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Using the Well-Poised Fractional Calculus Operator ${}_{g(z)}O_{\beta}^{\alpha}$ to obtain transformations of the Gauss hypergeometric function with higher level arguments

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Abstract

The main objective of this article is to add twelve new transformations formulas for the Gauss hypergeometric function having higher-order rational arguments than those recently obtained by Tremblay (R.Tremblay, New Quadratic Transformations of Hypergeometric Functions and Associated Summation Formulas Obtained with the Well-Poised Fractional Calculus Operator, Montes Taurus J. Pure Appl. Math. 2 (1), p. 36 - 62, 2020) and Tremblay and Gaboury (R.Tremblay and S. Gaboury, Well-posed fractional calculus: obtaining new transformations formulas involving Gauss hypergeometric functions with rational quadratic, cubic and higher degree arguments, Math. Meth. Appl. Sc., (13) (2018), p. 4967-4985). These transformation formulas are obtained with a new systematic method applied to known formulas, most of which come from the Goursat thesis published in 1881 (E. Goursat, Sur l'Équation différentielle linéaire qui admet pour intégrale la série hypergéométrique, *Annales scientifiques de l'É. N. S.*, 2e série tome 10 (1881), 3–142). The method used is based on the use of the fractional operator ${}_{g(z)}O_{\beta}^{\alpha}$ called 'well-poised fractional calculus operator' introduced a long time ago by Tremblay (R. Tremblay, Une contribution à la théorie de la dérivée fractionnaire, Doctoral thesis, Université Laval, Québec, Canada (1974)). After presenting the definition and a short list of these properties of the operator ${}_{g(z)}O_{\beta}^{\alpha}$, we give an detailed example of of calculations to obtain this type of transformation.

Keywords: Fractional derivatives, Well-poised fractional calculus operator, Special functions, Gauss hypergeometric function, Transformation formulas

2010 MSC: 26A33, 33C05, 33C20

1. Introduction

The fractional derivative of arbitrary order α ($\alpha \in \mathbb{C}$), is an extension of the familiar nth derivative $D_{g(z)}^{n} F(z) = d^{n}F(z)/d(g(z))^{n}$ of the function $F(z)$ with respect to $g(z)$ to non-integral values of n and is denoted by $D_{g(z)}^{\alpha} F(z)$. This concept has been introduced in different ways, by generalizing the classical definitions of the nth derivative, where the order n is replaced by an arbitrary complex number α. The best-known representation of the fractional derivative of order α of $f(z)$ is the Riemann-Liouville integral [10]

$$D_{z}^{\alpha} z^{p} f(z) = \frac{1}{\Gamma(-\alpha)} \int_{0}^{z} \xi^{p} f(\xi)(\xi - z)^{-\alpha-1}\, \mathrm{d}\xi. \tag{1.1}$$

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which holds for $\Re(\alpha) < 0$ and $\Re(p) > -1$.

In 1970, Osler [12] introduced a more general definition of the fractional derivative of a function $f(z)$ with respect to another function $g(z)$ based on Cauchy's integral formula

$$D^{\alpha}_{g(z)}\{[g(z)]^p f(z)\} = \frac{\Gamma(1+\alpha)}{2\pi i}\int_{g^{-1}(0)}^{z^+} \frac{f(\xi)[g(\xi)]^p g'(\xi)}{[g(\xi)-g(z)]^{\alpha+1}}\,\mathrm{d}\xi. \tag{1.2}$$

where $z \in \mathcal{R}\backslash\{g^{-1}(0)\}$, $\Re(\alpha) < 0$ and p not an integer. Using the representation (1.2), Osler generalized many classical differential calculus results.

Indeed, most of the properties of the classical calculus and a large number of formulas from the elementary calculus have been extended to fractional calculus. For instance, we have the composition rule and the relation $D^{\alpha}D^{\beta} = D^{\alpha+\beta}$ [13], the Leibniz rule [4, 11, 12, 14, 16, 17, 27], the chain rule [5, 11], the transformation formula [26], Taylor's and Laurent's series [15, 18, 29, 23] and the expansion in terms of a quadratic [29] and rational function [28]. All these fractional formulas illustrate that the fractional calculus exists as a natural extension of the elementary calculus. We can find many surveys and discussions on several of these approaches in literature [1, 2, 20, 21, 29]. Fractional calculus also provides tools that make it easier to study for new information about special functions of mathematical physics.

