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Argument Estimates for Carathéodory Functions with Applications

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Abstract

The purpose of the present paper is to investigate argument properties of Carathéodory functions applying the result obtained by Nunokawa *et al.*. We also obtain some geometric properties of analytic functions as special cases.

Keywords: Carathéodory functions, Argument estimates.

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1. Introduction

Let $\mathcal{P}$ be the class of all functions which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ with $p(0) = 1$. We say that $p \in \mathcal{P}$ is a Carathéodory function [2, 3] if it satisfies the condition $\text{Re}\, p(z) > 0$ in $\mathbb{U}$. We denote by $\mathcal{A}$ the class of analytic functions in $\mathbb{U}$ with the normalization $f(0) = f'(0) - 1 = 0$.

Nunokawa *et al.* ([5], also see [7], [8]) investigated an argument property of $p \in \mathcal{P}$ at extremal points on the boundary of the circle $|z| = r < 1$, which is the more extended the result earlier studied by Nunokawa [4]

In the present paper, we give some applications of the result obtained by Nunokawa *et al.* [5], which contain argument properties of Carathéodory functions. We also improve the results by Darus and Thomas [1], Nunokawa [4] and Nunokawa and Thomas [6] with some special cases.

2. Main results

To prove the main theorems, we need the following lemma due to Nunokawa *et al.* [7].

Lemma 2.1. *Let $p \in \mathcal{P}$ and $p(z) \neq 0$ in $\mathbb{U}$. If there exist two points $z_1, z_2 \in \mathbb{U}$ such that*

$$-\frac{\pi}{2}\alpha = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi}{2}\beta \tag{2.1}$$

for some $\alpha, \beta (\alpha, \beta > 0)$ and for all $z (|z| < |z_1| = |z_2|)$, then we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\left(\frac{\alpha+\beta}{2}\right)\left(\frac{1+s^2}{2s}\right)m \tag{2.2}$$

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and

$$\frac{z_2 p'(z_2)}{p(z_2)} = i\left(\frac{\alpha+\beta}{2}\right)\left(\frac{1+t^2}{2t}\right)m, \tag{2.3}$$

where

$$p(z_1)^{\frac{2}{\alpha+\beta}} = -is\exp\left(i\frac{\pi}{2}\left(\frac{\beta-\alpha}{\alpha+\beta}\right)\right) \quad (s>0) \tag{2.4}$$

and

$$p(z_2)^{\frac{2}{\alpha+\beta}} = it\exp\left(i\frac{\pi}{2}\left(\frac{\beta-\alpha}{\alpha+\beta}\right)\right) \quad (t>0) \tag{2.5}$$

when

$$m \geq \frac{1-|a|}{1+|a|}, \quad a = i\tan\frac{\pi}{4}\left(\frac{\beta-\alpha}{\alpha+\beta}\right). \tag{2.6}$$

At first, with the help of Lemma 2.1, we obtain the following result.

Theorem 2.2. *Let $k \in \mathbb{N} = \{1, 2, \cdots\}$, $\eta \in [0,\ 1]$ and $\alpha, \beta > 0$ with $(\alpha+\beta)(k-1) < 2$. If a function $p \in \mathcal{P}$ satisfies the condition*

$$-\frac{\pi}{2}c_1(a,k,\alpha,\beta,\eta) \ < \ \arg\left\{p(z)\left(1+\frac{zp'(z)}{p^k(z)}\right)^{\eta}\right\} \ < \ \frac{\pi}{2}c_2(a,k,\alpha,\beta,\eta)$$

where

$$c_1(a,k,\alpha,\beta,\eta) = \alpha + \frac{2\eta}{\pi}\tan^{-1}\left\{\frac{(\alpha+\beta)(1-|a|)\cos\frac{\pi}{2}\alpha(k-1)}{2(1+|a|)d(k,\alpha,\beta)+(\alpha+\beta)(1-|a|)\sin\frac{\pi}{2}\alpha(k-1)}\right\} \tag{2.7}$$

and

$$c_1(a,k,\alpha,\beta,\eta) = \beta + \frac{2\eta}{\pi}\tan^{-1}\left\{\frac{(\alpha+\beta)(1-|a|)\cos\frac{\pi}{2}\beta(k-1)}{2(1+|a|)d(k,\alpha,\beta)+(\alpha+\beta)(1-|a|)\sin\frac{\pi}{2}\beta(k-1)}\right\} \tag{2.8}$$

