Montes Taurus J. Pure Appl. Math. 3 (3), 140–146, 2021

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

A Note on Reciprocal Degenerate Bell Numbers and Polynomials

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Abstract

Recently, degenerate Bell numbers and polynomials were introduced as degenerate versions of the ordinary Bell numbers and polynomials. In this paper, we consider reciprocal degenerate Bell numbers and polynomials whose generating function is the reciprocal of that of the degenerate Bell polynomials. We investigate some properties for those numbers and polynomials, including their explicit expressions, recurrence relations and their connections with the degenerate Bell numbers and polynomials.

Keywords: Reciprocal degenerate Bell polynomials, Degenerate Bell polynomials, Degenerate Stirling numbers of the first kind, Degenerate Stirling numbers of the second kind

2010 MSC: 11B73, 11B83, 05A19

1. Introduction

It is Carlitz who initiated the study of degenerate versions of Bernoulli and Euler polynomials and numbers, namely the degenerate Bernoulli and Euler polynomials and numbers (see [4]). In recent years, some mathematicians began to explore various degenerate versions of some special polynomials and numbers and discovered many interesting arithmetic and combinatorial results about those polynomials and numbers [7-9,13,15 and the references therein]. Moreover, those degenerate versions of some special polynomials found some applications to other areas of mathematics such as differential equations, symmetric identities and probability theory [11,12,14]. They can be studied by various means like combinatorial methods, generating functions, differential equations, umbral calculus techniques, p-adic analysis and probability theory.

The Bell polynomials, also called Touchard or exponential polynomials, are natural extensions of Bell numbers B_n, which count the number of partitions of a set with n elements into disjoint nonempty subsets. As a degenerate version of these Bell polynomials and numbers, the degenerate Bell polynomials and numbers in (1.14) were introduced and studied under the different names of the partially degenerate Bell polynomials and numbers in [13,15]. Additional results were obtained in [15] by making use of umbral calculus. Here, we would like to note in passing that there are other types of degenerate Bell numbers and polynomials [7].

The present paper deals with the reciprocal degenerate Bell polynomials and numbers whose generating function is the reciprocal of that of the degenerate Bell polynomials. We investigate some properties for those numbers and polynomials which include their explicit expressions, recurrence relations and their connections with the degenerate

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Received:17 December 2020, Accepted:12 January 2021, Published:25 April 2021
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Bell numbers and polynomials. For the rest of this section, we will go over some necessary facts that will be needed throughout this paper.

For $n \geq 0$, the Stirling numbers of the first kind are given by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad \text{(see [8, 9, 13])}, \tag{1.1}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

As the inverse relation of (1.1), the Stirling numbers of the second kind are given by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (n \geq 0), \quad \text{(see [8, 9, 13])}. \tag{1.2}$$

From (1.1) and (1.2), we can easily derive the generating functions of the Stirling numbers of both kinds as in the following:

$$\frac{1}{k!}\big(\log(1+t)\big)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \tag{1.3}$$

and

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad (k \geq 0), \quad \text{(see [17])}. \tag{1.4}$$

For any $\lambda \in \mathbb{R}$, the λ-products are defined by

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \ (n \geq 1), \ \text{(see [7, 8, 13])}. \tag{1.5}$$

The degenerate exponential functions are defined by

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} \frac{(x)_{n,\lambda}}{n!}t^n, \quad \text{(see [7, 8, 13])}. \tag{1.6}$$

Thus, by (1.6), we get

$$\lim_{\lambda \to 0} e_\lambda^x(t) = e^{xt} \quad \text{and} \quad e_\lambda(t) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{n!}t^n. \tag{1.7}$$

As is well known, the Bell polynomials are defined by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad \text{(see [13])}. \tag{1.8}$$

Thus, by (1.4) and (1.8), we get

$$B_n(x) = \sum_{l=0}^{n} S_2(n,l)x^l = e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k, \quad (n \geq 0), \quad \text{(see [1-3, 5-10, 13.15-17])}. \tag{1.9}$$

When $x = 1$, $B_n = B_n(1)$, $(n \geq 0)$ are called the Bell numbers.

