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Fekete-Szegö Problems for the k^{th} Root Transform of Subclasses of Starlike and Convex Functions with Respect to Symmetric Points

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Abstract

In the present paper, sharp upper bounds for the coefficient functional $|b_{2k+1} - \mu b_{k+1}^2|$ corresponding to the k^{th} root transformation of certain normalized analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined on the unit disk Δ in the complex plane where the function $f(z)$ belong to certain subclasses of starlike and convex functions with respect to symmetric points are obtained. Further, Fekete-Szegö inequalities for the function $\frac{z}{f(z)}$ and the inverse function f for the above mentioned classes are investigated and pointed out the special cases as remark.

Keywords: Analytic functions, Subordination, k^{th}root transformation, Starlike functions, Convex functions, Fekete-Szegö inequality

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1. Introduction and Definitions

Denote by $\mathcal{A}$, the class of functions of the form

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ which are univalent in Δ. An analytic function f is subordinate to an analytic function g, written as $f(z) \prec g(z)$, if there exists an analytic self-map w of Δ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in \Delta)$. It follows from the Schwarz lemma that $f(z) \prec g(z)$ $(z \in \Delta) \Longrightarrow f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. If the function g is univalent in Δ then (see [10])

$$f(z) \prec g(z) \ \ (z \in \Delta) \Longleftrightarrow f(0) = g(0) \ \text{ and } \ f(\Delta) \subset g(\Delta).$$

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It is well-known that the n^{th} Taylor coefficient of an univalent function f in the class $\mathcal{A}$ is bounded by n (see [4]). The bounds for the coefficients of these functions give information about their geometric properties. The Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for normalized univalent function $f(z)$ of the form (1.1) is well-known for its rich history in the theory of geometric function theory. Several authors have investigated the Fekete-Szegö coefficient functional for functions in different subclasses of univalent and multivalent functions (for detail, see [1, 3, 5, 7, 9, 12]).

We recall the following definitions of Starlike and Convex functions with respect to symmetric points.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_s^*$, if it satisfies the condition

$$\Re\left[\frac{2zf'(z)}{f(z)-f(-z)}\right] > 0 \quad (\forall z \in \Delta). \tag{1.2}$$

The class $\mathcal{S}_s^*$ was introduced and studied by Sakaguchi [15].

Definition 1.2. ([6]) A function $f(z) \in \mathcal{A}$ is said to be in C_s, if it satisfies the condition:

$$\Re\left[\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right] > 0 \quad (\forall z \in \Delta). \tag{1.3}$$

It may be noted that $f \in C_s \Longleftrightarrow zf' \in \mathcal{S}_s^*$.
Motivated by aforecited works, we define the generalized subclasses of $\mathcal{S}_s^*$ and C_s as below:

Definition 1.3. [14] Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ be a univalent starlike functions with respect to 1 which maps the unit disk Δ onto a region in the right half plane which is symmetric with respect to the real axis and let $B_1 > 0$. The function $f \in \mathcal{A}$ is in the class $\mathcal{S}_s^*(\phi)$ if

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \phi(z), \tag{1.4}$$

and in the class $C_s(\phi)$ if

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} \prec \phi(z). \tag{1.5}$$

Remark 1.4. Putting $\phi(z) = \frac{1+z}{1-z}$ in (1.4) and (1.5), we get the classes $\mathcal{S}_s^*$ and C_s studied by Sakaguchi [15] and Das and Singh [6] respectively.

Let k be a positive integer. For an univalent function f of the form (1.1), the k^{th} root transform is defined by

$$F(z) = [f(z^k)]^{\frac{1}{k}} = z\left[\frac{f(z^k)}{z^k}\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{nk+1} z^{nk+1}, \tag{1.6}$$

where the initial coefficients are given by

$$\begin{aligned} b_{k+1} &= \frac{a_2}{k} \\ b_{2k+1} &= \frac{a_3}{k} + \frac{1-k}{2k^2}a_2^2, \\ b_{3k+1} &= \frac{a_4}{k} + \frac{1-k}{k^2}a_2 a_3 + \frac{(1-k)(1-2k)}{3!k^3}a_2^3, \end{aligned}$$

and so on. Since f is univalent, so $\frac{f(z^k)}{z^k}$ is non-vanishing in Δ implies that k^{th} root of f is analytic in Δ. The Fekete-Szegö coefficient functional of the associated function $F(z)$ is given by $|b_{2k+1} - \mu b_{k+1}^2|$. This quantity is known as Fekete-Szegö problem of the k^{th} root transform of f. The k^{th} root transform has been widely used in a variety of ways in complex function theory. In 2009, Ali et al. [2] have investigated the Fekete-Szegö coefficient functional for the k^{th}root transform of functions belonging to several classes of analytic functions defined by means of subordination (also see [11]).