when a is given by (2.6) *and*

$$d(k,\alpha,\beta) = \left(1+\frac{(\alpha+\beta)(k-1)}{2}\right)^{\frac{2+(\alpha+\beta)(k-1)}{4}}\left(1-\frac{(\alpha+\beta)(k-1)}{2}\right)^{\frac{2-(\alpha+\beta)(k-1)}{4}}, \tag{2.9}$$

then

$$-\frac{\pi}{2}\alpha \ < \ \arg\ p(z) \ < \ \frac{\pi}{2}\beta.$$

Proof. We note that $p(z) \neq 0$ for $z \in \mathbb{U}$. Otherwise, $p(z) = (z-z_1)^l p_1(z)$ $(z \in \mathbb{U})$ for some $l \geq 1$ and $z_1 \in \mathbb{U}$, where p_1 is an analytic function in $\mathbb{U}$ such that $p_1(z_1) \neq 0$. Then

$$\frac{zp'(z)}{p^k(z)} = \frac{1}{(z-z_1)^{l(k-1)+1}}\left(\frac{lz}{p_1^{k-1}(z)} + \frac{(z-z_1)zp_1'(z)}{p_1^k(z)}\right) \quad (z \in \mathbb{U}),$$

and so the above expression has a pole at the point z_1. This contradicts the assumptions of the theorem.

If there exist two points $z_1, z_2 \in \mathbb{U}$ such that the condition (2.1) is satisfied, then (by Lemma 2.1) we obtain (2.2) and (2.3) under the restrictions (2.4) and (2.5), respectively.

At first, we suppose that

$$p(z_2)^{\frac{2}{\alpha+\beta}} = it\exp\left(i\frac{\pi}{2}\left(\frac{\beta-\alpha}{\alpha+\beta}\right)\right) \quad (t>0).$$

Then, using (2.3), we have

$$p(z_2)\left(1+\frac{z_2p'(z_2)}{p^k(z_2)}\right)^{\eta} = t^{\frac{\alpha+\beta}{2}}e^{i\frac{\pi}{2}\beta}\left(1+me^{i\frac{\pi}{2}(1-\beta(k-1))}\frac{(\alpha+\beta)(1+t^2)}{4t^{\frac{(\alpha+\beta)(k-1)}{2}+1}}\right)^{\eta}.$$

Let

$$r(t)=\frac{(\alpha+\beta)(1+t^2)}{4t^{\frac{(\alpha+\beta)(k-1)}{2}+1}}\quad (t>0).$$

Noting that $(\alpha+\beta)(k-1)/2<1$, we can observe that the function r attains its minimum value $r(t_0)=(\alpha+\beta)/2d(k,\alpha,\beta)$, where $d(k,\alpha,\beta)$ is given by (2.9) and $t_0=\sqrt{(2+(\alpha+\beta)(k-1))/(2-(\alpha+\beta)(k-1))}$. Hence we obtain

$$\begin{aligned}\arg\left\{p(z_2)\left(1+\frac{z_2p'(z_2)}{p^k(z_2)}\right)^{\eta}\right\} &\geq \frac{\pi}{2}\beta+\eta\tan^{-1}\left\{\frac{mr(t_0)\cos\frac{\pi}{2}\beta(k-1)}{1+mr(t_0)\sin\frac{\pi}{2}\beta(k-1)}\right\}\\ &\geq \frac{\pi}{2}\beta+\eta\tan^{-1}\left\{\frac{(1-|a|)r(t_0)\cos\frac{\pi}{2}\beta(k-1)}{1+|a|+(1-|a|r(t_0)\sin\frac{\pi}{2}\beta(k-1)}\right\}\\ &= c_2(a,k,\alpha,\beta,\eta),\end{aligned}$$

where $c_2(a,k,\alpha,\beta,\eta)$ is given by (2.8). This contradicts the assumption of the theorem.