In [8], Kim considered the degenerate Stirling numbers of the second kind which are given by

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}. \tag{1.10}$$

By (1.10), we easily get

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l, \quad (n \geq 0). \tag{1.11}$$

As the inversion formula of (1.10), the degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda}, \quad (n \geq 0), \quad \text{(see [7])}. \tag{1.12}$$

Then, by (1.12), we get

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \quad \text{(see [7])}, \tag{1.13}$$

where $\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t$.

Recently, Kim-Kim introduced the degenerate Bell polynomials which are defined by

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [13])}. \tag{1.14}$$

When $x = 1, \beta_{n,\lambda} = \beta_{n,\lambda}(1)$ are called the degenerate Bell numbers. Further results on these polynomials were obtained in [15] by means of umbral calculus.

From (1.14), we note that

$$e^{x(e_\lambda(t)-1)} = e^{-x}\sum_{k=0}^{\infty} x^k \frac{1}{k!}e_\lambda^k(t) = \sum_{n=0}^{\infty}\Big(e^{-x}\sum_{k=0}^{\infty}\frac{(k)_{n,\lambda}}{k!}x^k\Big)\frac{t^n}{n!}. \tag{1.15}$$

From (1.10), (1.14) and (1.15), we note that

$$\beta_{n,\lambda}(x) = e^{-x}\sum_{k=0}^{\infty}\frac{(k)_{n,\lambda}}{k!}x^k = \sum_{k=0}^{n} S_{2,\lambda}(n,k)x^k, \quad (n \geq 0), \quad \text{(see [13])}. \tag{1.16}$$

Note that

$$\lim_{\lambda\to 0}\beta_{n,\lambda}(x) = e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k = B_n(x), \quad (n \geq 0).$$

In this paper, we consider reciprocal degenerate Bell numbers and polynomials and investigate some properties for those numbers and polynomials. In addition, we give some explicit formulas and identities of reciprocal Bell numbers and polynomials which are derived from the generating function of those polynomials and numbers. Finally, we will give some relation between the reciprocal Bell numbers and special numbers.

2. Reciprocal degenerate Bell numbers and polynomials

We first observe that $e^{x(1-e_\lambda(t))}$ is the reciprocal of the generating function of the degenerate Bell polynomials $\beta_{n,\lambda}(x)$ (see (1.14)). Now, we consider the reciprocal degenerate Bell polynomials $R_{n,\lambda}(x)$ whose generating function is given by

$$e^{x(1-e_\lambda(t))} = \sum_{n=0}^{\infty} R_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 1$, $R_{n,\lambda} = R_{n,\lambda}(1)$ are called the reciprocal degenerate Bell numbers.

From (2.1), we note that

$$\begin{aligned} e^{x(1-e_\lambda(t))} &= \sum_{k=0}^{\infty} x^k\frac{1}{k!}(1-e_\lambda(t))^k = \sum_{k=0}^{\infty}x^k(-1)^k\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}(-1)^k S_{2,\lambda}(n,k)x^k\Big)\frac{t^n}{n!}. \end{aligned} \tag{2.2}$$

On the other hand

$$\begin{aligned} e^{x(1-e_\lambda(t))} &= e^x \cdot e^{-xe_\lambda(t)} = e^x \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} e_\lambda^k(t) \\ &= e^x \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \sum_{n=0}^{\infty} \frac{(k)_{n,\lambda}}{n!} t^n \\ &= \sum_{n=0}^{\infty} \Big(e^x \sum_{k=0}^{\infty} (-1)^k \frac{(k)_{n,\lambda}}{k!} x^k \Big) \frac{t^n}{n!}. \end{aligned} \tag{2.3}$$

Therefore, by (2.1), (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$R_{n,\lambda}(x) = \sum_{k=0}^{n} (-1)^k S_{2,\lambda}(n,k) x^k = e^x \sum_{k=0}^{\infty} (-1)^k \frac{(k)_{n,\lambda}}{k!} x^k.$$

