *Remarks on γ–interior*, Acta Math. Hungar. **119**, 81-94, 2008.
[141] Yu. M. Smirnov, *On proximity spaces*, Math. Sb. **31**, 543-574, 1952, (Russian).
[142] B. M. R. Stadler and P. F. Stadler, *Basic properties of closure spaces*, J. Chem. Inf. Comput. Sci. **42**, 577-585, 2002.
[143] L. A. Steen and J. A. Seebach, *Counterexamples in Topology*, Springer-Verlag, New York, 1970.
[144] M. H. Stone, *Application of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. **41**, 374-481, 1937.
[145] Á. Száz, *Basic tools and mild continuities in relator spaces*, Acta Math. Hungar. **50**, 177-201, 1987.
[146] Á. Száz, *Directed, topological and transitive relators*, Publ. Math. Debrecen **35**, 179-196, 1988.
[147] Á. Száz, *The fat and dense sets are more important than the open and closed ones*, Abstracts of the Seventh Prague Topological Symposium, Inst. Math. Czechoslovak Acad. Sci., p. 106, 1991.
[148] Á. Száz, *Relators, Nets and Integrals*, Unfinished doctoral thesis, Debrecen, 126 pp., 1991.
[149] Á. Száz, *Inverse and symmetric relators*, Acta Math. Hungar. **60**, 157-176, 1992.
[150] Á. Száz, *Structures derivable from relators*, Singularité **3**, 14-30, 1992.
[151] Á. Száz, *Refinements of relators*, Tech. Rep., Inst. Math., Univ. Debrecen **76**, 19 pp., 1993.
[152] Á. Száz, *Neighbourhood relators*, Bolyai Soc. Math. Stud. **4**, 449-465, 1995.
[153] Á. Száz *Topological characterizations of relational properties*, Grazer Math. Ber. **327**, 37-52, 1996.
[154] Á. Száz, *Somewhat continuity in a unified framework for continuities of relations*, Tatra Mt. Math. Publ. **24**, 41-56, 2002.
[155] Á. Száz, *Upper and lower bounds in relator spaces*, Serdica Math. J. **29**, 239-270, 2003.
[156] Á. Száz, *Rare and meager sets in relator spaces*, Tatra Mt. Math. Publ. **28**, 75-95, 2004.
[157] Á. Száz, *Galois-type connections on power sets and their applications to relators*, Tech. Rep., Inst. Math., Univ. Debrecen **2005/2**, 38 pp., 2005.
[158] Á. Száz, *Supremum properties of Galois–type connections*, Comment. Math. Univ. Carolin. **47**, 569-583, 2006.
[159] Á. Száz, *Minimal structures, generalized topologies, and ascending systems should not be studied without generalized uniformities*, Filomat **21**, 87-97, 2007.
[160] Á. Száz, *Galois type connections and closure operations on preordered sets*, Acta Math. Univ. Comenian. (N.S.) **78**, 1-21, 2009.
[161] Á. Száz, *Foundations of the theory of vector relators*, Adv. Stud. Contemp. Math. **20**, 139-195, 2010.
[162] Á. Száz, *Inclusions for compositions and box products of relations*, J. Int. Math. Virt. Inst. **3**, 97-125, 2013.
[163] Á. Száz, *Lower semicontinuity properties of relations in relator spaces*, Adv. Stud. Contemp. Math. (Kyungshang) **23**, 107-158, 2013.
[164] Á. Száz, *A particular Galois connection between relations and set functions*, Acta Univ. Sapientiae, Math. **6**, 73-91, 2014.
[165] Á. Száz, *Generalizations of Galois and Pataki connections to relator spaces*, J. Int. Math. Virtual Inst. **4**, 43-75, 2014.
[166] Á. Száz, *Basic tools, increasing functions, and closure operations in generalized ordered sets*, In: P. M. Pardalos and Th. M. Rassias (Eds.), Contributions in Mathematics and Engineering: In Honor of Constantion Caratheodory, Springer, 551-616, 2016.
[167] Á. Száz, *Four general continuity properties, for pairs of functions, relations and relators, whose particular cases could be investigated by hundreds of mathematicians*, Tech. Rep., Inst. Math., Univ. Debrecen **2017/1**, 17 pp., 2017.
[168] Á. Száz, *Contra continuity properties of relations in relator spaces*, Tech. Rep., Inst. Math., Univ. Debrecen **2017/5**, 48 pp., 2017.
[169] Á. Száz, *The closure-interior Galois connection and its applications to relational equations and inclusions*, J. Int. Math. Virt. Inst. **8**, 181-224, 2018.
[170] Á. Száz, *Corelations are more powerful tools than relations*, In: Th. M. Rassias (Ed.), Applications of Nonlinear Analysis, Springer Optimization and Its Applications **134**, 711-779, 2018.
[171] Á. Száz, *Relationships between inclusions for relations and inequalities for corelations*, Math. Pannon. **26**, 15-31, 2018.
[172] Á. Száz, *Galois and Pataki connections on generalized ordered sets*, Earthline J. Math. Sci. **2**, 283-323, 2019.
[173] Á. Száz, *Birelator spaces are natural generalizations of not only bitopological spaces, but also ideal topological spaces*, In: Th. M. Rassias and P. M. Pardalos (Eds.), Mathematical Analysis and Applications, Springer Optimization and Its Applications **154**, Springer Nature Switzerland AG, 543-586, 2019.
[174] W. J. Thron, *Topological Structures*, Holt, Rinehart and Winston, New York, 1966.
[175] H. Tietze, *Beiträge zur allgemeinen Topologie I. Axiome für verschiedene Fassungen des Umgebungsbegriffs*, Math. Ann. **88**, 290-312, 1923.
[176] A. Weil, *Sur les espaces á structure uniforme et sur la topologie générale*, Actual. Sci. Ind. 551, Herman and Cie, Paris, 1937.
[177] S. Willard and R. M. Latif, *Semi-open and regularly closed sets in compact metric spaces*, Math. Japon. **46**, 157-161, 1997.