Montes Taurus J. Pure Appl. Math. 3 (3), 95–103, 2021

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Integrals with Two–Variable Generating Function in the Integrand

Musharraf Ali[a], Mohd Ghayasuddin[b], Tibor K. Pogány[c]

[a]Department of Mathematics, G.F. College, Shahjahanpur-242001, India
[b]Department of Mathematics, Integral University Campus, Shahjahanpur-242001, India
[c]Faculty of Maritime Studies, University of Rijeka, 51000 Rijeka, Croatia and Institute of Applied Mathematics, Óbuda University, 1034 Budapest, Hungary

Abstract

The main motive of this study is to present a new class of generalized integral formulae which involve a generating function of two variables $G(u, x)$. By this approach we deduce a set of new outcomes, which are integrals associated with generalized hypergeometric function, Laguerre, Hermite and Bessel polynomials, Kampé de Fériet hypergeometric series of two variables, Lauricella function and several special cases of our main results.

Keywords: Generating functions, Lauricella hypergeometric function $F_D^{(m)}$, Appell–Horn hypergeometric function F_1 of two variables, Kampé de Fériet series, Hypergeometric functions, Generalized Bessel polynomials, Hermite polynomials

2010 MSC: 33C05, 33C20, 33C45

1. Introduction

Recently, various applications of long familiar integral operators (in the domain of special functions) have been given by a number of researchers, [4, 7, 11, 13, 14]. To extend the literature of such type of works, in this research article, we deduce utilitarian results by means of a integrals finite integral operator involving generating functions of two variables.

The generating function of two variables $G(u, x)$ is defined as the power series in x in the following form [24, p. 79, Eq. (3)]:

$$G(u,x) = \sum_{r\geq 0} c_r\, g_r(u)\, x^r, \tag{1.1}$$

where each member of the generated sequence $(g_r(u))_{r\geq 0}$ is independent of x, and the coefficient sequence $(c_r)_{r\geq 0}$ may contain the parameters of the set $(g_r(u))_{r\geq 0}$, but it is independent of u and x.

For $b_j,\ j = \overline{1, q}$ different from non–positive integers the series

$${}_pF_q\Big[\begin{array}{c} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{array} \Big| z\Big] = \sum_{n\geq 0} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} = \sum_{n\geq 0} \frac{\prod\limits_{j=1}^{p} (a_j)_n}{\prod\limits_{j=1}^{q} (b_j)_n} \frac{z^n}{n!}$$

†*Article ID:* MTJPAM-D-20-00048
Email addresses: drmusharrafali@gmail.com (Musharraf Ali), ghayas.maths@gmail.com (Mohd Ghayasuddin), poganj@pfri.hr, pogany.tibor@uni-obuda.hu (Tibor K. Pogány)
Received:17 December 2020, Accepted:30 December 2020, Published:25 April 2021
⋆*Corresponding Author:* Tibor K. Pogány

is the generalized hypergeometric series, where the Pochhammer symbol

$$(a)_\mu := \frac{\Gamma(a+\mu)}{\Gamma(a)} = \begin{cases} 1, & \text{if } \mu = 0;\ a \in \mathbb{C} \setminus \{0\} \\ a(a+1)\cdots(a+n-1), & \text{if } \mu = n \in \mathbb{N};\ a \in \mathbb{C} \end{cases},$$

and by convention $(0)_0 = 1$. When $p \leq q$, the generalized hypergeometric function converges for all complex values of z; that is, ${}_pF_q[z]$ is an entire function. When $p > q+1$, the series converges only for $z = 0$, unless it terminates (as when one of the parameters $a_j,\ j = \overline{1,p}$ is a negative integer) in which case it is just a polynomial in z. When $p = q+1$, the series converges in the open unit disk $|z| < 1$, and also for $|z| = 1$ provided that

$$\Re\Big(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\Big) > 0\,.$$

In the case when $p = q = 1$, we have the Kummer (or confluent) hypergeometric function, while for $p = 2, q = 1$ we have the Gaussian hypergeometric function.

The famous Horn's list's [9] (corrected by his student Borngässer [3]) of double hypergeometric functions defined in the series form, confluent member

$$\Phi_2(\alpha, \alpha'; \gamma; x, y) = \sum_{m,n\geq 0} \frac{(\alpha)_m (\alpha')_n}{(\gamma)_{m+n}} \frac{x^m\, y^n}{m!\, n!}, \qquad x, y \in \mathbb{R}$$

turns out to be defined already in 1920 by Humbert [10] (also see [23, pp. 24–26]).