In particular,

$$R_{n,\lambda} = \sum_{k=0}^{n} (-1)^k S_{2,\lambda}(n,k) = e \sum_{k=0}^{\infty} (-1)^k \frac{(k)_{n,\lambda}}{k!}.$$

Let us take the derivative with respect to t on both sides of (2.1) . Then we have

$$-xe_\lambda^{1-\lambda}(t) e^{x(1-e_\lambda(t))} = \sum_{n=0}^{\infty} R_{n+1,\lambda}(x) \frac{t^n}{n!}. \tag{2.4}$$

On the other hand, we note that

$$\begin{aligned} -xe_\lambda^{1-\lambda}(t) e^{x(1-e_\lambda(t))} &= -x \sum_{m=0}^{\infty} (1-\lambda)_{m,\lambda} \frac{t^m}{m!} \sum_{l=0}^{\infty} R_{l,\lambda}(x) \frac{t^l}{l!} \\ &= -x \sum_{n=0}^{\infty} \Big(\sum_{l=0}^{n} \binom{n}{l} R_{l,\lambda}(x) (1-\lambda)_{n-l,\lambda} \Big) \frac{t^n}{n!}. \end{aligned} \tag{2.5}$$

Therefore, by (2.4) and (2.5), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$R_{n+1,\lambda}(x) = -x \sum_{l=0}^{n} \binom{n}{l} R_{l,\lambda}(x) (1-\lambda)_{n-l,\lambda}.$$

In particular,

$$R_{n+1,\lambda} = -\sum_{l=0}^{n} \binom{n}{l} R_{l,\lambda} (1-\lambda)_{n-l,\lambda}.$$

From (1.14) and (2.1), we note that

$$\begin{aligned} 1 = e^{x(e_\lambda(t)-1)} \cdot e^{x(1-e_\lambda(t))} &= \sum_{l=0}^{\infty} \beta_{l,\lambda}(x) \frac{t^l}{l!} \sum_{m=0}^{\infty} R_{m,\lambda}(x) \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \Big(\sum_{l=0}^{n} \binom{n}{l} \beta_{l,\lambda}(x) R_{n-l,\lambda}(x) \Big) \frac{t^n}{n!}. \end{aligned} \tag{2.6}$$

Therefore, by comparing the coefficients on both sides of (2.6), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda}(x)R_{n-l,\lambda}(x)=\delta_{n,0},$$

where $\delta_{n,0}$ is the Kronecker's symbol.

In particular,

$$\sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda}R_{n-l,\lambda}=\delta_{n,0}.$$

Let us take the derivative with respect to t on both sides of (1.14). Then we have

$$xe_{\lambda}^{1-\lambda}(t)\cdot e^{x(e_{\lambda}(t)-1)}=\sum_{n=0}^{\infty}\beta_{n+1,\lambda}(x)\frac{t^{n}}{n!}. \tag{2.7}$$

On the other hand,

$$\begin{aligned} xe_{\lambda}^{1-\lambda}(t)\cdot e^{x(e_{\lambda}(t)-1)} &= x\sum_{l=0}^{\infty}(1-\lambda)_{l,\lambda}\frac{t^{l}}{l!}\sum_{m=0}^{\infty}\beta_{m,\lambda}(x)\frac{t^{m}}{m!} \\ &= \sum_{n=0}^{\infty}\left(x\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}(x)(1-\lambda)_{n-m,\lambda}\right)\frac{t^{n}}{n!}. \end{aligned} \tag{2.8}$$

Thus, by (2.7) and (2.8), we obtain the following lemma.