In this paper, the authors obtain upper bounds for the Fekete-Szegö coefficient functional $|b_{2k+1} - \mu b_{k+1}^2|$ associated with the k^{th} root transform of the function f belonging to the classes $\mathcal{S}_s^*(\phi)$ and $C_s(\phi)$. Fekete-Szegö inequalities associated with the function $\frac{z}{f(z)}$ and inverse of the function $f(z)$ are also investigated.

2. Coefficient bounds for $f \in \mathcal{S}_s^*(\phi)$ and $f \in C_s(\phi)$

We recall the following lemmas to prove our main results:

Lemma 2.1. *[3] Let Ω be the class of analytic functions w, normalized by $w(0) = 0$ satisfying the condition $|w(z)| < 1$. If $w \in \Omega$ and*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad (z \in \Delta),$$

then for any real numbers t, we have

$$|w_2 - t w_1^2| \leq \begin{cases} -t, & t \leq -1 \\ 1, & -1 \leq t \leq 1 \\ t, & t \geq 1. \end{cases}$$

For $t < -1$ or $t > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. For $-1 < t < 1$, equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if

$$w(z) = z\left(\frac{\epsilon + z}{1 + \epsilon z}\right) \quad (0 \leq \epsilon \leq 1)$$

or one of its rotations, while for $t = 1$, equality holds if and only if

$$w(z) = -z\left(\frac{\epsilon + z}{1 + \epsilon z}\right) \quad (0 \leq \epsilon \leq 1)$$

or one of its rotations.

Lemma 2.2. *[8] If $w \in \Omega$, then*

$$|w_2 - t w_1^2| \leq \max\{1, |t|\},$$

for any complex number t. The result is sharp for the function $w(z) = z^2$ or $w(z) = z$.

Now we obtain the bounds for the functional $|b_{2k+1} - \mu b_{k+1}^2|$ corresponding to the k^{th}root transform for the function f in the classes $\mathcal{S}_s^*(\phi)$ and $C_s(\phi)$.

Theorem 2.3. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0,\ B_2 \geq 0$ and B_n's real. If $f \in \mathcal{S}_s^*(\phi)$ and F is the k^{th}root transformation of f given by (1.6), then for any complex number μ, we have*

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \frac{B_1}{2k}\, max\left\{1, \left|\frac{k-1}{4k}B_1 - \frac{B_2}{B_1} + \frac{\mu}{2k}B_1\right|\right\}. \tag{2.1}$$

The result is sharp.

Proof. Let $f \in \mathcal{S}_s^*(\phi)$. Then by (1.4) of Definition 1.3, there exists a Schwarz's function $w(z) \in \Omega$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = \phi(w(z)). \tag{2.2}$$

Now, it follows from (1.1) that

$$\frac{2zf'(z)}{f(z) - f(-z)} = 1 + 2a_2 z + 2a_3 z^2 + (4a_4 - 2a_2 a_3)z^3 + \cdots. \tag{2.3}$$

Also,

$$\phi(w(z)) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2)z^2 + \cdots. \tag{2.4}$$

Making use of (2.3) and (2.4) in (2.2), we obtain

$$2a_2 = B_1 w_1 \Longrightarrow a_2 = \frac{B_1 w_1}{2} \tag{2.5}$$

$$2a_3 = B_1w_2 + B_2w_1^2 \Longrightarrow a_3 = \frac{B_1w_2 + B_2w_1^2}{2}. \tag{2.6}$$

If $F(z)$ is the k^{th}root transformation of $f(z)$, then from (1.6), (2.5) and (2.6) we get

$$b_{k+1} = \frac{a_2}{k} = \frac{B_1w_1}{2k}, \tag{2.7}$$

and

$$b_{2k+1} = \frac{a_3}{k} + \frac{1-k}{2k^2}a_2^2 = \frac{B_1w_2}{2k} + \frac{B_2}{2k}w_1^2 + \frac{1-k}{8k^2}B_1^2w_1^2. \tag{2.8}$$