Next, for the exposition we need the Kampé de Fériet generalized hypergeometric function of two variables defined by the double–series [2]

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix} (a_p) : (b_q); (c_k) \\ (\alpha_l) : (\beta_m); (\gamma_n) \end{matrix} \,\middle|\, x, y\right] = \sum_{r,s\geq 0} \frac{\prod\limits_{j=1}^{p} (a_j)_{r+s} \prod\limits_{j=1}^{q} (b_j)_r \prod\limits_{j=1}^{k} (c_j)_s}{\prod\limits_{j=1}^{l} (\alpha_j)_{r+s} \prod\limits_{j=1}^{m} (\beta_j)_r \prod\limits_{j=1}^{n} (\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!},$$

which converges when [22]

(i) $p+q < l+m+1,\ p+k < l+n+1;\ \max\{|x|, |y|\} < \infty,$ or

(ii) $p+q = l+m+1,\ p+k = l+n+1;$ and

$$\begin{cases} |x|^{\frac{1}{p-l}} + |y|^{\frac{1}{p-l}} < 1, & l < p \\ \max(|x|, |y|) < 1, & l > p \end{cases}.$$

Finally, the related power series definition of the Lauricella hypergemoetric function of m variables is [17, p. 113] (also consult [23, p. 33, Eq. (1)])

$$F_D^{(m)}[\alpha; \boldsymbol{\beta}; \gamma; \boldsymbol{x}] = \sum_{\boldsymbol{k}\geq 0} \frac{(\alpha)_{k_1+\cdots+k_m}}{(\gamma)_{k_1+\cdots+k_m}} \prod_{j=1}^{m} \frac{(b_j)_{k_j}\, x_j^{k_j}}{k_j!}\,.$$

The convergence domain is $\max_{1\leq j\leq m} |x_j| < 1$, reported in [17, p. 119]. Here, and in what follows, we write the shorthands

$$\boldsymbol{a} = (a_1, \cdots, a_m), \quad \text{and} \quad \lambda \cdot \boldsymbol{a} = (\lambda a_1, \cdots, \lambda a_m)\,.$$

The generating function of the hypergeometric function ${}_1F_0$ is defined by [24, p. 44, Eq. (8)]

$$G_{hyp}(u, x) = (1-ux)^{-\lambda} = \sum_{r\geq 0} \frac{(\lambda)_r (ux)^r}{r!} = {}_1F_0[\lambda; -; ux], \qquad |ux| < 1, \tag{1.2}$$

while the following series generate the Laguerre polynomials [24, p. 84, Eq. (15)]

$$G_L(u,x) := G_L(u,x,\lambda) = (1+x)^\lambda\, \mathrm{e}^{-ux} = \sum_{r\geq 0} L_r^{(\lambda-r)}(u)x^r; \tag{1.3}$$

here $L_r^{(\lambda)}(u)$ are the generalized Laguerre polynomials. The generating function of generalized Bessel polynomials is

$$G_B(u,x) := G_B(u,x,\alpha,\beta) = \mathrm{e}^x\Big(1-\frac{ux}{\beta}\Big)^{1-\alpha} = \sum_{r\geq 0} y_r(u,\alpha-r,\beta)\frac{x^r}{r!}, \tag{1.4}$$

where $y_r(u,a,b)$ denotes the generalized Bessel polynomials defined by Krall and Frink [15]. The display (1.4) is also mentioned in [8] and [24]. Now, in our setting it reads

$$y_r(u,a,b) = \sum_{k=0}^r (-r)_k\, \frac{(a+r-1)_k}{k!}\Big(-\frac{u}{b}\Big)^k = {}_2F_0\Big[\begin{matrix} -r, a+r-1 \\ - \end{matrix}\Big| -\frac{u}{b}\Big]. \tag{1.5}$$

The generating function of Hermite polynomials is described by [24, p. 83, Eq. (11)]:

$$G_H(u,x) = \mathrm{e}^{2ux-x^2} = \sum_{r\geq 0} H_r(u)\frac{x^r}{r!}, \qquad \max\{|u|,|x|\} < \infty. \tag{1.6}$$

Also, we recall here the following generating function for the Humbert confluent hypergeometric Φ_2 function of two variables [21, p.409, Eq. (2)]:

$$G_{\Phi_2}(u,x) = \sum_{r\geq 0} \frac{(\alpha)_r}{(\beta)_r}\, {}_1F_1[\alpha;\beta+r;u]\frac{x^r}{r!} = \Phi_2(\alpha,\alpha;\beta;u,x), \tag{1.7}$$

for all $|u|$, $|x| < \infty$.

2. Integrals Involving Generating Functions

We start with the so–called Beta integral transform method developed by Krattenthaler and Srinivasa Rao [16] which concerns the Euler's integral of the first kind

$$\mathrm{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x, \qquad \min\{\Re(p),\Re(q)\} > 0$$

and maps a suitable input function $h_{\boldsymbol\theta}(x)$ into a multiparameter function

$$\int_0^1 x^{p-1}(1-x)^{q-1}\, h_{\boldsymbol\theta}(x)\mathrm{d}x\,.$$

The polynomial case

$$h_{\boldsymbol\theta}(x) = \prod_{j=1}^n (1-\theta_j x)^{s_j-1}$$

in the previous integral, which is of particular interest, is expressible in a closed form in terms of the Lauricella generalized hypergeometric function $F_D^{(m)}$ of m variables.