In 1974, the author [23] introduced the following more general definition of the fractional derivative of a function $f(z)$ with respect to another function $g(z)$ based on Pochhammer's integral formula:

Definition 1.1. Let $f(z)$ be analytic in a simply connected region of $\mathcal{R}$. Let $g(z)$ be regular and univalent on $\mathcal{R}$, and let $g^{-1}(0)$ be an interior point of $\mathcal{R}$. Let $F(a) = f(a)g(a)^p(g(a)-g(z))^{-\alpha-1}$ denote the principal value. Then if α is not a negative integer, p is not an integer, and $z \in \mathcal{R} \setminus \{g^{-1}(0)\}$, we define the fractional derivative of order α of $[g(z)]^p f(z)$ with respect to $g(z)$ to be

$$D^{\alpha}_{g(z)}\{[g(z)]^p f(z)\} = \frac{\mathrm{e}^{-\mathrm{i}\pi p}\Gamma(1+\alpha)}{4\pi\sin(\pi p)}\int_{C(z+,g^{-1}(0)+,z-,g^{-1}(0)-;F(a),F(a))} \frac{f(\xi)[g(\xi)]^p g'(\xi)}{[g(\xi)-g(z)]^{\alpha+1}}\,\mathrm{d}\xi. \tag{1.3}$$

For non-integers α and p, the functions $[g(\xi)]^p$ and $[g(\xi)-g(z)]^{-\alpha-1}$ in the integrand have two branch lines which begin, respectively, at $\xi = z$ and $\xi = g^{-1}(0)$, and both branches pass through the point $\xi = a$ without crossing the Pochhammer contour $P(a) = \{C_1 \cup C_2 \cup C_3 \cup C_4\}$ at any other point as shown in Figure 1 (contour starting and ending at a and made of four loops C_1, C_2, C_3, C_4 running around the two branch points z and $g^{-1}(0)$). Here $F(a)$ denotes the principal value of the integrand in (1.3) at the beginning and ending point of the Pochhammer contour $P(a)$, which is closed on the Riemann surface of the multiple-valued function $F(\xi)$.

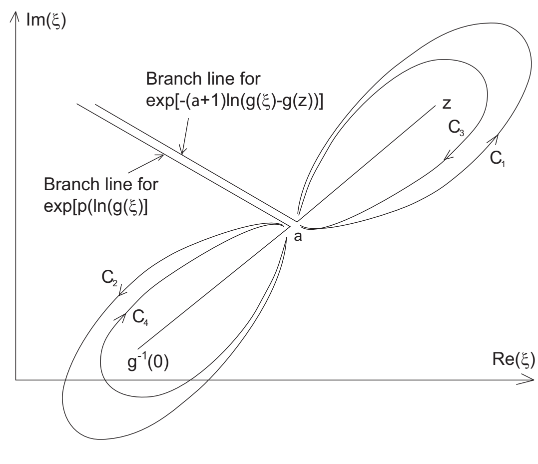

Figure 1. Pochhammer's contour.

Remark 1.2. It is interesting to note here that we could also allow $f(z)$ to have an essential singularity at $\xi = g^{-1}(0)$, and (1.3) would still be valid. For a complete study on the analyticity of the fractional derivative operator $D^{\alpha}_{g(z)}\{[g(z)]^p f(z)\}$ based on Pochhammer's contour integral, the reader is referred to [8].

Remark 1.3. If the Pochhammer contour never crosses the singularities at $\xi = g^{-1}(0)$ and $\xi = z$ in (1.3), then we know that the integral is analytic for all p and for all α and for $z \in \mathcal{R} \setminus \{g^{-1}(0)\}$. Indeed, the only possible singularities of $D^{\alpha}_{g(z)}\{[g(z)]^p f(z)\}$ are $\alpha = -1, -2, \ldots$, and $p = 0, \pm 1, \pm 2, \ldots$, which can be directly identified from the coefficient $\dfrac{e^{-i\pi p}\Gamma(1+\alpha)}{4\pi \sin(\pi p)}$ of the integral (1.3). However, integrating by parts N times (where N is a positive integer) the integral in (1.3) into two different ways, we can show that $\alpha = -1, -2, \ldots$, and $p = 0, 1, 2, \ldots$ are removable singularities [9].

