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Frankl-Type Problem for a Mixed Type Equation Associated Hyper-Bessel Differential Operator

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Abstract

The main target of the present research is the Frankl-type problem for mixed type equation with the Caputo-like counter part hyper-Bessel fractional derivative. We prove a unique solvability of this problem under certain conditions on given data. For this aim we use energy integrals (for the uniqueness) and method of integral equations (for the existence).

Keywords: Frankl-type problem, Mixed equation, Caputo-like counterpart hyper-Bessel operator, Integral equation

2010 MSC: 35M10, 35M12

1. Introduction

Boundary value problems (BVPs) involving fractional differential operators (FDOs) have become an important field in mathematics with various applications in all related fields of science and engineering. Doing research on partial differential (PDEs) equations has found increasing interest because of its applications, especially models of gas movement in a channel and underground water flows. Parabolic-hyperbolic type equations with fractional order in a mixed domain are mainly used for describing the models of those processes more precisely.

It can be presented as an example a number of works devoted to studying BVPs [7]-[12]. For the first time, many authors took an essential results of studying BVPs with Riemann-Liouville and Caputo FDOs [4]-[28]. Moreover, other FDOs were popular in terms of modeling of real-life problems, in particular Bessel-type operator introduced by Dimovski [6] (called later as hyper-Bessel operator see [19]). Thereafter, majority of articles were reported regarding to unique solvability of the Cauchy and some BVPs involving hyper-Bessel FDO [3], [17], also it can be found some papers for nonlinear equations with regularized Caputo-like counterpart hyper-Bessel operator (see [32], [21]). So far, non-local problems for fractional derivative are seeing an interesting target by many authors [18]-[8]. Frankl-type problems for mixed hyperbolic-parabolic type equations were studied in [20]-[25] which are considered a good example for non-local problems. Recently, E.T.Karimov investigated Frankl-type problem with the Caputo FDO and found a unique solution of the problem [15]. Also it is useful to indicate for investigation, the recent work related to solving on a terminal value problem for a generalization of the fractional diffusion equation with hyper-Bessel operator [33] and new results for generalization of the time-fractional diffusion equation with variable coefficients [29], [26] and we also recommend for readers some useful papers with further developments of those operators (see [31]-[10]).

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In this paper, we aim to study a unique solvability of a Frankl-type problem for partial differential equation of mixed type involving sub-diffusion equation with hyper-Bessel fractional derivative and wave equation. We have used method of energy integrals for proving uniqueness of the solution and also method of integral equations for showing existence of the solution.

We have to note that obtained result can be used in mathematical models of the gas movement in a porous medium. The usage of special fractional derivative can be justified by memory effect. For details we refer E.Karimov's DSc thesis [14].

2. Formulation of a Problem

Let $\Omega = \Omega_1 \cup \Omega_2 \cup AB$ be a simple-connected domain, where $AB = \{(x,t) : t = 0, 0 < x < 1\}$, $\Omega_1 = \{(x,t) : 0 < t < 1, 0 < x < 1\}$, $\Omega_2 = \{(x,t) : -t < x < t+1, -1/2 < t < 0\}$.

Problem B. To find a function $u(x,t)$, which is

1) $u(x,t) \in C\left(\overline{\Omega}\right) \cap C^1\left(\overline{\Omega_2}\right) \cap C^2(\Omega_2)$, ${}^C(t^\theta \frac{\partial}{\partial t})^\alpha u \in C(\Omega_1)$, $u_{xx} \in C(\Omega_1)$;

2) satisfies equation

$$0 = \begin{cases} u_{xx}(x,t) - {}^C(t^\theta \frac{\partial}{\partial t})^\alpha u(x,t), & (x,t) \in \Omega_1, \\ u_{xx}(x,t) - u_{tt}(x,t), & (x,t) \in \Omega_2; \end{cases} \tag{2.1}$$

3) satisfies non-local conditions

$$a_1(z)\, u(0,z) + b_1(z)\, u(z/2, -z/2) = c_1(z), \quad 0 \le z \le 1, \tag{2.2}$$

$$a_2(z)\, u(1,z) + b_2(z)\, u((z+1)/2, (z-1)/2) = c_2(z), \quad 0 \le z \le 1, \tag{2.3}$$

$$a_3(z)\, u(0,z) + b_3(z)\, u(1,z) = c_3(z), \quad 0 \le z \le 1; \tag{2.4}$$

4) $u(x,t)$ satisfies the following conjugating condition

$$\lim_{t \to +0} t^{1-\alpha(1-\theta)} u_t(x,t) = \lim_{t \to -0} u_t(x,t). \tag{2.5}$$