Theorem 2.1. *For all $\boldsymbol\theta$, $\max_j|\theta_j| < 1$, and for $\min\{\Re(p),\Re(q)\} > 0$, we have*

$$\int_0^1 x^{p-1}(1-x)^{q-1} h_{\boldsymbol\theta}(x)\mathrm{d}x = \mathrm{B}(p,q)\, F_D^{(m)}[p, \mathbf{1}-\mathbf{s}; p+q; \boldsymbol\theta]\,. \tag{2.1}$$

Proof. Applying the binomial series expansion to all terms of $h_{\boldsymbol{\theta}}(x)$ and by legitimate change of the order of summation and integration we obtain

$$\begin{aligned}\mathscr{I}(p,q;\boldsymbol{\theta}) &= \int_0^1 x^{p-1}(1-x)^{q-1}\prod_{j=1}^m (1-\theta_j x)^{s_j-1}\,\mathrm{d}x \\ &= \sum_{\boldsymbol{k}\geq 0}\prod_{j=1}^m \binom{s_j-1}{k_j}(-\theta_j)^{k_j}\int_0^1 x^{p+|k|-1}(1-x)^{q-1}\,\mathrm{d}x \\ &= \sum_{\boldsymbol{k}\geq 0}\frac{\Gamma(p+|k|)\Gamma(q)}{\Gamma(p+q+|k|)}\prod_{j=1}^m \frac{(1-s_j)_{k_j}\,\theta_j^{k_j}}{k_j!} \\ &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\sum_{\boldsymbol{k}\geq 0}\frac{(p)_{|k|}}{(p+q)_{|k|}}\prod_{j=1}^m \frac{(1-s_j)_{k_j}\,\theta_j^{k_j}}{k_j!}\,, \end{aligned} \tag{2.2}$$

which is equivalent with the asserted formula (2.1). Here, and throughout the shorthand symbol $|k| = k_1 + \cdots + k_m$ is used.

Now, it remains enough to show that there are no constraints upon the parameters s_j, except its finiteness. Indeed, by the triangle inequality we have

$$|\mathscr{I}(p,q;\boldsymbol{\theta})| \leq \mathrm{B}(p,q)\prod_{j=1}^m (1+|\theta_j|)^{s_j-1}\,,$$

since $x \in [0,1]$. So, the integral $\mathscr{I}(p,q,\boldsymbol{s};\boldsymbol{\theta})$ possesses finite modulus for the permitted values of the Beta function which are $\Re(p) > 0, \Re(q) > 0$. Next, we precise the remaining convergence conditions in (2.1), since this formula is equivalent to (2.2). The binomial expansion of $(1-\theta x)^{s-1}$ requires $|\theta x| < 1$. On the other hand the integration domain is the unit interval $x \in [0,1]$, which implies $|\theta x| < |\theta| < 1$, and by the definition of $F_D^{(m)}$ the convergence condition of the Lauricella function in (2.1) is fulfilled. □

Remark 2.2. The same formula can be deduced by using the Beta–integral or Euler type single integral representation of the Lauricella function $F_D^{(m)}$ appeared in [17, p. 146, Eq. (25)], *viz.*

$$F_D^{(m)}[\alpha;\boldsymbol{\beta};\gamma;\boldsymbol{x}] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}\int_0^1 \frac{t^{\alpha-1}(1-t)^{\gamma-\alpha-1}}{(1-x_1t)^{\beta_1}\cdots(1-x_mt)^{\beta_m}}\,\mathrm{d}t, \tag{2.3}$$

provided $\Re(\gamma) > \Re(\alpha) > 0$. Equating the parameters in the integral $\mathscr{I}(p,q,\boldsymbol{s};\boldsymbol{\theta})$ from one, and in (2.3) from another side, we reach (2.2).

The two–variable Lauricella function $F_D^{(2)} \equiv F_1$, the Appell–Horn function of two variables having power series definition [6, p. 224, Eq. (6)]

$$F_1(\alpha;\beta,\beta';\gamma;x,y) = \sum_{m,n\geq 0}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \qquad |x|<1,\ |y|<1\,.$$

Thus, applying Theorem 2.1 we get the following result.