Remark 1.4. The value of the contour integral depends on the points $\xi = g^{-1}(0)$ and z, but does not depend on the point a, which can be arbitrarily chosen between them (see Figure 1).

The main objective of this article is to illustrate a new method making it possible to systematically obtain new transformations formulas, from the Gauss hypergeometric transformations obtained by Goursat in 1881. Using the usual methods, discover new transformation formulas of the Gauss hypergeometric function implicating order arguments greater than 2 is a very difficult task. There are very few in the literature. We limit the list to twelve new transformation formulas obtained from the transformations of the Gauss hypergeometric function [7, Eq.135 and Eq.137, p. 142] having complex functions as argument $\dfrac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}$ and $-\dfrac{108z(1-z)^4}{(1+z)^2(1-34z+z^2)^2}$. However, the method can be applied to most of the transformations contained in Goursat's thesis. To do this, in Section 2, we introduce Tremblay's fractional calculation operator ${}_{g(z)}O^{\alpha}_{\beta}$ and some of its properties. Section 3 is devoted to the detailed demonstration of three transformations which demonstrate the efficiency of this operator for calculations involving fractional derivatives. The other nine listed in section 4 are obtained in a similar way.

The number and complexity of the twelve transformation formulas presumably new unequivocally demonstrate the effectiveness of the operator ${}_{g(z)}O^{\alpha}_{\beta}$ in the execution of calculations allowing the discovery of new relations involving special functions.

2. The well poised fractional calculus operator ${}_{g(z)}O^{\alpha}_{\beta}$

The operator ${}_{g(z)}O^{\alpha}_{\beta}$ has been introduced by Tremblay [23] and is defined in terms of fractional calculus operator $D^{\alpha}_{g(z)}$ as

$$ {}_{g(z)}O^{\alpha}_{\beta}\, f(z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, g(z)^{1-\beta}\, D^{\alpha-\beta}_{g(z)}\, g(z)^{\alpha-1}\, f(z). \tag{2.1} $$

It seems to be simply a rewrite of the fractional derivative. This operator is motivated because of its analytical properties that are simpler than those of the fractional derivative operator $D^{\alpha}_{g(z)}$. However, it has an important list of easy-to-demonstrate properties that simplify calculations while having simpler analytical properties than the fractional derivative itself. It is more appropriate to explore special functions as it has already been demonstrated in [6, 5, 25, 24, 23]. These are the reasons why we called it 'well poised fractional calculus operator'.

The well-posed fractional calculus operator ${}_{g(z)}O^{\alpha}_{\beta}$ possesses the following integral representation with regard of Pochhammer's integral contour.

Definition 2.1. Let $f(z)$ be analytic in a simply connected region of $\mathcal{R}$. Let $g(z)$ be regular and univalent on $\mathcal{R}$, and let $g^{-1}(0)$ be an interior point of $\mathcal{R}$. Let $F(a) = f(a)g(a)^{\alpha-1}(g(a)-g(z))^{\beta-\alpha-1}$ denote the principal value. Then if $\alpha \neq 1, 2, \ldots, \beta - \alpha \neq 1, 2, \ldots, \beta \neq 0, -1, -2 \ldots$ and $z \in \mathcal{R} \setminus \{g^{-1}(0)\}$, we define the fractional operator ${}_{g(z)}O^{\alpha}_{\beta}$ with parameters α and β on $f(z)$ with respect to $g(z)$ to be

$$ {}_{g(z)}O^{\alpha}_{\beta} f(z) = -\frac{e^{-i\pi\beta}\Gamma(\beta)}{4\pi\,\sin(\pi\alpha)\sin(\pi(\beta-\alpha))} \cdot \int_{C(z^+,g^{-1}(0)^+,z^-,g^{-1}(0)^-;F(a),F(a))} \frac{f(\xi)\,[g(\xi)]^{\alpha-1}\, g'(\xi)}{[g(\xi)-g(z)]^{\alpha-\beta+1}}\, d\xi. \tag{2.2} $$

Remark 2.2. The restrictions on the parameters in Definition 2.2 come essentially from the product of the three gamma functions $\Gamma(\beta)$, $\Gamma(1-\alpha)$, and $\Gamma(1+\alpha-\beta)$ appearing in (2.2). However, as mentioned in Definition 1.3 of the fractional derivative, integrating by parts N times the integral in (2.2) into two different ways, we can show that $\alpha = 1, 2, \ldots$, and $\beta - \alpha = 1, 2, \ldots$ are removable singularities [9]. Moreover, if we consider the following expression $\frac{{}_{g(z)}O_\beta^\alpha}{\Gamma(\beta)}$ than there are no longer restrictions on the parameter β.