Next, we suppose that

$$p(z_1)^{\frac{2}{\alpha+\beta}}=-is\exp\left(i\frac{\pi}{2}\left(\frac{\beta-\alpha}{\alpha+\beta}\right)\right)\quad (s>0),$$

applying the same method as the above and using (2.2) and (2.6), we have

$$\begin{aligned}\arg\left\{p(z_1)\left(1+\frac{z_1p'(z_1)}{p^k(z_1)}\right)^{\eta}\right\} &\leq -\frac{\pi}{2}\alpha-\eta\tan^{-1}\left\{\frac{(1-|a|)r(t_0)\cos\frac{\pi}{2}\alpha(k-1)}{1+|a|+(1-|a|r(t_0)\sin\frac{\pi}{2}\alpha(k-1)}\right\}\\ &= -c_1(a,k,\alpha,\beta,\eta),\end{aligned}$$

where $c_1(a,k,\alpha,\beta,\eta)$ is given by (2.7), which contradiction to the assumption of the theorem. Therefore we complete the proof of Theorem 2.2. □

If we let $\alpha=\beta$ in Theorem 2.2, then we see easily the following corollary.

Corollary 2.3. *Let $k\in\mathbb{N}$, $\eta\in[0,\ 1]$ and $\alpha>0$ with $\alpha(k-1)<1$. If a function $p\in\mathcal{P}$ satisfies the condition*

$$\left|\arg\left\{p(z)\left(1+\frac{zp'(z)}{p^k(z)}\right)^{\eta}\right\}\right| < \frac{\pi}{2}c(k,\alpha,\eta)$$

where

$$c(k,\alpha,\eta)=\alpha+\frac{2\eta}{\pi}\tan^{-1}\left\{\frac{\alpha\cos\frac{\pi}{2}\alpha(k-1)}{(1+\alpha(k-1))^{\frac{1+\alpha(k-1)}{2}}(1-\alpha(k-1))^{\frac{1-\alpha(k-1)}{2}}+\alpha\sin\frac{\pi}{2}\alpha(k-1)}\right\},$$

then

$$|\arg\ p(z)|<\frac{\pi}{2}\alpha.$$

Remark 2.4. For $k=2$ and $\eta=1$, Corollary 2.3 is the result obtained by Nunokawa and Thomas [6].

Taking $p(z)=zf'(z)/f(z)$ in Theorem 2.2, we have

Corollary 2.5. *Let $k\in\mathbb{N}$, $\eta\in[0,\ 1]$ and $\alpha,\beta>0$ with $(\alpha+\beta)(k-1)<2$. If a function $f\in\mathcal{A}$ satisfies the conditon*

$$\begin{aligned}-\frac{\pi}{2}c_1(a,k,\alpha,\beta,\eta) < \arg\left\{\frac{zf'(z)}{f(z)}\left(1+\left(\frac{f(z)}{zf'(z)}\right)^{k-1}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right)^{\eta}\right\}\\ < \frac{\pi}{2}c_2(a,k,\alpha,\beta,\eta),\end{aligned}$$

where $c_1(a,k,\alpha,\beta,\eta)$ and $c_2(a,k,\alpha,\beta,\eta)$ are given by (2.7) *and* (2.8)*, respectively, then*

$$-\frac{\pi}{2}\alpha < \arg\frac{zf'(z)}{f(z)} < \frac{\pi}{2}\beta.$$

Remark 2.6. (i) If we take $k = 2$ and $\alpha = \beta$ in Corollary 2.5, we have the corresponding result obtained by Darus and Thomas [1]. (ii) For the case of $k = 2,\ \eta = 1$ and $\alpha = \beta$, Corollary 2.5 is the result studied by Nunokawa [4].

Letting $p(z) = f(z)/z$ in Theorem 2.2, we have the following result.

Corollary 2.7. *Let $k \in \mathbb{N},\ \eta \in [0,\ 1]$ and $\alpha, \beta > 0$ with $(\alpha + \beta)(k - 1) < 2$. If a function $f \in \mathcal{A}$ satisfies the condition*

$$-\frac{\pi}{2}c_1(a, k, \alpha, \beta, \eta) < \arg\left\{\frac{f(z)}{z}\left(1 + \left(\frac{z}{f(z)}\right)^{k-1}\left(\frac{zf'(z)}{f(z)} - 1\right)\right)^{\eta}\right\} < \frac{\pi}{2}c_2(a, k, \alpha, \beta, \eta),$$

where $c_1(a, k, \alpha, \beta, \eta)$ and $c_1(a, k, \alpha, \beta, \eta)$ are given by (2.7) *and* (2.8)*, respectively, then*

$$-\frac{\pi}{2}\alpha < \arg\frac{f(z)}{z} < \frac{\pi}{2}\beta.$$

Setting $p(z) = f'(z)$ in Theorem 2.2, we have the following corollary.