Corollary 2.3. *For all* $a, b,$ $\max\{|a|,|b|\} < 1$*, and for* $\min\{\Re(p),\Re(q)\} > 0$*, we have*

$$\int_0^1 x^{p-1}(1-x)^{q-1}(1-ax)^{s-1}(1-bx)^{t-1}\mathrm{d}x = \mathrm{B}(p,q)\,F_1(p;1-s,1-t;p+q;a,b)\,. \tag{2.4}$$

Proof. Along the lines of Remark 2.2 we arrive at the same formula considering the integral expression for F_1 which for all $\Re(\gamma) > \Re(\alpha) > 0$ reads [6, p. 231, Eq. (5)]

$$F_1(\alpha;\beta,\beta';\gamma;x,y) = \frac{\Gamma(\alpha)}{\Gamma(\gamma)\Gamma(\alpha-\gamma)}\int_0^1 \frac{u^{\alpha-1}(1-u)^{\gamma-\alpha-1}\,\mathrm{d}u}{(1-xu)^{\beta}(1-yu)^{\beta'}}\,;$$

now, we conclude (2.4) in a straightforward way. □

Remark 2.4. Lavoie and Trottier proved in [18] among other related formulae that

$$F_1\Big(\rho; 1-2\rho, 1-\delta; \rho+2\delta; \frac{1}{3}, \frac{1}{4}\Big) = \Big(\frac{4}{9}\Big)^{\rho}\, \frac{\mathrm{B}(\rho+2\delta,\delta)}{\mathrm{B}(\rho+\delta, 2\delta)}\,, \tag{2.5}$$

by which, with the aid of Corollary 2.3 we conclude the following formula

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1}\Big(1-\frac{x}{3}\Big)^{2\rho-1}\Big(1-\frac{x}{4}\Big)^{\delta-1}\mathrm{d}x = \Big(\frac{4}{9}\Big)^{\rho}\mathrm{B}(\rho,\delta), \tag{2.6}$$

provided $\min\{\Re(\rho), \Re(\delta)\} > 0$.

Now, we devote the rest of this section to Beta–transforms of the product of a weight–function $h_{\boldsymbol\theta}(x)$ and a two–variable generating functions $G(u,\cdot)$ of (1.1) shape with a general second argument, namely

$$G(u, \widetilde{h}_{\boldsymbol\theta}(x)) = \sum_{r\geq 0} c_r\, g_r(u)\, \widetilde{h}^r_{\boldsymbol\theta}(x), \tag{2.7}$$

where

$$\widetilde{h}_{\boldsymbol\theta}(x) = x^{\tau}(1-x)^{\eta}\prod_{j=1}^m (1-\theta_j x)^{t_j}\,,$$

provided $\boldsymbol\theta = (\theta_1, \cdots, \theta_m) \in (-1,1)^m$ and the powers τ, η, t_j do not disturb the uniform convergence requirements of the series (2.7).

Theorem 2.5. *Consider the generating function $G(u, \widetilde{h}_{\boldsymbol\theta}(x))$ which uniformly converges for all $x \in [0,1]$ and the coordinates of the vector $\boldsymbol\theta$ satisfy $\max\limits_{1\leq j\leq m}|\theta_j| < 1$. Then, for all $\Re(p+\tau), \Re(q+\eta) > 0$ we have*

$$\begin{aligned}\int_0^1 x^{p-1}(1-x)^{q-1}\, h_{\boldsymbol\theta}(x)\, G(u, \widetilde{h}_{\boldsymbol\theta}(x))\,\mathrm{d}x &= \mathrm{B}(p,q)\sum_{r\geq 0} c_r\, g_r(u)\frac{(p)_{\tau r}\,(q)_{\eta r}}{(p+q)_{(\tau+\eta)r}} \\ &\quad\times F_D^{(m)}[p+\tau r\,; \mathbf{1}-\boldsymbol{s}-r\cdot\boldsymbol{t}\,; p+q+(\tau+\eta)r\,; \boldsymbol\theta\,].\end{aligned} \tag{2.8}$$

Proof. Denote the Beta–transform under consideration with

$$\mathscr{I}_G := \mathscr{I}_G(p,q;\boldsymbol\theta) = \int_0^1 x^{p-1}(1-x)^{q-1}\, h_{\boldsymbol\theta}(x)\, G(u, \widetilde{h}_{\boldsymbol\theta}(x))\,\mathrm{d}x\,.$$

Since the generating function uniformly converges for all $x \in [0,1]$ the presumption for integration– summation order exchange is fulfilled, hence

