





New Decomposition Formulas Associated with the Lauricella Multivariable Hypergeometric Functions

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Abstract

Certain decomposition formulas associated with a number of simple and multiple Gaussian hypergeometric functions including the multiple Lauricella hypergeometric functions have been presented. In this paper, we aim to establish further new decomposition formulas involving the multiple Lauricella hypergeometric functions $F_A^{(n)}$ and $F_B^{(n)}$.

Keywords: Lauricella functions, Multiple hypergeometric functions, Decomposition formula

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1. Introduction

A great deal of attraction has been drawn to the theory of multiple hypergeometric functions, mainly because they can provide solutions of diverse applied problems involving (for example) partial differential equations (see, for details, [26, p.47 et seq. Section 1.7]; see also the works [20, 21] and the references cited therein). For instance, the energy absorbed by some nonferromagnetic conductor sphere included in an internal magnetic field can be calculated with the help of such functions [18]. Hypergeometric functions of several variables are used in physical and quantum chemical applications as well [19, 20, 22, 23, 24, 25, 27]. Especially, many problems in gas dynamics lead to solutions of degenerate second-order partial differential equations which are then solvable in terms of multiple hypergeometric functions. Among examples, we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow [12].

We note that Riemann's functions, Green's functions and the fundamental solutions of the degenerate second-order partial differential equations are expressible by means of hypergeometric functions of several variables [8, 9, 10, 11, 13, 14, 28, 29]. In investigation of the boundary-value problems for these partial differential equations, we need decompositions for hypergeometric functions of several variables in terms of simpler hypergeometric functions of (for example) the Gauss and Appell types.

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The familiar operator method of Burcnall and Chaundy [2, 3, 4] has been used by them rather extensively for finding decomposition formulas for hypergeometric functions of two variables in terms of the classical Gauss hypergeometric function of one variable.

Following the works [2, 3], Hasanov and Srivastava [15, 16] introduced operators generalizing the Burcnall-Chaundy operators and found expansion formulas for many triple hypergeometric functions, and they proved recurrent formulas when the dimension of hypergeometric function exceeds three. However, due to the recurrence, additional difficulties may arise in the applications of those decomposition formulas.

Other generalizations of the Burcnall-Chaundy operators and their applications can be found in the works [5, 6].

In this paper for the two Lauricella hypergeometric functions in several variables we prove new decomposition formulas which are free from the recurrence .

The plan of this paper is as follows. In Section 2 we briefly give some preliminary information, which will be used later. In Section 3, we present the well-known decomposition formulas associated with the two and more dimensional Lauricella hypergeometric functions. In Section 4, we will prove new decomposition and summation formulas.

2. Preliminaries

Below we give some formulas for Euler gamma-function, Gauss hypergeometric function, Lauricella hypergeometric functions of three and more variables, which will be used in the next sections.

Let be N set of the natural numbers : $N = \{1, 2, 3, \dots\}$.

It is known that the Euler gamma-function $\Gamma(a)$ has property [7, p.17, (2)]

$$\Gamma(a + m) = \Gamma(a)(a)_m.$$

Here $(a)_m$ is a Pochhammer symbol, for which the equality $(a)_{m+n} = (a)_m(a + m)_n$ and its particular case $(a)_{2m} = (a)_m(a + m)_m$ are true [7, p.67,(5)].

A function

$$F(a, b; c; x) \equiv F \left[\begin{matrix} a, b; \\ c; \end{matrix} x \right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \quad c \neq 0, -1, -2, \dots$$

is known as the Gaussian hypergeometric function and an equality

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad c \neq 0, -1, -2, \dots, \operatorname{Re}(c - a - b) > 0 \tag{2.1}$$

holds [7, p.73, (73)]. Moreover, the following autotransformation formula [7, p.76, (22)]

$$F(a, b; c; x) = (1 - x)^{-b} F\left(c - a, b; c; \frac{x}{x - 1}\right) \tag{2.2}$$

is valid.