Here $0 < \alpha < 1$, $\theta < 1$, $a_i(z)$, $b_i(z)$, $c_i(z)$ $\left(i = \overline{1,3}\right)$ are given continuous functions such that

$$a_1^2(z) + a_2^2(z) \neq 0, \;\; a_1^2(z) + a_3^2(z) \neq 0, \;\; b_1^2(z) + b_2^2(z) \neq 0,$$
$$b_1^2(z) + b_3^2(z) \neq 0, \;\; a_j^2(z) + b_j^2(z) + c_j^2(z) \neq 0, \;\; j = 1, 2,$$

$${}^C\left(t^\theta \frac{\partial}{\partial t}\right)^\alpha f(t) = \frac{(1-\theta)^{\alpha+1} t^{-\alpha(1-\theta)}}{\Gamma(1-\alpha)} \left(1 - \alpha + \frac{t}{1-\theta}\frac{d}{dt}\right) t^{(1-\theta)(\alpha-1)} \int_0^t \left(t^{1-\theta} - \tau^{1-\theta}\right)^{-\alpha} \tau^{-\theta} \left[f(\tau) - f(0)\right] d\tau$$

is a Caputo-like counterpart hyper-Bessel fractional differential operator of order α $(0 < \alpha < 1)$ (see, for instance [3]) and when $\theta = 0$, $\alpha = 1$ it coincides with the first derivative of $f(t)$ (see the definition of Erdelyi-Kober fractional integral [19]). Also Caputo fractional operator can be considered as the special case of the Caputo-like counterpart hyper-Bessel fractional differential operator for $\theta = 0$.

The similar problem for mixed type equation with the Caputo fractional derivative has been considered in the recent work by E.T.Karimov [15].

3. Main Result

The main result we formulate as the following

Theorem 3.1. *If* $\frac{d}{dz}\overline{d_2(z)} > 0$ *and* $a_i(z)$, $b_i(z)$, $c_i(z) \in C[0,1] \cap C^1(0,1)$, *such that* $a_i(z) \neq 0$, $b_i(z) \neq 0$, $z \in [0,1]$ *then, a unique solution of the Problem B exists, where*

$$\overline{d_2}(z) = \frac{1 + d_2(z)}{1 - d_2(z)}, \quad d_2(z) = \frac{a_1(z)\, b_2(z)\, b_3(z)}{a_2(z)\, a_3(z)\, b_1(z)}.$$

Proof. First, we prove the uniqueness of the solution.

3.1. The uniqueness of a solution.

We assume that the solution of the problem exists then, we introduce the following notations and assumptions:

$$u(x,0)=\tau(x),\ x\in[0,1],\quad \tau(x)\in C[0,1]\cap C^2(0,1), \tag{3.1}$$

$$\lim_{t\to-0} u_t(x,t)=\nu_2(x),\ x\in(0,1),\ \ \nu_2(x)\in C^1(0,1)\cap L^1(0,1). \tag{3.2}$$

Solutions of the Cauchy problem for the equation (2.1) in Ω_2 can be represented by the D'Alembert's formula

$$u(x,t)=\frac{1}{2}\left[\tau(x+t)+\tau(x-t)+\int_{x-t}^{x+t}\nu_2(z)\,dz\right]. \tag{3.3}$$

We assume that $a_i,\ b_i\neq 0\ \left(i=\overline{1,3}\right)$ and from (2.2) and (2.3), we find that

$$u(0,z)=\frac{c_1(z)-b_1(z)\psi(z)}{a_1(z)}=g_1(z), \tag{3.4}$$

$$u(1,z)=\frac{c_2(z)-b_2(z)\varphi(z)}{a_2(z)}=g_2(z), \tag{3.5}$$

where

$$u((z+1)/2,(z-1)/2)=\varphi(z),\ 0\le z\le 1, \tag{3.6}$$

$$u(z/2,-z/2)=\psi(z),\ 0\le z\le 1. \tag{3.7}$$

We note that $g_1(z)$, $g_2(z)$, $\psi(z)$, $\varphi(z)$ are not known yet.

In short, we can write main functional relation in Ω_2 by doing evaluations considered in [15]:

$$\nu_2(z)=\overline{d_2}(z)\tau'(z)+\overline{d_2}'(z)\tau(z)+e(z),\ \ 0<z<1, \tag{3.8}$$

where

$$e(z)=\frac{d_2'(z)}{1-d_2(z)}\left[\frac{d_1(0)-A_1}{d_2(0)}-d_1(z)\right]-\frac{2d_1'(z)}{1-d_2(z)}+$$
$$+\frac{d_2'(z)(1+d_2(z))}{[1-d_2(z)]^2}\left[d_1(z)-\frac{d_1(0)-A_1}{d_2(0)}\right],$$
$$d_1(z)=\frac{a_2(z)a_3(z)c_1(z)-a_1(z)a_2(z)c_3(z)+a_1(z)b_3(z)c_2(z)}{a_2(z)a_3(z)b_1(z)},$$
$$A_1=\tau(0)=\frac{c_1(0)}{a_1(0)+b_1(0)},\ \ A_2=\tau(1)=\frac{c_2(0)}{a_2(0)}-\frac{b_2(0)}{a_2(0)}\frac{d_1(0)-A_1}{d_2(0)}, \tag{3.9}$$