Lemma 2.4. *For $n \geq 0$, we have*

$$\beta_{n+1,\lambda}(x)=x\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}(x)(1-\lambda)_{n-m,\lambda}.$$

In particular,

$$\beta_{n+1,\lambda}=\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}(1-\lambda)_{n-m,\lambda}.$$

By mutiplying $e^{x(1-e_{\lambda}(t))}$ on both sides of (2.7), we get

$$\begin{aligned} xe_{\lambda}^{1-\lambda}(t) &= \sum_{m=0}^{\infty}\beta_{m+1,\lambda}(x)\frac{t^{m}}{m!}\sum_{l=0}^{\infty}R_{l,\lambda}(x)\frac{t^{l}}{l!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}R_{l,\lambda}(x)\beta_{n-l+1,\lambda}(x)\right)\frac{t^{n}}{n!}. \end{aligned} \tag{2.9}$$

On the other hand, we also have

$$xe_{\lambda}^{1-\lambda}(t)=x\sum_{n=0}^{\infty}(1-\lambda)_{n,\lambda}\frac{t^{n}}{n!}. \tag{2.10}$$

Therefore, by (2.9) and (2.10), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$\sum_{l=0}^{n}\binom{n}{l}R_{l,\lambda}(x)\beta_{n-l+1,\lambda}(x)=x(1-\lambda)_{n,\lambda}. \tag{2.11}$$

In particular,

$$\sum_{l=0}^{n}\binom{n}{l}R_{l,\lambda}\beta_{n-l+1,\lambda}=(1-\lambda)_{n,\lambda}. \tag{2.12}$$

By multiplying $e^{x(e_\lambda(t)-1)}$ on both sides of (2.4), we get

$$\begin{aligned}-xe_\lambda^{1-\lambda}(t) &= \sum_{l=0}^{\infty} R_{l+1}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}\beta_{m,\lambda}(x)\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}(x)R_{n-m+1,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.13}$$

On the other hand,

$$-xe_\lambda^{1-\lambda}(t) = -x\sum_{n=0}^{\infty}(1-\lambda)_{n,\lambda}\frac{t^n}{n!}. \tag{2.14}$$

Therefore, by comparing the coefficients on both sides of (2.13) and (2.14), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}(x)R_{n-m+1,\lambda}(x) = -x(1-\lambda)_{n,\lambda}.$$

In particular,

$$\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}R_{n-m+1,\lambda} = -(1-\lambda)_{n,\lambda}.$$

3. Conclusion

In recent years, it has been demonstrated that it is worthwhile to consider various degenerate versions of some special polynomials and numbers. Indeed, to name a few, these include some degenerate versions of Bernoulli numbers and polynomials of the second kind, central Bell numbers and polynomials, central factorial numbers of the second kind, Cauchy numbers, Eulerian numbers and polynomials, Fubini polynomials, Stirling numbers of the first kind, Stirling polynomials of the second kind, central complete Bell polynomials, Bell numbers and polynomials, Changhee numbers of the second kind, Daehee numbers of the second kind and Bernstein polynomials.

As degenerate versions of the usual Bell polynomials and numbers, the degenerate Bell polynomials and numbers were introduced under the different names of the partially degenerate Bell polynomials and numbers. The present paper dealt with the reciprocal degenerate Bell polynomials and numbers whose generating function is the reciprocal of that of the degenerate Bell polynomials. We investigated some properties for those numbers and polynomials, including their explicit expressions, recurrence relations and their connections with the degenerate Bell numbers and polynomials.

We have three immediate applications of our results. The first is to differential equations. For instance, in [11] an identity, involving the degenerate Bernoulli and higher-order degenerate Bernoulli numbers, was obtained from an infinite family of nonlinear differential equations. The second is to identities of symmetry. Indeed, in [12] many symmetric identities in three variables were obtained for degenerate Euler polynomials and alternating generalized falling factorial sums. The third is to probability theory. For example, in [14] it was shown that certain expressions of the probability distributions of appropriate random variables involve both the degenerate λ-Stirling polynomials of the second and the r-truncated degenerate λ-Stirling polynomials of the second kind.

Finally, it is one of our future projects to continue to study various degenerate versions of some special polynomials and to find their applications to mathematics, science and engineering.

Acknowledgements

This paper is dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th Birthday.

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