$$\begin{aligned}\mathscr{I}_G &= \sum_{r\geq 0} c_r\, g_r(u)\int_0^1 x^{p+\tau r-1}(1-x)^{q+\eta r-1}\prod_{j=1}^m (1-\theta_j x)^{s_j+rt_j-1}\,\mathrm{d}x \\ &= \sum_{r\geq 0} c_r\, g_r(u)\sum_{\boldsymbol{k}\geq 0}(-1)^{|k|}\prod_{j=1}^m \binom{s_j+rt_j-1}{k_j}\theta_j^{k_j}\int_0^1 x^{p+\tau r+|k|-1}(1-x)^{q+\eta r-1}\,\mathrm{d}x \\ &= \sum_{r\geq 0} c_r\, g_r(u)\sum_{\boldsymbol{k}\geq 0}\prod_{j=1}^m (1-s_j-rt_j)_{k_j}\frac{\Gamma(p+\tau r+|k|)\,\Gamma(q+\eta r)}{\Gamma(p+q+(\tau+\eta)r+|k|)}\,\frac{\theta_j^{k_j}}{k_j!} \\ &= \mathrm{B}(p,q)\sum_{r\geq 0} c_r\, g_r(u)\frac{(p)_{\tau r}\,(q)_{\eta r}}{(p+q)_{(\tau+\eta)r}}\sum_{\boldsymbol{k}\geq 0}\frac{(p+\tau r)_{|k|}}{(p+q+(\tau+\eta)r)_{|k|}}\prod_{j=1}^m (1-s_j-rt_j)_{k_j}\frac{\theta_j^{k_j}}{k_j!}.\end{aligned}$$

In turn, this is exactly the stated formula (2.8).

The parameter constraint $|\theta_j| < 1$ required for the binomial series expansion $\widetilde{h}^r_{\boldsymbol\theta}(x)$ we prove by the same argumentation as in the proof of Theorem 2.1. □

Now, it is enough to take $m = 2$, that is to consider the input pair

$$h_{a,b}(x) = (1-ax)^{s_1-1}(1-bx)^{s_2-1} \tag{2.9}$$

$$\widetilde{h}_{a,b}(x) = x^{\tau}(1-x)^{\eta}(1-ax)^{t_1}(1-bx)^{t_2}, \qquad \max\{|a|,|b|\} < 1\,, \tag{2.10}$$

in Theorem 2.5 to reach the related Beta–transform.

Corollary 2.6. *Let $a, b \in (-1,1)$ and $G(u,\widetilde{h}_{a,b}(x))$ be uniformly convergent for $x \in [0,1]$. Then, for all $\Re(p+\tau), \Re(q+\eta) > 0$ we have*

$$\int_0^1 x^{p-1}(1-x)^{q-1} h_{a,b}(x) G(u,\widetilde{h}_{a,b}(x))\mathrm{d}x = \mathrm{B}(p,q)\sum_{r\ge 0} c_r\, g_r(u)$$
$$\times \frac{(p)_{\tau r}(q)_{\eta r}}{(p+q)_{(\tau+\eta)r}} F_1(p+\tau r\,; \mathbf{1}-\mathbf{s}-r\cdot\mathbf{t}\,; p+q+(\tau+\eta)r\,; a, b\,),$$

where $\mathbf{1}-\mathbf{s}-r\cdot\mathbf{t} = (1-s_1-rt_1, 1-s_2-rt_2)$.

To take advantage of the result (2.5), we have to specify the parameters getting

$$h_{\frac13,\frac14}(x) = \Big(1-\frac{x}{3}\Big)^{2\rho-1}\Big(1-\frac{x}{4}\Big)^{\delta-1}; \quad \widetilde{h}_{\frac13,\frac14}(x) = (1-x)^2\Big(1-\frac{x}{4}\Big). \tag{2.11}$$

The related result follows in a straightforward way.

Corollary 2.7. *Let $G(u,\widetilde{h}_{\frac13,\frac14}(x))$ be uniformly convergent for $x \in [0,1]$. Then for all $\Re(\rho) > 0, \Re(\delta) > -1$ we have*

$$\int_0^1 \frac{x^{\rho-1}}{(1-x)^{1-2\delta}}\, h_{\frac13,\frac14}(x)\, G(u,\widetilde{h}_{\frac13,\frac14}(x))\mathrm{d}x = \Big(\frac49\Big)^{\rho}\mathrm{B}(\rho,\delta)\sum_{r\ge0}\frac{c_r\,(\delta)_r}{(\rho+\delta)_r}\, g_r(u). \tag{2.12}$$

Proof. Variant A. Consider the integral I_A, say, in (2.12). Specify the parameters in Corollary 2.6 in the following way:

$$p=\rho, q=2\rho, s_1=2\delta, s_2=\delta; a=\frac13, b=\frac14; \tau=0, \eta=2, t_1=0, t_2=1\,,$$

by which we earn

$$I_A = \sum_{r\ge0} c_r g_r(u)\frac{\mathrm{B}(\rho,2\delta)(2\delta)_{2r}}{(\rho+2\delta)_{2r}} F_1\Big(\rho; 1-2\rho, 1-(\delta+r); \rho+2(\delta+r); \frac13, \frac14\Big).$$

Now, by virtue of (2.5) we confirm the asserted right–hand side expression in (2.12).