For obtaining main functional relation in Ω_1, we need to write the solution of (2.1) satisfied (3.1), (3.4), (3.5) conditions. In this case, we considered $u(x,t)=\omega(x,t)+v(x,t)$, in other words $\omega(x,t)$ and $v(x,t)$ are the solutions of the following problems

$$(*)\begin{cases}\omega_{xx}(x,t)-{}^{C}(t^{\theta}\frac{\partial}{\partial t})^{\alpha}\omega(x,t)=0,\ (x,t)\in\Omega_1,\\ \omega(x,0)=0,\ \ x\in[0,1],\\ \omega(0,t)=g_1(t),\ \ \omega(1,t)=g_2(t),\ t\in[0,1],\end{cases}$$

and

$$(**)\begin{cases}v_{xx}(x,t)-{}^{C}(t^{\theta}\frac{\partial}{\partial t})^{\alpha}v(x,t)=0,\ (x,t)\in\Omega_1,\\ v(x,0)=\tau(x)-\omega(x,0),\ \ x\in[0,1],\\ v(0,t)=v(1,t)=0,\ \ t\in[0,1],\end{cases}$$

respectively.

The problem (**) was investiagted in [3] and proved the existence of the solution.

Considering above problems we can write the solution of the problem (2.1), (3.1), (3.4), (3.5) as follows:

$$u(x,t) = \sum_{k=1}^{\infty} \left[\tau_{0k}\, E_{\alpha}\left(-\frac{(k\pi)^2}{p^{\alpha}}\, t^{p\alpha} \right) \right] \sin k\pi x + \omega(x,t) \tag{3.10}$$

where $p = 1 - \theta$ and

$$\omega(x,t) = g_1(t) + x[g_2(t) - g_1(t)],$$

$$\tau_{0k} = 2 \int_0^1 \tau(x) \sin k\pi x dx.$$

We introduce another notation:

$$\nu_1(x) = \lim_{t \to +0} t^{1-p\alpha} u_t(x,t). \tag{3.11}$$

Using representation (3.10), we evaluate $t^{1-p\alpha}u_t(x,t)$:

$$t^{1-p\alpha}u_t(x,t) = \sum_{k=1}^{\infty} \tau_{0k} \left(-\frac{(k\pi)^2}{p^{\alpha}} \right) E_{\alpha,\alpha}\left(-\frac{(k\pi)^2}{p^{\alpha}} t^{\alpha p} \right) \sin k\pi x + t^{1-p\alpha}\omega_t(x,t).$$

Considering above evaluation, from $1 - p\alpha > 0$ we obtain the following functional relation on AB deduced from Ω_1 as $t \to +0$:

$$\nu_1(x) = \frac{1}{p^{\alpha}\Gamma(\alpha)} \tau''(x). \tag{3.12}$$

Here, we have used $2\int_0^1 \tau(x) \sin k\pi x dx = -\frac{2}{(k\pi)^2} \int_0^1 \tau''(x) \sin k\pi x dx$, which is true due to $\tau(0) = \tau(1) = 0$ when we consider homogeneous case.

Clearly, for showing the uniqueness of the solution of the considered problem, it is enough to prove that homogeneous problem has only trivial solution.

In this case, we will have homogeneous problem at

$$c_1(z) = c_2(z) = c_3(z) = 0.$$

Then, from (3.9) it follows that

$$\tau(0) = \tau(1) = 0. \tag{3.13}$$

Also, it is comprehensible that $e(x) = 0$ when we consider homogeneous problem.

We multiply equation (3.12) to the function $\tau(x)$ and integrate along AB. Then, using integration by parts and equality (3.13), we get

$$\int_0^1 [\tau'(x)]^2 dx + p^{\alpha}\Gamma(\alpha) \int_0^1 \tau(x)\, \nu_1(x)\, dx = 0. \tag{3.14}$$

Let us evaluate the sign of the integral below

$$\overline{I} = \int_0^1 \tau(x)\, \nu_1(x)\, dx. \tag{3.15}$$

If $c_j(z) \equiv 0$, $\left(j = \overline{1,3}\right)$ and considering $\nu_1(x) = \nu_2(x)$, (3.15) can be written as follows:

$$\overline{I} = \int_0^1 \tau(x)\, \nu_2(x)\, dx = \int_0^1 \tau(x) \left[\overline{d_2(x)} \tau' x + \overline{d_2'(x)} \tau x \right] dx = \frac{1}{2} \int_0^1 [\tau(x)]^2\, \overline{d_2'(x)} dx.$$