Multiple Lauricella hypergeometric functions $F_A^{(n)}$ and $F_B^{(n)}$ in $n \in N$ (real or complex) variables are defined as following ([17] and [1, p.33])

$$F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty} \frac{(a)_{|\mathbf{k}|} (b_1)_{k_1} \dots (b_n)_{k_n}}{k_1! \dots k_n! (c_1)_{k_1} \dots (c_n)_{k_n}} x_1^{k_1} \dots x_n^{k_n} \\ \left[c_i \neq 0, -1, -2, \dots; i = \overline{1, n}; |x_1| + \dots + |x_n| < 1 \right],$$

$$F_B^{(n)}(\mathbf{a}, \mathbf{b}; c; \mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty} \frac{(a_1)_{k_1} \dots (a_n)_{k_n} (b_1)_{k_1} \dots (b_n)_{k_n}}{k_1! \dots k_n! (c)_{|\mathbf{k}|}} x_1^{k_1} \dots x_n^{k_n}, \\ [c \neq 0, -1, -2, \dots; |x_1| < 1, \dots, |x_n| < 1],$$

where

$$\mathbf{a} := (a_1, \dots, a_n), \quad \mathbf{b} := (b_1, \dots, b_n), \quad \mathbf{c} := (c_1, \dots, c_n); \\ \mathbf{x} := (x_1, \dots, x_n); \quad |\mathbf{k}| := k_1 + \dots + k_n, \quad k_1 \geq 0, \dots, k_n \geq 0.$$

3. Decomposition formulas associated with the Lauricella functions $F_A^{(n)}$ and $F_B^{(n)}$

For a given multiple hypergeometric function, it is useful to find a decomposition formula which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables.

Burchnall and Chaundy [2, 3] systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. For example, the Appell function

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j} (b_1)_i (b_2)_j x^i y^j}{(c_1)_i (c_2)_j i! j!}$$

$$[c_1, c_2 \neq 0, -1, -2, \dots; |x| + |y| < 1]$$

has the expansion [2]

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = \sum_{i=0}^{\infty} \frac{(a)_i (b_1)_i (b_2)_i}{i! (c_1)_i (c_2)_i} x^i y^i F(a+i, b_1+i; c_1+i; x) F(a+i, b_2+i; c_2+i; y). \tag{3.1}$$

The Burchnall–Chaundy method, which is limited to functions of two variables, is based on the following mutually inverse symbolic operators [2]

$$\nabla(h) = \frac{\Gamma(h)\Gamma(\delta_1 + \delta_2 + h)}{\Gamma(\delta_1 + h)\Gamma(\delta_2 + h)}, \quad \Delta(h) = \frac{\Gamma(\delta_1 + h)\Gamma(\delta_2 + h)}{\Gamma(h)\Gamma(\delta_1 + \delta_2 + h)}, \tag{3.2}$$

where $\delta_1 = x \frac{\partial}{\partial x}$ and $\delta_2 = y \frac{\partial}{\partial y}$.

In order to generalize the operators $\nabla(h)$ and $\Delta(h)$, defined in (3.2), Hasanov and Srivastava [15, 16] introduced the operators

$$\tilde{\nabla}_{x_1, x_2, \dots, x_n}(h) = \frac{\Gamma(h)\Gamma(\delta_1 + \dots + \delta_n + h)}{\Gamma(\delta_1 + h)\Gamma(\delta_2 + \dots + \delta_n + h)},$$

$$\tilde{\Delta}_{x_1, x_2, \dots, x_n}(h) = \frac{\Gamma(\delta_1 + h)\Gamma(\delta_2 + \dots + \delta_n + h)}{\Gamma(h)\Gamma(\delta_1 + \dots + \delta_n + h)},$$

where $\delta_k = x_k \frac{\partial}{\partial x_k}$ ($k = \overline{1, n}$), with the help of which they managed to find decomposition formulas for a whole class of hypergeometric functions in several variables.

Following the works [2, 3] Hasanov and Srivastava [15] found following decomposition formulas for the Lauricella functions of three variables

$$F_A^{(3)}(a, b_1, b_2, b_3; c_1, c_2, c_3; x_1, x_2, x_3) = \sum_{i,j,k=0}^{\infty} \frac{(a)_{i+j+k} (b_1)_{j+k} (b_2)_{i+k} (b_3)_{i+j}}{i! j! k! (c_1)_{j+k} (c_2)_{i+k} (c_3)_{i+j}} x_1^{j+k} F(a+j+k, b_1+j+k; c_1+j+k; x_1) \times$$

$$\times x_2^{i+k} F(a+i+j+k, b_2+i+k; c_2+i+k; x_2) \times$$

$$\times x_3^{i+j} F(a+i+j+k, b_3+i+j; c_3+i+j; x_3), \tag{3.3}$$

$$F_B^{(3)}(a_1, a_2, a_3; b_1, b_2, b_3; c; x_1, x_2, x_3) = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k} (a_1)_{j+k} (b_1)_{j+k} (a_2)_{i+k} (b_2)_{i+k} (a_3)_{i+j} (b_3)_{i+j}}{(c-1+j+k)_{j+k} (c-1+2(j+k)+i)_i (c)_{2(i+j+k)} i! j! k!} \times$$