Variant B. Direct calculations imply

$$\begin{aligned} I_A &= \int_0^1 x^{\rho-1}\Big(1-\frac{x}{3}\Big)^{2\rho-1}\sum_{r\ge0} c_r g_r(u)(1-x)^{2(\delta+r)-1}\Big(1-\frac{x}{4}\Big)^{\delta+r-1}\mathrm{d}x \\ &= \sum_{r\ge0} c_r g_r(u)\int_0^1 x^{\rho-1}(1-x)^{2(\delta+r)-1}\Big(1-\frac{x}{3}\Big)^{2\rho-1}\Big(1-\frac{x}{4}\Big)^{(\delta+r)-1}\mathrm{d}x. \end{aligned}$$

Finally, applying (2.6) we conclude the required result. □

By similar argumentation we establish the following result.

Corollary 2.8. *Let $h_{\frac13,\frac14}(x)$ as in* Corollary 2.7 *and $\widehat{h}_{\frac13,\frac14}(x) = x(1-\frac{x}{3})^2$. Assume that $G(u,\widehat{h}_{\frac13,\frac14}(x))$ is uniformly convergent for $x \in [0,1]$. Then,*

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1}\, h_{\frac13,\frac14}(x)\, G(u,\widehat{h}_{\frac13,\frac14}(x))\,\mathrm{d}x = \mathrm{B}(\rho,\delta)\sum_{r\ge0}\frac{c_r(\rho)_r}{(\rho+\delta)_r} g_r(u)\Big(\frac49\Big)^{\rho+r}. \tag{2.13}$$

Manipulating with the parameters we can derive a whole set of different expression which origins are now completely known; the master formulae are given in Theorems 2.1 and 2.5.

Next, specifying the suitable values of c_r and $g_r(u)$, we can derive further results of interest, setting for instance

$$c_r = \frac{(a_1)_r \cdots (a_{p-1})_r}{(b_1)_r \cdots (b_{q-1})_r\, r!}$$

and $g_r(u) = u^r$ in (2.12) and (2.13), respectively, we infer the following generalized hypergeometric fashion expressions:

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1}\Big(1-\frac{x}{3}\Big)^{2\rho-1}\Big(1-\frac{x}{4}\Big)^{\delta-1} G(u, \widetilde{h}_{\frac13,\frac14}(x))\mathrm{d}x = \Big(\frac{4}{9}\Big)^{\rho} \mathrm{B}(\rho,\delta)\, {}_pF_q\Big[\begin{array}{c} \delta, a_1, \cdots, a_{p-1} \\ \rho+\delta, b_1, \cdots, b_{q-1} \end{array} \Big| u\Big],$$

and

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1}\Big(1-\frac{x}{3}\Big)^{2\rho-1}\Big(1-\frac{x}{4}\Big)^{\delta-1} G(u, \widehat{h}_{\frac13,\frac14}(x))\mathrm{d}x = \Big(\frac{4}{9}\Big)^{\rho} \mathrm{B}(\rho,\delta)\, {}_pF_q\Big[\begin{array}{c} \rho, a_1, \cdots, a_{p-1} \\ \rho+\delta, b_1, \cdots, b_{q-1} \end{array} \Big| \frac{4u}{9}\Big].$$

Similarly, for suitably different choice of c_r and $g_r(u)$, we can establish various other integral formulas.

3. Special Cases

In this section some special cases of our main results are presented by using the concept of generating function of certain known special functions. The functions $h_{a,b}(x), \widetilde{h}_{a,b}(x), h_{\frac13,\frac14}(x), \widetilde{h}_{\frac13,\frac14}(x), \widehat{h}_{\frac13,\frac14}(x)$ have been defined above in (2.9), (2.10) and (2.11), respectively. We notice that the parameters could differ from case to case, but this does not harm the exposition's clarity.

1. With the help of (1.2), choosing $u = 1$ and (2.6), for all $\Re(\rho) > 0, \Re(\delta) > 0$ we get

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1} h_{\frac13,\frac14}(x) G_{hyp}(1, \widetilde{h}_{\frac13,\frac14}(x))\, \mathrm{d}x = \Big(\frac{4}{9}\Big)^{\rho} \mathrm{B}(\rho,\delta)_2F_1\Big[\begin{array}{c} a,\, \delta \\ \rho+\delta \end{array} \Big| 1\Big].$$

Obviously, the Gauss summation formula for the unit argument ${}_2F_1$ cannot be employed, being the hypergeometric term zero-balanced.