Therefore, for any complex number μ, we have

$$|b_{2k+1} - \mu b_{k+1}^2| = \frac{B_1}{2k}|w_2 - tw_1^2|, \tag{2.9}$$

where

$$t = \frac{\mu}{2k}B_1 - \frac{B_2}{B_1} - \frac{1-k}{4k}B_1.$$

An application of Lemma 2.2 to the right hand side of (2.9) gives the desire result as mentioned in (2.1). The result is sharp and followed by

$$|b_{2k+1} - \mu b_{k+1}^2| = \begin{cases} \frac{B_1}{2k}, & w(z) = z^2 \\ \frac{B_1}{2k}|\frac{\mu}{2k}B_1 - \frac{B_2}{B_1} - \frac{1-k}{4k}B_1|, & w(z) = z. \end{cases}$$

This complete the proof of Theorem 2.3. □

We note that the result is sharp for the extremal functions given by $p(z) = \frac{z}{1+z^2}$.
Taking $\phi(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots$ and $k = 1$, we obtain the Fekete-Szegö coefficient functional for the class $\mathcal{S}_s^*$ as follows:

Corollary 2.4. *Let the function $f \in \mathcal{A}$ given by (1.1) be in the class $\mathcal{S}_s^*$. Then for any complex number μ, we have*

$$|a_3 - \mu a_2^2| \leq max\{1, |\mu - 1|\}.$$

The estimate is sharp.

Remark 2.5. Putting $\mu = 1$ in Corollary 2.4 we obtain the result $|a_3 - a_2^2| \leq 1$ for the class $\mathcal{S}_s^*$ due to Shanmugam et al. [16].

Theorem 2.6. *Let $\phi(z) = 1 + B_1z + B_2z^2 + \cdots$ with $B_1 > 0,\ B_2 \geq 0$ and B_n's real. If $f \in C_s(\phi)$ and F is the k^{th}root transformation of f given by (1.6), then for any complex number μ, we have*

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \frac{B_1}{6k}max\left\{1, \left|\frac{3(k-1)}{16k}B_1 - \frac{B_2}{B_1} + \frac{3\mu}{8k}B_1\right|\right\}. \tag{2.10}$$

The result is sharp.

Proof. Let $f \in C_s(\phi)$. Then by (1.5) of Definition 1.3, it follows that

$$\frac{2(zf'(z))'}{[f(z) - f(-z)]'} = \phi(w(z)). \tag{2.11}$$

A simple calculation shows

$$\frac{2(zf'(z))'}{[f(z) - f(-z)]'} = 1 + 4a_2z + 6a_3z^2 + \cdots. \tag{2.12}$$

Proceeding as in Theorem 2.3, we obtain

$$a_2 = \frac{B_1 w_1}{4}, \tag{2.13}$$

and

$$a_3 = \frac{B_1 w_2 + B_2 w_1^2}{6}. \tag{2.14}$$

If $F(z)$ is the k^{th}root transform of $f(z)$, then

$$b_{k+1} = \frac{a_2}{k} = \frac{B_1 w_1}{4k}, \tag{2.15}$$

and

$$b_{2k+1} = \frac{a_3}{k} + \frac{1-k}{2k^2} a_2^2 = \frac{B_1 w_2}{6k} + \frac{B_2 w_1^2}{6k} + \frac{1-k}{32k^2} B_1^2 w_1^2. \tag{2.16}$$

Thus, for any complex number μ, we obtain

$$|b_{2k+1} - \mu b_{k+1}^2| = \frac{B_1}{6k} |w_2 - t w_1^2|, \tag{2.17}$$

where

$$t = \frac{3(k-1)}{16k} B_1 - \frac{B_2}{B_1} + \frac{3\mu}{8k} B_1.$$

By an application of Lemma 2.2 we obtain the desire result as stated in Theorem 2.6. The result is sharp and followed by

$$|b_{2k+1} - \mu b_{k+1}^2| = \begin{cases} \frac{B_1}{6k}, & w(z) = z^2 \\ \frac{B_1}{6k} \left| \frac{3(k-1)}{16k} B_1 - \frac{B_2}{B_1} + \frac{3\mu}{8k} B_1 \right|, & w(z) = z. \end{cases}$$

The proof of Theorem 2.6 is thus completed. □

Taking $\phi(z) = \frac{1+z}{1-z}$ and $k = 1$, we get the following result.