Corollary 2.8. *Let $k \in \mathbb{N},\ \eta \in [0,\ 1]$ and $\alpha, \beta > 0$ with $(\alpha + \beta)(k - 1) < 2$. If a function $f \in \mathcal{A}$ satisfies the condition*

$$-\frac{\pi}{2}c_1(a, k, \alpha, \beta, \eta) < \arg\left\{f'(z)\left(1 + \frac{zf''(z)}{(f'(z))^k}\right)^{\eta}\right\} < \frac{\pi}{2}c_2(a, k, \alpha, \beta, \eta),$$

where $c_1(a, k, \alpha, \beta, \eta)$ and $c_1(a, k, \alpha, \beta, \eta)$ are given by (2.7) *and* (2.8)*, respectively, then*

$$-\frac{\pi}{2}\alpha < \arg f'(z) < \frac{\pi}{2}\beta.$$

By using the similar method as in the proof of Theorem 2.2, we have the following three theorems below. The proof is much akin to that of Theorem 2.2 and so the details may be omitted.

Theorem 2.9. *Let $\alpha, \beta \in (0, 1]$. If a function $f \in \mathcal{A}$ satisfies the condition*

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \tan^{-1}\frac{(\alpha + \beta)(1 - |a|)}{2(1 + |a|)},$$

where a is given by (2.6)*, then*

$$-\frac{\pi}{2}\alpha < \arg\frac{f(z)}{z} < \frac{\pi}{2}\beta.$$

Remark 2.10. For the case $\alpha = \beta$, Theorem 2.9 is the result obtained by Nunokawa and Thomas [6].

Theorem 2.11. *Let $\alpha, \beta \in (0, 1/2]$. If a function $f \in \mathcal{A}$ satisfies the condition*

$$-\pi\alpha - \tan^{-1}\frac{(\alpha + \beta)(1 - |a|)}{2(1 + |a|)} < \arg\frac{f(z)f'(z)}{z} < \pi\beta + \tan^{-1}\frac{(\alpha + \beta)(1 - |a|)}{2(1 + |a|)},$$

where a is given by (2.6)*, then*

$$-\frac{\pi}{2}\alpha < \arg\frac{f(z)}{z} < \frac{\pi}{2}\beta.$$

Theorem 2.12. *Let $\alpha, \beta \in (0, 1)$. If a function $f \in \mathcal{A}$ satisfies the condition*

$$-\frac{\pi}{2}d_1(a, \alpha, \beta) < \arg\left\{\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} < \frac{\pi}{2}d_2(a, \alpha, \beta),$$

where

$$d_1(a, \alpha, \beta) = \frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha + \beta)(1 - |a|)\cos\frac{\pi}{2}\alpha}{2(1 + |a|)\left(1 + \frac{\alpha+\beta}{2}\right)^{\frac{2+\alpha+\beta}{4}}\left(1 - \frac{\alpha+\beta}{2}\right)^{\frac{2-\alpha-\beta}{4}} + (\alpha + \beta)(1 - |a|)\sin\frac{\pi}{2}\alpha}\right\}$$

and

$$d_2(a,\alpha,\beta) = \frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha+\beta)(1-|a|)\cos\frac{\pi}{2}\beta}{2(1+|a|)\left(1+\frac{\alpha+\beta}{2}\right)^{\frac{2+\alpha+\beta}{4}}\left(1-\frac{\alpha+\beta}{2}\right)^{\frac{2-\alpha-\beta}{4}}+(\alpha+\beta)(1-|a|)\sin\frac{\pi}{2}\beta}\right\},$$

when where a is given by (2.6)*, then*

$$-\frac{\pi}{2}\alpha < \arg\frac{zf'(z)}{f(z)} < \frac{\pi}{2}\beta.$$

Taking $\alpha=\beta$ in Theorem 2.12, we have the following result.

Corollary 2.13. *Let $\alpha\in(0,1)$. If a function $f\in\mathcal{A}$ satisfies the condition*

$$\left|\arg\left\{\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}\right| < \tan^{-1}\left\{\frac{\alpha\cos\frac{\pi}{2}\alpha}{(1+\alpha)^{\frac{1+\alpha}{2}}(1-\alpha)^{\frac{1-\alpha}{2}}+\alpha\sin\frac{\pi}{2}\alpha}\right\},$$

then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\alpha.$$

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