2. Similar reasoning leads us to the next application valid for the same parameter space as above

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1} h_{\frac13,\frac14}(x) G_{hyp}(1, \widehat{h}_{\frac13,\frac14}(x))\mathrm{d}x = \Big(\frac{4}{9}\Big)^{\rho} \mathrm{B}(\rho,\delta)_2F_1\Big[\begin{array}{c} a,\, \rho \\ \rho+\delta \end{array} \Big| \frac{4}{9}\Big].$$

3. The generating function (1.3) for the Laguerre polynomials $L_r^{(\alpha-r)}(u)$ produces the following relation:

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1} h_{\frac13,\frac14}(x) G_L(u, \widetilde{h}_{\frac13,\frac14}(x))\mathrm{d}x = \Big(\frac{4}{9}\Big)^{\rho} \sum_{r\geq 0} \mathrm{B}(\rho,\delta+r) L_r^{(\lambda-r)}(u);$$

here the definition (1.3) and the Lavoie–Trottier formula (2.6) are the tools in establishing this result.

4. The generating function (1.3) for the Laguerre polynomials $L_r^{(\alpha-r)}(u)$ gives another relation, when we change the input function:

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1} h_{\frac13,\frac14}(x) G_L(u, \widehat{h}_{\frac13,\frac14}(x))\mathrm{d}x = \Big(\frac{4}{9}\Big)^{\rho} \sum_{r\geq 0} \mathrm{B}(\rho,\delta+r) L_r^{(\lambda-r)}(u).$$

5. The generalized Bessel polynomials $y_r(u, a, b)$ (see (1.5)) are defined by the generating function (1.4). Taking now this function's Beta–transform, we arrive at

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1} h_{\frac13,\frac14}(x)\, G_B(u, \widetilde{h}_{\frac13,\frac14}(x))\mathrm{d}x = \Big(\frac{4}{9}\Big)^{\rho} \sum_{r\geq 0} \mathrm{B}(\rho,\delta+r) \frac{y_r(u, a-r, b)}{r!},$$

which also follows by (2.6).

6. The next employing the Bessel generating function produces the following summation formula obtained again with the ad of (1.4) and (2.6). It reads

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1}h_{\frac{1}{3},\frac{1}{4}}(x)\,G_B(u,\widehat{h}_{\frac{1}{3},\frac{1}{4}}(x))\mathrm{d}x = \sum_{r\geq 0}\mathrm{B}(\rho+r,\delta)\Big(\frac{4}{9}\Big)^{r+\rho}\frac{y_r(u,a-r,b)}{r!}.$$

7. The Hermite polynomial's generating function $G_H(u,x)$, described in (1.6), has a Beta–transform in the form

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1}h_{\frac{1}{3},\frac{1}{4}}(x)\,G_H(u,\widetilde{h}_{\frac{1}{3},\frac{1}{4}}(x))\mathrm{d}x = \Big(\frac{4}{9}\Big)^{\rho}\sum_{r\geq 0}\mathrm{B}(\rho,\delta+r)\frac{H_r(u)}{r!}$$

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1}h_{\frac{1}{3},\frac{1}{4}}(x)\,G_H(u,\widehat{h}_{\frac{1}{3},\frac{1}{4}}(x))\mathrm{d}x = \sum_{r\geq 0}\mathrm{B}(\rho+r,\delta)\Big(\frac{4}{9}\Big)^{r+\rho}\frac{H_r(u)}{r!},$$

and the cornerstone derivation relation is again the one by Lavoie and Trottier, (2.6).

8. Finally, the input of Humbert generating function $G_{\Phi_2}(u,x)$ defined by (1.7) results in the following two outcome series

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1}h_{\frac{1}{3},\frac{1}{4}}(x)\,G_{\Phi_2}(u,\widetilde{h}_{\frac{1}{3},\frac{1}{4}}(x))\mathrm{d}x = \Big(\frac{4}{9}\Big)^{\rho}\mathrm{B}(\rho,\rho+\delta)F^{0:3;1}_{1:2;0}\left[\begin{array}{c} -: \\ b: \end{array}\begin{array}{c} a,\,\delta,\,b; \\ \rho+\delta,\,b; \end{array}\begin{array}{c} a \\ - \end{array}\Big|\,1,\,u\right]$$

$$\int_0^1 x^{\rho-1}(1-x)^{2\delta-1}h_{\frac{1}{3},\frac{1}{4}}(x)\,G_{\Phi_2}(u,\widehat{h}_{\frac{1}{3},\frac{1}{4}}(x))\mathrm{d}x = \Big(\frac{4}{9}\Big)^{\rho}\mathrm{B}(\rho,\delta)F^{0:3;1}_{1:2;0}\left[\begin{array}{c} -: \\ b: \end{array}\begin{array}{c} a,\,\delta,\,b; \\ \rho+\delta,\,b; \end{array}\begin{array}{c} a \\ - \end{array}\Big|\,\frac{4}{9},\,u\right],$$

where $F^{p:q;k}_{l:m;n}$ stands for the Kampé de Fériet function.

This completes our exposition of special cases.