$$\times x_1^{j+k} F(a_1+j+k, b_1+j+k; c+2(j+k); x_1) \times$$

$$\times x_2^{i+k} F(a_2+i+k, b_2+i+k; c+2(i+j+k); x_2) \times$$

$$\times x_3^{i+j} F(a_3+i+j, b_3+i+j; c+2(i+j+k); x_3)$$

and they proved that for all $n \in \mathbb{N} \setminus \{1\}$ are true the recurrence formulas [16]

$$F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) = \sum_{|\mathbf{k}'|=0}^{\infty} \frac{(a)_{|\mathbf{k}'|} (b_1)_{|\mathbf{k}'|} (b_2)_{k_2} \dots (b_n)_{k_n}}{k_2! \dots k_n! (c_1)_{|\mathbf{k}'|} (c_2)_{k_2} \dots (c_n)_{k_n}} x_1^{|\mathbf{k}'|} x_2^{k_2} \dots x_n^{k_n} F(a + |\mathbf{k}'|, b_1 + |\mathbf{k}'|; c_1 + |\mathbf{k}'|; x_1) \times \\ \times F_A^{(n-1)}(a + |\mathbf{k}'|, b_2 + k_2, \dots, b_n + k_n; c_2 + k_2, \dots, c_n + k_n; x_2, \dots, x_n), \tag{3.4}$$

$$F_B^{(n)}(\mathbf{a}, \mathbf{b}; c; \mathbf{x}) = \sum_{|\mathbf{k}'|=0}^{\infty} \frac{(-1)^{|\mathbf{k}'|} (a_1)_{|\mathbf{k}'|} (b_1)_{|\mathbf{k}'|} (a_2)_{k_2} (b_2)_{k_2} \dots (a_n)_{k_n} (b_n)_{k_n}}{(c - 1 + |\mathbf{k}'|)_{|\mathbf{k}'|} (c)_{2|\mathbf{k}'|} k_2! \dots k_n!} x_1^{|\mathbf{k}'|} x_2^{k_2} \dots x_n^{k_n} F(a_1 + |\mathbf{k}'|, b_1 + |\mathbf{k}'|; c + 2|\mathbf{k}'|; x_1) \times \\ \times F_B^{(n-1)}(a_2 + k_2, \dots, a_n + k_n, b_2 + k_2, \dots, b_n + k_n; c + 2|\mathbf{k}'|; x_2, \dots, x_n), \tag{3.5}$$

where $|\mathbf{k}'| := k_2 + \dots + k_n$.

However, due to the recurrence of formula (3.4) and (3.5), additional difficulties may arise in the applications of this expansion. Further study of the properties of the Lauricella functions $F_A^{(n)}$ and $F_B^{(n)}$ showed that formulas (3.4) and (3.5) can be reduced to a more convenient forms.

4. New decomposition formulas associated with the Lauricella functions $F_A^{(n)}$ and $F_B^{(n)}$

Before proceeding to the presentation of the main result of this article, we introduce the notations

$$A(k, n) = \sum_{i=2}^{k+1} \sum_{j=i}^n m_{i,j}, \quad B(k, n) = \sum_{i=2}^k m_{i,k} + \sum_{i=k+1}^n m_{k+1,i}, \\ |\mathbf{m}_n| := \sum_{i=2}^n \sum_{j=i}^n m_{i,j}, \quad M_n! := \prod_{i=2}^n \prod_{j=i}^n m_{i,j}!$$

where $k, n \in \mathbb{N}, k \leq n; m_{i,j} \in \mathbb{N} \cap \{0\} (2 \leq i \leq j \leq n)$.

It should be noted here that the sum $B(2, n) + B(3, n) + \dots + B(n, n)$ has the parity property, which plays an important role in the calculation of the some values of hypergeometric functions. In fact, by virtue of equality

$$\sum_{k=2}^n \sum_{i=2}^k m_{i,k} = \sum_{k=1}^{n-1} \sum_{i=k+1}^n m_{k+1,i}$$

we obtain

$$\sum_{k=1}^n B(k, n) = 2 \sum_{k=2}^n \sum_{i=2}^k m_{i,k} = 2 \sum_{k=1}^{n-1} \sum_{i=k+1}^n m_{k+1,i}. \tag{4.1}$$

We present other simple properties of the functions $A(k, n)$ and $B(k, n)$:

$$A(n + 1, n + 1) - B(n + 1, n + 1) = A(n, n), \tag{4.2}$$

$$A(k + 1, k + 1) - B(k + 1, k + 1) = A(k, n) - B(k, n) + m_{2,n+1} + \dots + m_{k,n+1}. \tag{4.3}$$

Those properties are easily proved if we proceed from the definitions of functions $A(k, n)$ and $B(k, n)$.