We recall some of the important properties of the fractional calculus operator ${}_{g(z)}O_\beta^\alpha$. We choose to simply list them since the proofs are readily obtainable:
- Effect on a hypergeometric function

$$ {}_{g(z)}O_\beta^\alpha \; {}_pF_q\left[\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix}\middle| g(z)\right] = {}_{p+1}F_{q+1}\left[\begin{matrix} \alpha, a_1, a_2, \cdots, a_p \\ \beta, b_1, b_2, \cdots, b_q \end{matrix}\middle| g(z)\right] \tag{2.3} $$

where

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix}\middle| z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}. $$

- Linearity

$$ {}_{g(z)}O_\beta^\alpha \{\lambda_1 f(z) + \lambda_2 h(z)\} = \lambda_1 \, {}_{g(z)}O_\beta^\alpha f(z) + \lambda_2 \, {}_{g(z)}O_\beta^\alpha h(z), \tag{2.4} $$

- Useful cases

$$ {}_{g(z)}O_\beta^\alpha \, [g(z)]^\lambda f(z) = \frac{\Gamma(\beta)\Gamma(\alpha+\lambda)}{\Gamma(\alpha)\Gamma(\beta+\lambda)} \, [g(z)]^\lambda \, {}_{g(z)}O_{\beta+\lambda}^{\alpha+\lambda} \, f(z), \tag{2.5} $$

$$ {}_{g(z)}O_\beta^\alpha \, [g(w) - g(z)]^\theta f(z)\Big|_{w=z} = \frac{\Gamma(\beta)\Gamma(\beta-\alpha+\theta)}{\Gamma(\beta-\alpha)\Gamma(\beta+\theta)} \, [g(z)]^\theta \, {}_{g(z)}O_{\beta+\theta}^{\alpha} \, f(z), \tag{2.6} $$

$$ {}_{g(z)}O_\beta^\alpha \, [g(z)]^\lambda [g(w) - g(z)]^\theta f(z)\Big|_{w=z} = \frac{\Gamma(\beta)\Gamma(\alpha+\lambda)\Gamma(\beta-\alpha+\theta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)\Gamma(\beta+\lambda+\theta)} \, [g(z)]^{\lambda+\theta} \, {}_{g(z)}O_{\beta+\lambda+\theta}^{\alpha+\lambda} \, f(z). \tag{2.7} $$

A more complete list of the properties of this operator can be found in ([23, 24]).

3. Illustrations of the method used to obtain the transformations of the Gauss hypergeometric function

The method is based on the following theorem allowing to go from operator ${}_{g(z)}O_\beta^\alpha$ to operator ${}_{h(z)}O_\beta^\alpha$.

Theorem 3.1. *Let $f(g^{-1}(z))$ and $f(h^{-1}(z))$ be defined and analytic on the simply connected region $\mathcal{R}$. Let $f(z)$ be a function that satisfies the conditions for the existence of ${}_{g(z)}O_\beta^\alpha f(z)$ and ${}_{h(z)}O_\beta^\alpha f(z)$ listed in Definition 2.1 and using a Pochhammer contour $P(a) = \{C_1 \cup C_2 \cup C_3 \cup C_4\}$ laid out around the points $\xi = g^{-1}(0)$, $\xi = h^{-1}(0)$ and z (see Figure 2). For $z \in \mathcal{R} \setminus \{g^{-1}(0), h^{-1}(0)\}$ we have*

$$ {}_{g(z)}O_\beta^\alpha f(z) = \left(\frac{g(z)}{h(z)}\right)^{1-\beta} \left\{ {}_{h(z)}O_\beta^\alpha \left(\frac{g(z)}{h(z)}\right)^{\alpha-1} \frac{g'(z)}{h'(z)} \left(\frac{g(z)-g(w)}{h(z)-h(w)}\right)^{\beta-\alpha-1} f(z) \right\}\Bigg|_{w=z}. \tag{3.1} $$