4. Concluding remarks

In the present article, we have established generalized integral formulae involving generating function of two variables. Also, we have pointed out some special cases of our main results which are given in terms of hypergeometric type functions like the Gaussian ${}_2F_1$, generalized hypergeometric function ${}_pF_q$, Humbert Φ_2, Laguerre, Hermite and Bessel polynomials. It is remarkably noticed that a number of authors have been given a list of integral formulae (involving some well known special functions) by means of some standard results. For example, by using the results of Oberhettinger [20], Edward [5], MacRobert [19], Lavoie and Trottier [18] for instance, numerous authors have presented a large number of integral expressions see for details [1], [4], [7], [11], [13], [12] and [14]. Therefore, if we consider the concept of generating functions in conjunction of the results due to Oberhettinger [20], Edward [5] and MacRobert [19], we can establish various another integral formulae of summation purposes as well.

Acknowledgements

This paper is dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th Birthday.

The research of Tibor K. Pogány has been supported in part by the University of Rijeka, Croatia under the projects labeled `uniri-pr-prirod-19-16` and `uniri-tehnic-18-66`.

References

[1] S. Ahmed and M. A. Khan, *Euler type integral involving generalized Mittag-Leffler function*, Commun. Korean Math. Soc. **29 (3)**, 479-487, 2014.

[2] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques: Polynomes d'Hermite*, Gauthier-Villars, Paris, 1926.

[3] L. Borngässer, *Über hypergeometrische Funktionen zweier Veränderlichen*. Dissertation der Technischen Hochschuhle Darmstadt, Universität Darmstadt, 1933.

[4] J. Choi, P. Agarwal, S. Mathur and S. D. Purohit, *Certain new integral formulas involving the generalized Bessel function*, Bull. Korean Math. Soc. **51 (4)**, 995-1003, 2014.

[5] J. Edward, *A Treatise on the Integral Calculus*, Vol. II, Chelsea Pub. Com., New York, 1922.

[6] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill Book Company, Inc., New York, Toronto and London, 1953.

[7] M. Ghayasuddin, N. U. Khan and S. W. Khan, *Some finite integrals involving the product of Bessel function with Jacobi and Laguerre polynomials*, Commun. Korean Math. Soc. **33 (3)**, 1013-1024, 2018.

[8] E. Grasswald, *Bessel Polynomials*, Springer-Verlag, Berlin, Heidelberg and New York, 1978.

[9] J. Horn, *Hypergeometrische funktionen zweier veränderlichen*, Math. Ann. **105**, 381-407, 1931.

[10] P. Humbert, *The confluent hypergeometric functions of two variables*, Proc. Roy. Soc. Edinburgh **41**, 73-96, 1922.

[11] N. U. Khan and M. Ghayasuddin, *Some unified integrals associated with Whittaker function*, J. Frac. Calc. Appl. **9 (1)**, 153-159, 2018.

[12] N. U. Khan, M. Ghayasuddin and M. Shadab, *A new class of Euler type integral operators involving multiindex Mittag-Leffler function*, Honam Math. J. **40 (4)**, 691-700, 2018.

[13] N. U. Khan, M. Ghayasuddin and T. Usman, *On certain integral formulas involving the product of Bessel function and Jacobi polynomial*, Tamkang J. Math. **47 (3)**, 339–349, 2016.

[14] N. U. Khan, S. W. Khan and M. Ghayasuddin, *Some new results associated with the Bessel–Struve kernel function*, Acta Univ. Apulensis **48**, 89-101, 2016.

[15] H.L. Krall and O. Frink, *A new class of orthogonal polynomials, the Bessel polynomials*, Trans. Amer. Math. Soc. **65**, 100-115, 1949.

[16] C. Krattenthaler and K. Srinivasa Rao, *Automatic generation of hypergeometric identities by the beta integral method*, in R. Jagannathan, S. Kanemitsu, G. Vanden Berghe, W. Van Assche (Eds.) *Proceedings of the International Conference on Special Functions and their Applications* (Chennai, 2002). J. Comput. Appl. Math. **160 (1-2)**, 159-173, 2003.

[17] G. Lauricella, *Sulle funzioni ipergeometriche a più variabili*, Rendiconti del Circolo Matematico di Palermo **7**, 111-158, 1893.

[18] J. L. Lavoie and G. Trottier, *On the sum of certain Appell series*, Gaṇita **20 (1)**, 43-46, 1969.

[19] T. M. MacRobert, *Beta functions formulae and integrals involving E-function*, Math. Ann. **142**, 450-452, 1960/61.

[20] F. Oberhettinger, *Tables of Mellin transforms*, Springer, New York, 1974.

[21] A. P. Prudnikov, A. Yu. Brychkov and O. I. Marichev, *Integrals and sreies*, Vol. III, Gordon and Breach Science Publishers, New York, 1990.

[22] H. M. Srivastava and M. C. Daous, *A note on the convergence of Kampé de Fériet double hypergeometric series*, Math. Nachr. **53**, 151-159, 1972.

[23] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1985.

[24] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Ltd., Chichester, England, 1984.