Lemma 4.1. *The following decomposition formulas hold true at $n \in \mathbb{N}$*

$$F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_n!} \prod_{k=1}^n \frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}} \prod_{k=1}^n x_k^{B(k,n)} F \left[\begin{matrix} a + A(k, n), b_k + B(k, n); \\ c_k + B(k, n); \end{matrix} x_k \right], \tag{4.4}$$

$$F_B^{(n)}(\mathbf{a}, \mathbf{b}; c; \mathbf{x}) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(-1)^{A(n,n)}}{(c)_{2A(n,n)}} \frac{1}{M_n!} \prod_{k=1}^n \frac{(a_k)_{B(k,n)} (b_k)_{B(k,n)}}{(c - 1 + A(k, n) - A(k - 1, n))_{A(k,n) - A(k - 1, n)}} \times \\ \times \prod_{k=1}^n x_k^{B(k,n)} F \left[\begin{matrix} a_k + B(k, n), b_k + B(k, n); \\ c + 2A(k, n); \end{matrix} x_k \right]. \tag{4.5}$$

Proof. We carry out the proof by the method mathematical induction. First, we prove the validity of the equality (4.4). For clarity of the course of the proof, we introduce the notations

$$N_l(k, n) = \sum_{i=l}^{k+1} \sum_{j=i}^n m_{i,j}, M_l(k, n) = \sum_{i=l}^k m_{i,k} + \sum_{i=k+1}^n m_{k+1,i}, l \in N.$$

It's obvious that

$$N_2(k, n) = A(k, n), M_2(k, n) = B(k, n).$$

So we have to prove the fairness of equality

$$F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) = \sum_{|\mathbf{m}|=0}^{\infty} \frac{(a)_{N_2(n,n)}}{M_n!} \prod_{k=1}^n \frac{(b_k)_{M_2(k,n)}}{(c_k)_{M_2(k,n)}} x_k^{M_2(k,n)} F \left[\begin{matrix} a + N_2(k, n), b_k + M_2(k, n); \\ c_k + M_2(k, n); \end{matrix} x_k \right]. \tag{4.6}$$

In the case $n = 1$ the equality (4.6) is obvious.

Let $n = 2$. Since $M_2(1, 2) = M_2(2, 2) = N_2(1, 2) = N_2(2, 2) = m_{2,2} := i$, we obtain the formula (3.1).

For the sake of interest, we will check the formula (4.6) in yet another value of n .

Let $n = 3$. In this case

$$\begin{aligned} M_2(1, 3) &= m_{2,2} + m_{2,3}, & M_2(2, 3) &= m_{2,2} + m_{3,3}, \\ M_2(3, 3) &= m_{2,3} + m_{3,3}, & N_2(1, 3) &= m_{2,2} + m_{2,3}, \\ N_2(2, 3) &= N_2(3, 3) &= m_{2,2} + m_{2,3} + m_{3,3}. \end{aligned}$$

For brevity, making the substitutions $m_{2,2} := i, m_{2,3} := j, m_{3,3} := k$, we obtain the formula (3.3).

So the formula (4.6), that is formula (4.4), works for $n = 1, n = 2$ and $n = 3$.

Now we assume that for $n = s$ equality (4.6) holds; that is, that

$$F_A^{(s)}(a, b_1, \dots, b_s; c_1, \dots, c_s; x_1, \dots, x_s) = \sum_{|\mathbf{m}|=0}^{\infty} \frac{(a)_{N_2(s,s)}}{M_s!} \prod_{k=1}^s \frac{(b_k)_{M_2(k,s)}}{(c_k)_{M_2(k,s)}} x_k^{M_2(k,s)} F \left[\begin{matrix} a + N_2(k, s), b_k + M_2(k, s); \\ c_k + M_2(k, s); \end{matrix} x_k \right]. \tag{4.7}$$

Let $n = s + 1$. We prove that following formula

$$\begin{aligned} F_A^{(s+1)}(a, b_1, \dots, b_{s+1}; c_1, \dots, c_{s+1}; x_1, \dots, x_{s+1}) &= \sum_{|\mathbf{m}_{s+1}|=0}^{\infty} \frac{(a)_{N_2(s+1,s+1)}}{M_{s+1}!} \prod_{k=1}^{s+1} \frac{(b_k)_{M_2(k,s+1)}}{(c_k)_{M_2(k,s+1)}} \times \\ &\times \prod_{k=1}^{s+1} x_k^{M_2(k,s+1)} F \left[\begin{matrix} a + N_2(k, s + 1), b_k + M_2(k, s + 1); \\ c_k + M_2(k, s + 1); \end{matrix} x_k \right] \end{aligned} \tag{4.8}$$

is valid.