Corollary 2.7. *Let $f \in \mathcal{A}$ given by (1.1) be in the class C_s. Then, for any complex number μ, we have*

$$|a_3 - \mu a_2^2| \leq \frac{1}{3} max \left\{ 1, \left| \frac{3\mu}{8} - 1 \right| \right\}.$$

The estimate is sharp.

Remark 2.8. Taking $\mu = 1$ in Corollary 2.7, we get the result $|a_3 - a_2^2| \leq \frac{1}{3}$ for the class C_s due to Shanmugam et al. [16].

Now, we determine the bounds for the functional $|b_{2k+1} - \mu b_{k+1}^2|$ for real μ for the classes $\mathcal{S}_s^*(\phi)$ and $C_s(\phi)$.

Theorem 2.9. *If $f \in \mathcal{S}_s^*(\phi)$ and F is the k^{th}root transform of the function f defined by (1.6), then for any real number μ and for*

$$l_1 = \frac{4k(B_2 - B_1) + (1-k)B_1^2}{2B_1^2}, \tag{2.18}$$

$$l_2 = \frac{4k(B_1 + B_2) + (1-k)B_1^2}{2B_1^2}, \tag{2.19}$$

we have

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \begin{cases} \frac{B_1}{2k} \left(\frac{B_2}{B_1} - \frac{k-1}{4k} B_1 - \frac{\mu}{2k} B_1 \right), & \mu \leq l_1 \\ \frac{B_1}{2k}, & l_1 \leq \mu \leq l_2 \\ \frac{B_1}{2k} \left(\frac{k-1}{4k} B_1 - \frac{B_2}{B_1} + \frac{\mu}{2k} B_1 \right), & \mu \geq l_2. \end{cases} \tag{2.20}$$

Each of the estimate (2.20) is sharp.

Proof. Let $f \in \mathcal{S}_s^*(\phi)$. From (2.9) we have

$$|b_{2k+1} - \mu b_{k+1}^2| = \frac{B_1}{2k}|w_2 - tw_1^2|, \tag{2.21}$$

where

$$t = \frac{\mu B_1}{2k} - \frac{B_2}{B_1} - \frac{1-k}{4k}B_1.$$

An applications of Lemma 2.1 on right hand side of (2.21) give the following cases:

Case-1: If $\mu \leq l_1$ then

$$\mu \leq \frac{4k(B_2 - B_1) + (1-k)B_1^2}{2B_1^2}.$$

Which implies, $t \leq -1$ and $|w_2 - tw_1^2| \leq -t$, hence we get

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \frac{B_1}{2k}\left[\frac{B_2}{B_1} - \frac{(k-1)}{4k}B_1 - \frac{\mu}{2k}B_1\right]. \tag{2.22}$$

Case 2: If $l_1 \leq \mu \leq l_2$ then

$$\frac{4k(B_2 - B_1) + (1-k)B_1^2}{2B_1^2} \leq \mu \leq \frac{4k(B_1 + B_2) + (1-k)B_1^2}{2B_1^2}.$$

Which implies, $-1 \leq t \leq 1$ and $|w_2 - tw_1^2| \leq 1$, thus

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \frac{B_1}{2k}. \tag{2.23}$$

Case 3: If $\mu \geq l_2$, then

$$\mu \geq \frac{4k(B_1 + B_2) + (1-k)B_1^2}{2B_1^2}$$

which implies $t \geq 1$ and $|w_2 - tw_1^2| \leq t$, hence we get

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \frac{B_1}{2k}\left(\frac{\mu}{2k}B_1 - \frac{B_2}{B_1} - \frac{1-k}{4k}B_1\right). \tag{2.24}$$

The result in (2.20) follows from (2.22), (2.23) and (2.24).
We also note that when

1. $\mu \leq l_1$, the equality holds if and only if $w(z) = z$ or one of its rotation.

2. $l_1 \leq \mu \leq l_2$, then the equality holds if and only if $w(z) = z^2$ or one of its rotation.

3. $\mu \geq l_2$, then the equality holds when $w(z) = \frac{z(\epsilon + z)}{1 + \epsilon z}$ $(0 \leq \epsilon \leq 1)$ or one of its rotation.