Proof. Using 2.1, the formula 3.1 becomes

$$ \frac{\Gamma(\beta)}{\Gamma(\alpha)} \, g(z)^{1-\beta} D_{g(z)}^{\alpha-\beta} g(z)^{\alpha-1} f(z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \, h(z)^{1-\beta} \left(\frac{g(z)}{h(z)}\right)^{1-\beta} D_{h(z)}^{\alpha-\beta} h(z)^{\alpha-1} \left(\frac{g(z)}{h(z)}\right)^{\alpha-1} \frac{g'(z)}{h'(z)} \left(\frac{g(z)-g(w)}{h(z)-h(w)}\right)^{\beta-\alpha-1} f(z) \Bigg\}\Bigg|_{w=z}. \tag{3.2} $$

Replacing $\beta - \alpha$ by α and $f(z)(g(z))^{\alpha-1}$ by $f(z)$, we get

$$ D_{g(z)}^\alpha f(z) = D_{h(z)}^\alpha \left\{ \frac{g'(z)}{h'(z)} \left(\frac{g(z)-g(w)}{h(z)-h(w)}\right)^{\alpha-1} f(z) \right\}\Bigg|_{w=z}. \tag{3.3} $$

which is equivalent to the Osler's theorem concerning the generalized chain rule [13, Eq.(3.1), p. 290]. □

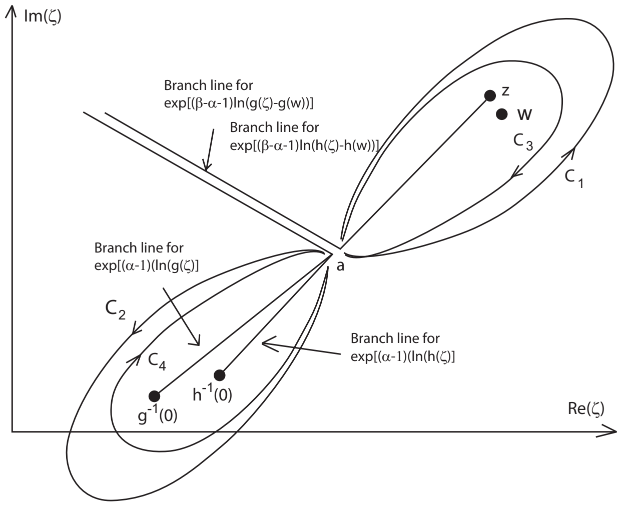

Figure 2. *The contour used in integral* (3.1).

It would be possible to use (3.1) for any values of α and β (except $\beta = 0, -1, -2, \ldots$). The general case will also be considered in a future paper. For the moment, we limit our study only to the cases $\beta - \alpha = n$ with $n = 1, 2$ because the number of terms increases considerably when n is very large.

Corollary 3.2. *With the same hypothesis in Theorem 2, we have:*

$$ {}_{g(z)}O^{\alpha}_{\alpha+1} f(z) = \left(\frac{g(z)}{h(z)}\right)^{-\alpha} \left\{ {}_{h(z)}O^{\alpha}_{\alpha+1} \left(\frac{g(z)}{h(z)}\right)^{\alpha-1} \frac{g'(z)}{h'(z)} f(z) \right\} \tag{3.4} $$

and

$$ {}_{g(z)}O^{\alpha}_{\alpha+2} f(z) = \left(\frac{g(z)}{h(z)}\right)^{-\alpha-1} \left\{ {}_{h(z)}O^{\alpha}_{\alpha+2} \left(\frac{g(z)}{h(z)}\right)^{\alpha-1} \frac{g'(z)}{h'(z)} \left(\frac{g(z)-g(w)}{h(z)-h(w)}\right) f(z) \right\}\Bigg|_{w=z}. \tag{3.5} $$

Proof. With $\beta = \alpha + 1$ and $\beta = \alpha + 2$ in relation (3.1), we obtain respectively (3.4) and (3.5). ☐

Remark 3.3. It is important to note that these formulas (3.4) and (3.5) can be applied also recursively to all cases.