We write the Hasanov-Srivastava's formula (3.4) in the form

$$\begin{aligned} F_A^{(s+1)}(a, b_1, \dots, b_{s+1}; c_1, \dots, c_{s+1}; x_1, \dots, x_{s+1}) &= \sum_{m_{2,2}, \dots, m_{2,s+1}=0}^{\infty} \frac{(a)_{N_2(1,s+1)}(b_1)_{M_2(1,s+1)}(b_2)_{m_{2,2}} \cdots (b_{s+1})_{m_{2,s+1}}}{m_{2,2}! \cdots m_{2,s+1}! (c_1)_{M_2(1,s+1)}(c_2)_{m_{2,2}} \cdots (c_{s+1})_{m_{2,s+1}}} \times \\ &\times x_1^{M_2(1,s+1)} x_2^{m_{2,2}} \cdots x_{s+1}^{m_{2,s+1}} F \left[\begin{matrix} a + N_2(1, s + 1), b_1 + M_2(1, s + 1); \\ c_1 + M_2(1, s + 1); \end{matrix} x_1 \right] \times \\ &\times F_A^{(s)} \left[\begin{matrix} a + N_2(1, s + 1), b_2 + m_{2,2}, \dots, b_{s+1} + m_{2,s+1}; \\ c_2 + m_{2,2}, \dots, c_{s+1} + m_{2,s+1}; \end{matrix} x_2, \dots, x_{s+1} \right]. \end{aligned} \tag{4.9}$$

By virtue of the formula (4.7) we have

$$F_A^{(s)} \left[\begin{matrix} a + N_2(1, s + 1), b_2 + m_{2,2}, \dots, b_{s+1} + m_{2,s+1}; \\ c_2 + m_{2,2}, \dots, c_{s+1} + m_{2,s+1}; \end{matrix} x_2, \dots, x_{s+1} \right] = \sum_{\substack{m_{i,j}=0 \\ (3 \leq i \leq s+1)}}^{\infty} \frac{(a + N_2(1, s + 1))_{N_3(s+1, s+1)}}{m_{ij}!} \times \\ \times \prod_{k=2}^{s+1} \frac{(b_k + m_{2,k})_{M_3(k, s+1)}}{(c_k + m_{2,k})_{M_3(k, s+1)}} x_k^{M_3(k, s+1)} F \left[\begin{matrix} a + N_2(1, s + 1) + N_3(k, s + 1), b_k + m_{2,k} + M_3(k, s + 1); \\ c_k + m_{2,k} + M_3(k, s + 1); \end{matrix} x_k \right]. \quad (4.10)$$

Substituting from (4.10) into (4.9) we obtain

$$F_A^{(s+1)}(a, b_1, \dots, b_{s+1}; c_1, \dots, c_{s+1}; x_1, \dots, x_{s+1}) = \sum_{\substack{m_{i,j}=0 \\ (2 \leq i \leq s+1)}}^{\infty} \frac{(a)_{N_2(1, s+1) + N_3(s+1, s+1)}}{m_{ij}!} \prod_{k=1}^{s+1} \frac{(b_k)_{m_{2,k} + M_3(k, s+1)}}{(c_k)_{m_{2,k} + M_3(k, s+1)}} x_k^{m_{2,k} + M_3(k, s+1)} \times \\ \times F \left[\begin{matrix} a + N_2(1, s + 1) + N_3(k, s + 1), b_k + m_{2,k} + M_3(k, s + 1); \\ c_k + m_{2,k} + M_3(k, s + 1); \end{matrix} x_k \right].$$

Further, by virtue of the following obvious equalities

$$N_2(1, s + 1) + N_3(k, s + 1) = N_2(k, s + 1), \quad 1 \leq k \leq s + 1, s \in \mathbb{N}, \\ m_{2,k} + M_3(k, s + 1) = M_2(k, s + 1), \quad 1 \leq k \leq s + 1, s \in \mathbb{N},$$

we finally find the equality (4.8).