This complete the proof of Theorem 2.9. □

Theorem 2.10. *If $f \in C_s(\phi)$ and F is the k^{th}root transform of the function f defined by (1.6), then for any real number μ and for*

$$e_1 = \frac{16k(B_2 - B_1) + 3(1-k)B_1^2}{6B_1^2}, \tag{2.25}$$

$$e_2 = \frac{16k(B_1 + B_2) + 3(1-k)B_1^2}{6B_1^2}, \tag{2.26}$$

we have

$$|b_{2k+1} - \mu b_{k+1}^2| \le \begin{cases} \frac{B_1}{6k}\left(\frac{B_2}{B_1} - \frac{3(k-1)}{16k}B_1 - \frac{3\mu}{8k}B_1\right) & \mu \le e_1 \\ \frac{B_1}{6k} & e_1 \le \mu \le e_2 \\ \frac{B_1}{6k}\left(\frac{3(k-1)}{16k}B_1 - \frac{B_2}{B_1} + \frac{3\mu}{8k}B_1\right) & \mu \ge e_2. \end{cases} \tag{2.27}$$

Each of the estimate (2.27) is sharp.

The proof of theorem is much akin to proof of Theorem 2.9 and hence we omit it.

Remark 2.11. The result is sharp for the extremal functions $\frac{2zf'(z)}{f(z)-f(-z)} = p(z) = \frac{z(z-t)}{1-zt}$ $|t| < 1$ and the result is sharp for the functions given by $\frac{2(zf'(z))'}{[f(z)-f(-z)]'} = p(z) = \frac{z}{1+z^2}$. Further it is of interest to note that, by taking $k = 1$ in Theorem 2.9 and Theorem 2.10 we obtain the result due to Shanmugam et al. ([16], Theorem 2.1) and ([16], Corollary 2.4) respectively.

3. Coefficient functional associated with $\frac{z}{f(z)}$

In this section, we determine Fekete-Szegö coefficient functional bound associated with the function $M(z)$ defined by

$$M(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} u_n z^n \quad (z \in \Delta) \tag{3.1}$$

where f belongs to the classes $\mathcal{S}_s^*(\phi)$ and $C_s(\phi)$.

Theorem 3.1. *Let $f \in \mathcal{S}_s^*(\phi)$ and $M(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} u_n z^n$. Then for any complex number μ, we have*

$$|u_2 - \mu u_1^2| \le \frac{B_1}{2} max\left\{1, \left|\frac{B_2}{B_1} - \frac{B_1}{2} + \frac{\mu}{2}B_1\right|\right\}. \tag{3.2}$$

The result is sharp.

Proof. The proof of the theorem is similar to those of given in Theorem 2.3. We sketch the main steps and omit the details.
By simple computation, we have

$$M(z) = \frac{z}{f(z)} = 1 - a_2 z + (a_2^2 - a_3)z^2 + \cdots \tag{3.3}$$

From (3.1) and (3.3) we have

$$u_1 = -a_2 \tag{3.4}$$

$$u_2 = a_2^2 - a_3. \tag{3.5}$$

Using (2.5)and (2.6) in (3.4) and (3.5), we obtain

$$u_1 = -\frac{B_1 w_1}{2},$$

and

$$u_2 = \frac{B_1^2 w_1^2}{4} - \frac{B_1 w_2 + B_2 w_1^2}{2}$$

For any complex number μ, we get

$$|u_2 - \mu u_1^2| = \frac{B_1}{2}\left|w_2 - \left(\frac{B_1}{2} - \frac{B_2}{B_1} - \frac{\mu}{2}B_1\right)w_1^2\right|.$$

Applying Lemma 2.2, we get the desire result. This completes the proof of Theorem 3.1. □

Proceeding similar manner we obtain the following result $f \in C_s(\phi)$.

Theorem 3.2. *Let $f \in C_s(\phi)$ and $M(z) = 1 + \sum_{n=1}^{\infty} u_n z^n$. Then for any complex number μ, we obtain*

$$|u_2 - \mu u_1^2| \leq \frac{B_1}{6} max \left\{1, \left|\frac{B_2}{B_1} - \frac{3B_1}{8} + \frac{3\mu}{8} B_1\right|\right\}.$$

The result is sharp.