According to Theorem, $\overline{d_2'(x)} > 0$ hence, $\overline{I} \geq 0$. Then, by means of (3.14) we find $\tau'(x) = 0$ or $\tau(x) = Const$. Taking into account (3.13), we determine that $\tau(x) \equiv 0$ at $0 \leq x \leq 1$. From (3.8) and (3.12) we get $\nu_1(x) = \nu_2(x) \equiv 0$. Then according to (3.3) one can see that $u(x,t) \equiv 0$ in Ω_2. Moreover, while considering homogeneous problem, it is straightforward to find $u(0,z) = u(1,z) = 0$, or $g_1(z) = g_2(z) = 0$, $0 \leq z \leq 1$. Then, from (3.10) it is inferred that $u(x,t) \equiv 0$ in Ω_2. Consequently, $u(x,t) \equiv 0$ in Ω, which completes the proof of uniqueness of the solution of the main problem.

3.2. The existence of the solution.

Consider the problem {(3.12), (3.9)}, i.e.

$$\begin{cases} \tau''(x) = p^\alpha \Gamma(\alpha)\nu_1(x), \\ \tau(0) = A_1, \quad \tau(1) = A_2. \end{cases}$$

Introducing a notation

$$\overline{\tau}(x) = \tau(x) - A_1 - (A_2 - A_1)x, \tag{3.16}$$

we get

$$\begin{cases} \overline{\tau}''(x) = p^\alpha \Gamma(\alpha)\nu_1(x), \\ \overline{\tau}(0) = 0, \quad \overline{\tau}(1) = 0. \end{cases} \tag{3.17}$$

We write the solution of this problem by Green's function

$$\overline{\tau}(x) = p^\alpha \Gamma(\alpha) \int_0^1 \nu_1(\xi) G_0(x,\xi) d\xi, \tag{3.18}$$

where

$$G_0(x,\xi) = \begin{cases} (\xi - 1)x, & x < \xi, \\ (x-1)\xi, & \xi < x \end{cases}$$

is Green's function of (3.17).

Considering the notation (3.16) and from (3.18), we get

$$\tau(x) = A_1 + (A_2 - A_1)x + p^\alpha \Gamma(\alpha) \int_0^1 \nu_1(\xi) G_0(x,\xi) d\xi. \tag{3.19}$$

Now considering (2.5) and (3.8), after some evaluations, from (3.19) we obtain the second kind Fredholm integral equation with respect to $\tau(x)$, which is equivalent to the formulated problem:

$$\tau(x) - \int_0^1 \tau(\xi) \tilde{K}(x,\xi) d\xi = \tilde{F}(x), \tag{3.20}$$

where

$$\tilde{K}(x,\xi) = \frac{d_2(\xi)}{(1-\theta)^\alpha \Gamma(\alpha)} \frac{d}{d\xi} G_0(x,\xi),$$

$$\tilde{F}(x) = A_1 + (A_2 - A_1)x + \frac{1}{(1-\theta)^\alpha \Gamma(\alpha)} \int_0^1 e(\xi) G_0(x,\xi) d\xi.$$

If

$$a_i(z), b_i(z), c_i(z) \in C[0,1] \cap C^1(0,1), \tag{3.21}$$

then the kernel and the right side of the integral equation (3.20) will be continuous in their domain. Since equation (3.20) is equivalent to the considered problem, then from the uniqueness of the problem, it follows that equation (3.20) is uniquely solvable in the class of continuous functions.

Further, finding function $\nu_2(x)$ by formula (3.8), we find a solution of the considered problem in the domain Ω_2 as the solution of the Cauchy problem by formula (3.3). Then, from (3.6), (3.7) $\varphi(z)$, $\psi(z)$ are found and this opens a way for finding unknown functions $g_1(z)$, $g_2(z)$. After that we rewrite $\omega(x,t)$ and $\phi(x)$ with known functions. Finally, in the domain Ω_1, solution can be represented by the formula (3.10) problem for equation (2.1) at $t > 0$. The theorem is proved. □

4. Conclusion

Our main aim was to show a unique solvability of the considered problem. In [3], the solution was found for homogeneous boundary condition in Ω_1 only. Therefore, we needed to rewrite that solution for non-homogeneous boundary conditions. First, we use them formally, then we will find them lately. The considered problem is equivalently reduced to the second kind Fredholm integral equations. Therefore, a uniqueness of the problem we had to prove additionally. For this aim we use the method of energy integrals. All in all, sufficient conditions for given functions are presented in terms of uniqueness and existence of the solution of the Frankl type problem in the domain Ω.

Practical value of this result can be estimated by possible application in mathematical modeling of gas movement in a channel surrounded by porous medium (see [14]).

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