The equality (4.5) is proved similarly as proof of the equality (4.4). Q.E.D. □

Lemma 4.2. Let a, b_1, \dots, b_n are real numbers with $a \neq 0, -1, -2, \dots$ and $a > |\mathbf{b}|$, where $|\mathbf{b}| := b_1 + \dots + b_n$. Then the following summation formulas hold true at $n \in \mathbb{N}$

$$\sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_n!} \prod_{k=1}^n \frac{(b_k)_{B(k,n)} (a - b_k)_{A(k,n) - B(k,n)}}{(a)_{A(k,n)}} = \frac{\Gamma(a - |\mathbf{b}|)}{\Gamma(a)} \prod_{k=1}^n \frac{\Gamma(a)}{\Gamma(a - b_k)}, \quad (4.11)$$

$$\sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(-1)^{A(n,n)}}{(a)_{2A(n,n)}} M_n! \prod_{k=1}^n \frac{(b_k)_{B(k,n)} (a)_{2A(k,n)}}{(a - b_k)_{2A(k,n) - B(k,n)}} \prod_{k=1}^n \frac{(a - 1)_{A(k,n) - A(k-1,n)}}{(a - 1)_{2A(k,n) - 2A(k-1,n)}} = \frac{\Gamma(a)}{\Gamma(a - |\mathbf{b}|)} \prod_{k=1}^n \frac{\Gamma(a - b_k)}{\Gamma(a)}. \quad (4.12)$$

Proof. We carry out the proof by the method mathematical induction. First, we prove the validity of the equality (4.11).

In the case $n = 1$ the equality (4.11) is obvious.

Let $n = 2$. Since $A(1, 2) = A(2, 2) = B(1, 2) = B(2, 2) = m_{2,2} := i$, we obtain well-known summation formula (3.1):

$$\sum_{m_{22}=0}^{\infty} \frac{(b_1)_i (b_1)_i}{(c)_i i!} := F(b_1, b_2; a; 1) = \frac{\Gamma(a - b_1 - b_2) \Gamma(a)}{\Gamma(a - b_1) \Gamma(a - b_2)}.$$

So the formula (4.11) works for $n = 1$ and $n = 2$.

Now we denote the left side of the formula (4.11) by

$$T_n(a, \mathbf{b}) := \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_n!} \prod_{k=1}^n \frac{(b_k)_{B(k,n)} (a - b_k)_{A(k,n) - B(k,n)}}{(a)_{A(k,n)}}$$

and considering fair equality

$$T_n(a, \mathbf{b}) = \frac{\Gamma(a - |\mathbf{b}|)}{\Gamma(a)} \prod_{k=1}^n \frac{\Gamma(a)}{\Gamma(a - b_k)},$$

we will prove that

$$T_{n+1}(a, \mathbf{b}, b_{n+1}) = \frac{\Gamma(a - |\mathbf{b}| - b_{n+1})}{\Gamma(a)} \prod_{k=1}^{n+1} \frac{\Gamma(a)}{\Gamma(a - b_k)}. \tag{4.13}$$

For this aim we will put

$$T_{n+1}(a, \mathbf{b}, b_{n+1}) = \sum_{|\mathbf{m}_{n+1}|=0}^{\infty} \frac{(a)_{A(n+1, n+1)}}{M_{n+1}!} \prod_{k=1}^{n+1} \frac{(b_k)_{B(k, n+1)} (a - b_k)_{A(k, n+1) - B(k, n+1)}}{(a)_{A(k, n+1)}} \tag{4.14}$$

and show the validity of the recurrence relation

$$T_{n+1}(a, \mathbf{b}, b_{n+1}) = T_n(a - b_{n+1}, \mathbf{b}) \prod_{k=1}^n \frac{\Gamma(a) \Gamma(a - b_k - b_{n+1})}{\Gamma(a - b_{n+1}) \Gamma(a - b_k)}. \tag{4.15}$$

This process consists of n steps. A detailed look at the first step.

By virtue of the equality

$$\sum_{|\mathbf{m}_{n+1}|=0}^{\infty} f(\dots) = \sum_{|\mathbf{m}_n|=0}^{\infty} \sum_{\substack{m_{i, n+1}=0 \\ (2 \leq i \leq n+1)}}^{\infty} f(\dots)$$

and the properties of functions $A(k, n)$ and $B(k, n)$ (see formulas (4.2) and (4.3)), the right side of equality (4.14) it is easy to convert to the form

$$\begin{aligned} T_{n+1}(a, \mathbf{b}, b_{n+1}) &= \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a - b_{n+1})_{A(n, n)} (b_n)_{B(n, n)}}{M_n!} \sum_{\substack{m_{i, n+1}=0 \\ (2 \leq i \leq n)}}^{\infty} \frac{(b_{n+1})_{|\tilde{\mathbf{m}}_n|} (a - b_n)_{A(n, n) - B(n, n) + |\tilde{\mathbf{m}}_n|}}{m_{i, n+1}! (a)_{A(n, n) + |\tilde{\mathbf{m}}_n|}} \times \\ &\times \prod_{k=1}^{n-1} \frac{(b_k)_{B(k, n) + m_{k+1, n+1}} (a - b_k)_{A(k, n) - B(k, n) + |\tilde{\mathbf{m}}_{k+1}|}}{(a)_{A(k, n) + |\tilde{\mathbf{m}}_{k+1}|}} S(k, n), \end{aligned}$$