Assuming μ to be real number, we state the following coefficient inequalities:

Theorem 3.3. *If $f \in \mathcal{S}_s^*(\phi)$ and $M(z) = 1 + \sum_{n=1}^{\infty} u_n z^n$, then for any real number μ and for*

$$\eta_1 = \frac{B_1^2 - 2(B_1 + B_2)}{B_1^2} \qquad and \qquad \eta_2 = \frac{B_1^2 - 2(B_2 - B_1)}{B_1^2}$$

we have

$$|u_2 - \mu u_1^2| \leq \begin{cases} \frac{B_1}{2}\left(\frac{B_1}{2} - \frac{B_2}{B_1} - \frac{\mu}{2} B_1\right) & (\mu \leq \eta_1) \\ \frac{B_1}{2} & (\eta_1 \leq \mu \leq \eta_2) \\ \frac{B_1}{2}\left(\frac{B_2}{B_1} - \frac{B_1}{2} + \frac{\mu}{2} B_1\right) & (\mu \geq \eta_2). \end{cases}$$

Theorem 3.4. *If $f \in C_s(\phi)$ and $M(z) = 1 + \sum_{n=1}^{\infty} u_n z^n$, then for any real number μ and for*

$$\eta_3 = \frac{3B_1^2 - 8(B_1 + B_2)}{3B_1^2},$$

and

$$\eta_4 = \frac{3B_1^2 - 8(B_2 - B_1)}{3B_1^2}$$

we have

$$|u_2 - \mu u_1^2| \leq \begin{cases} \frac{B_1}{6}\left(\frac{3B_1}{8} - \frac{B_2}{B_1} - \frac{3\mu}{8} B_1\right) & (\mu \leq \eta_3) \\ \frac{B_1}{6} & (\eta_3 \leq \mu \leq \eta_4) \\ \frac{B_1}{6}\left(\frac{B_2}{B_1} - \frac{3B_1}{8} + \frac{3\mu}{8} B_1\right) & (\mu \geq \eta_4). \end{cases}$$

4. Coefficient inequality for the inverse of the function $f(z)$

Theorem 4.1. *If $f \in \mathcal{S}_s^*(\phi)$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ is the inverse function of f with $|w| < r_0$, where r_0 is the greater than the radius of the Koebe domain of the class $\mathcal{S}_s^*(\phi)$, then for any complex number μ, we have*

$$|d_3 - \mu d_2^2| \leq \frac{B_1}{2} max \left\{1, \left|B_1 - \frac{B_2}{B_1} - \frac{\mu}{2} B_1\right|\right\}. \tag{4.1}$$

The result is sharp.

Proof. Since

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n \tag{4.2}$$

is the inverse function of f, we have

$$f^{-1}(f(z)) = f(f^{-1}(z)) = z. \tag{4.3}$$

From (4.2) and (4.3) we obtain

$$f^{-1}(f(z)) = f^{-1}(z + \sum_{n=2}^{\infty} a_n z^n) = z \tag{4.4}$$

Equating the coefficients of z and z^2 from equation (4.2) and (4.4) we get

$$d_2 = -a_2 = -\frac{B_1 w_1}{2}$$

and

$$d_3 = 2a_2^2 - a_3 = \frac{B_1^2 w_1^2 - B_1 w_2 - B_2 w_1^2}{2}.$$

For any complex number μ, we have

$$|d_3 - \mu d_2^2| = \frac{B_1}{2}|w_2 - \lambda w_1^2| \tag{4.5}$$

where

$$\lambda = B_1 - \frac{B_2}{B_1} - \frac{\mu}{2}B_1.$$

Application of Lemma 2.2 in (4.5) gives the desire result as stated in (4.1). The result is sharp and followed by

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{B_1}{2} & (w(z) = z^2) \\ \frac{B_1}{2}|B_1 - \frac{B_2}{B_1} - \frac{\mu}{2}B_1| & (w(z) = z). \end{cases}$$

□

Proceeding as in Theorem 4.1 we state the following result $f \in C_s(\phi)$ without proof.

Theorem 4.2. *If $f \in C_s(\phi)$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ is the inverse function of f with $|w| < r_0$, where r_0 is the greater than the radius of the Koebe domain of the class $C_s(\phi)$, then for any complex number μ, we have*

$$|d_3 - \mu d_2^2| \leq \frac{B_1}{6} max\left\{1, \left|\frac{3B_1}{4} - \frac{B_2}{B_1} - \frac{3\mu}{8}B_1\right|\right\}. \tag{4.6}$$

The result is sharp.

Remark 4.3. The result is sharp for the extremal functions $p(z) = \frac{1-z^2}{1+z^2} = 1 + 2z^2 + 2z^4 + \cdots$ (see[13]) or functions defined of the form $p(z) = \frac{z(z-t)}{1-zt}$ $\quad |t| < 1$.

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