where

$$|\tilde{\mathbf{m}}_j| := m_{2, n+1} + m_{3, n+1} + \dots + m_{j, n+1}, \quad j = 2, \dots, n;$$

$$S(k, n) = \sum_{m_{n+1, n+1}=0}^{\infty} \frac{(b_n + B(n, n))_{m_{n+1, n+1}} (b_{n+1} + |\tilde{\mathbf{m}}_n|)_{m_{n+1, n+1}}}{m_{n+1, n+1}! (a + A(n, n) + |\tilde{\mathbf{m}}_n|)_{m_{n+1, n+1}}}.$$

It is easy to notice that

$$S(k, n) = F(b_n + B(n, n), b_{n+1} + |\tilde{\mathbf{m}}_n|; a + A(n, n) + |\tilde{\mathbf{m}}_n|; 1).$$

Applying now the summation formula (2.1) to the last equality after elementary transformations we get

$$\begin{aligned} T_{n+1}^{(1)}(a, \mathbf{b}, b_{n+1}) &= \frac{\Gamma(a - b_n - b_{n+1}) \Gamma(a)}{\Gamma(a - b_n) \Gamma(a - b_{n+1})} \sum_{|\mathbf{m}_{n+1}|=0}^{\infty} \frac{(b_n)_{B(n, n)} (a - b_n - b_{n+1})_{A(n, n) - B(n, n)}}{M_{n+1}!} \times \\ &\times \sum_{\substack{m_{i, n+1}=0 \\ (2 \leq i \leq n)}}^{\infty} \frac{(b_{n+1})_{|\tilde{\mathbf{m}}_n|}}{m_{i, n+1}!} \prod_{k=1}^{n-1} \frac{(b_k)_{B(k, n) + m_{k+1, n+1}} (a - b_k)_{A(k, n) - B(k, n) + |\tilde{\mathbf{m}}_{k+1}|}}{(a)_{A(k, n) + |\tilde{\mathbf{m}}_{k+1}|}}. \end{aligned}$$

For definiteness, we denoted the result of the first step of the process under consideration by $T_{n+1}^{(1)}(a, \mathbf{b}, b_{n+1})$. We continue the process of proving the recurrence relation (4.15). In each next step, having consistently repeated the

reasoning carried out in the first step, we get

$$T_{n+1}^{(s)}(a, \mathbf{b}, b_{n+1}) = \frac{\Gamma^s(a)}{\Gamma^s(a - b_{n+1})} \prod_{k=n-s+1}^n \frac{\Gamma(a - b_k - b_{n+1})}{\Gamma(a - b_k)} \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{1}{M_n!} \prod_{k=n-s+1}^n \frac{(b_k)_{B(k,n)} (a - b_k - b_{n+1})_{A(k,n)-B(k,n)}}{(a - b_{n+1})_{A(k,n)}} \times$$

$$\times \sum_{\substack{m_{i,n+1}=0 \\ (2 \leq i \leq n-s+1)}}^{\infty} \frac{(a - b_{n+1})_{N(n,n)} (b_{n+1})_{|\tilde{\mathbf{m}}_{n-s+1}|}}{m_{ij}!} \prod_{k=1}^{n-s} \frac{(b_k)_{B(k,n)+m_{k+1,n+1}} (a - b_k)_{A(k,n)-B(k,n)+|\tilde{\mathbf{m}}_{k+1}|}}{(a)_{A(k,n)+|\tilde{\mathbf{m}}_{k+1}|}}$$

and in the last step

$$T_{n+1}^{(n)}(a, \mathbf{b}, b_{n+1}) = \frac{\Gamma^n(a)}{\Gamma^n(a - b_{n+1})} \prod_{k=1}^n \left[\frac{\Gamma(a - b_{n+1} - b_k)}{\Gamma(a - b_k)} \right] \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a - b_{n+1})_{A(n,n)}}{m_{ij}!} \prod_{k=1}^n \left[\frac{(b_k)_{B(k,n)} (a - b_{n+1} - b_k)_{A(k,n)-B(k,n)}}{(a - b_{n+1})_{A(k,n)}} \right],$$

that is

$$T_{n+1}^{(n)}(a, \mathbf{b}, b_{n+1}) = \frac{\Gamma^n(a)}{\Gamma^n(a - b_{n+1})} \prod_{k=1}^n \left[\frac{\Gamma(a - b_{n+1} - b_k)}{\Gamma(a - b_k)} \right] T_n(a - b_{n+1}, \mathbf{b}).$$

Thus, the validity of the ratio (4.15) is established. By the induction hypothesis, from the (4.15) follows the equality

$$T_n(a - b_{n+1}, \mathbf{b}) = \Gamma(a - b_{n+1} - |\mathbf{b}|) \frac{\Gamma^{n-1}(a - b_{n+1})}{\prod_{k=1}^n \Gamma(a - b_{n+1} - b_k)}.$$

Substituting the last expression in (4.15) we get the equality (4.13). Therefore, the equality (4.11) is true. The equality (4.12) is proved similarly as proof of the equality (4.11). Q.E.D. □

Lemma 4.3. *Let a, b_1, \dots, b_n are real numbers with $a, c_k, c_k - b_k \neq 0, -1, -2, \dots$ and $a > |\mathbf{b}|$. Then the following limit formulas hold true at $n \in \mathbb{N}$*

$$\lim_{\mathbf{x} \rightarrow 0} \left\{ \mathbf{x}^{-\mathbf{b}} F_A^{(n)} \left(a, \mathbf{b}; \mathbf{c}; 1 - \frac{1}{\mathbf{x}} \right) \right\} = \frac{\Gamma(a - |\mathbf{b}|)}{\Gamma(a)} \prod_{k=1}^n \frac{\Gamma(c_k)}{\Gamma(c_k - b_k)}; \tag{4.16}$$

$$\lim_{\mathbf{x} \rightarrow 0} \left\{ \mathbf{x}^{-\mathbf{b}} F_B^{(n)} \left(\mathbf{a}; \mathbf{b}; c; 1 - \frac{1}{\mathbf{x}} \right) \right\} = \frac{\Gamma(c)}{\Gamma(c - |\mathbf{b}|)} \prod_{k=1}^n \frac{\Gamma(a_k - b_k)}{\Gamma(a_k)}; \tag{4.17}$$

where

$$\mathbf{x}^{-\mathbf{b}} := x_1^{-b_1} \dots x_n^{-b_n}; \quad \frac{1}{\mathbf{x}} := \left(\frac{1}{x_1}, \dots, \frac{1}{x_n} \right).$$

Proof. By virtue of the decomposition formula (4.4) we obtain

$$F_A^{(n)} \left(a, \mathbf{b}; \mathbf{c}; 1 - \frac{1}{\mathbf{x}} \right) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_n!} \prod_{k=1}^n \frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}} \prod_{k=1}^n \left(1 - \frac{1}{x_k} \right)^{B(k,n)} F \left[\begin{matrix} a + A(k,n), b_k + B(k,n); \\ c_k + B(k,n); \end{matrix} 1 - \frac{1}{x_k} \right]. \tag{4.18}$$

Applying now the familiar autotransformation formula (2.2) to each hypergeometric function included in the sum (4.18), we get

$$F_A^{(n)} \left(a, \mathbf{b}; \mathbf{c}; 1 - \frac{1}{\mathbf{x}} \right) = \mathbf{x}^{\mathbf{b}} \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_n!} \prod_{k=1}^n \frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}} (x_k - 1)^{B(k,n)} \prod_{k=1}^n F \left[\begin{matrix} c_k - a + B(k,n) - A(k,n), b_k + B(k,n); \\ c_k + B(k,n); \end{matrix} 1 - x_k \right].$$

Using the parity property of the sum $B(2, n) + B(3, n) + \dots + B(n, n)$ (see formula (4.1)), we calculate the limit

$$\lim_{x \rightarrow 0} \left\{ x^{-b} F_A^{(n)} \left(a, \mathbf{b}; \mathbf{c}; 1 - \frac{1}{x} \right) \right\} = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n,n)}}{M_n!} \prod_{k=1}^n \frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}} \prod_{k=1}^n F \left[\begin{matrix} c_k - a + B(k, n) - A(k, n), b_k + B(k, n); \\ c_k + B(k, n); \end{matrix} 1 \right]$$

and applying the summation formula (2.1) to the Gauss hypergeometric functions in the last sum, by virtue of the equality (4.11), we obtain the equality (4.16). The equality (4.17) is proved similarly as proof of the equality (4.16). Q.E.D. \square

Conflicts of Interest

The authors declare no conflicts of interest.

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