We find an example in [24, Cases 5.10 and 5.14]. Starting from the quadratic transformation [19, Th. 25, p. 67]

$$ {}_2F_1\left[\begin{matrix} a, b \\ a+b+1/2 \end{matrix}\middle| 4z(1-z)\right] = {}_2F_1\left[\begin{matrix} 2a, 2b \\ a+b+1/2 \end{matrix}\middle| z\right], \tag{3.6} $$

the repeated application of (3.5) with $g(z) = z(1-z)$ and $h(z) = z/(1-z)$) makes it possible to successively obtain the transformation formulas transformation

$$ \begin{aligned} (1-z)^{2a} {}_3F_2\left[\begin{matrix} a+1/2, a-1, b+1/2 \\ a+1, a+b+1/2 \end{matrix}\middle| 4z(1-z)\right] &= (1-z)^2 \, {}_3F_2\left[\begin{matrix} 2a, a-1, 1/2+a-b \\ a+1, a+b+1/2 \end{matrix}\middle| -\frac{z}{1-z}\right] \\ &\quad - \frac{(a-1)}{(a+1)} z^2 \, {}_3F_2\left[\begin{matrix} 2a, a, 1/2+a-b \\ a+2, a+b+1/2 \end{matrix}\middle| -\frac{z}{1-z}\right]. \end{aligned} \tag{3.7} $$

and

$$ (1-z)^{c+2} {}_2F_1\left[\begin{matrix} c/2-a/2, c/2+a/2-1/2 \\ c+3 \end{matrix}\middle| 4z(1-z)\right] = (1-z) \, {}_2F_1\left[\begin{matrix} a, 1-a \\ c+3 \end{matrix}\middle| z\right] \tag{3.8} $$

$$- 2z(1-z)\frac{c}{(c+3)}\,{}_3F_2\left[\begin{matrix}a, 1-a, c+1\\ c, c+4\end{matrix}\middle| z\right]$$
$$- z(3-2z)\frac{(c+1)}{(c+3)}\,{}_3F_2\left[\begin{matrix}a, 1-a, c+2\\ c+1, c+4\end{matrix}\middle| z\right]$$
$$+ 2z^2\frac{(c+1)(c+2)}{(c+3)(c+4)}\,{}_3F_2\left[\begin{matrix}a, 1-a, c+3\\ c+1, c+5\end{matrix}\middle| z\right]$$
$$+ 2z^2(3-2z)\frac{c(c+2)}{(c+3)(c+4)}\,{}_4F_3\left[\begin{matrix}a, 1-a, c+1, c+3\\ c, c+2, c+5\end{matrix}\middle| z\right]$$
$$- 4z^3\frac{c(c+2)}{(c+4)(c+5)}\,{}_4F_3\left[\begin{matrix}a, 1-a, c+1, c+4\\ c, c+2, c+6\end{matrix}\middle| z\right].$$

Theorem 3.4. *If* $\left|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right| < 1$ *and* $a \neq 1/2$ *and* $-5/12$ *then*

$$\frac{(1-z)^{a-1/2}}{(1+30z-96z^2+64z^3)^{2a-1}(2a-1)}\,{}_2F_1\left[\begin{matrix}a, a-1/2\\ 2a+5/6\end{matrix}\middle| \frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] \tag{3.9}$$
$$= \sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(2a-1+2n)}z^n\,{}_3F_2\left[\begin{matrix}a+n-1/2, 6a, 2/3-2a\\ a+n+1/2, 2a+5/6\end{matrix}\middle| z\right]$$
$$- 32z\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(2a+1+2n)}z^n\,{}_3F_2\left[\begin{matrix}a+n+1/2, 6a, 2/3-2a\\ a+n+3/2, 2a+5/6\end{matrix}\middle| z\right]$$
$$+ 288z^2\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(2a+3+2n)}z^n\,{}_3F_2\left[\begin{matrix}a+n+3/2, 6a, 2/3-2a\\ a+n+5/2, 2a+5/6\end{matrix}\middle| z\right]$$
$$- 512z^3\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(2a+5+2n)}z^n\,{}_3F_2\left[\begin{matrix}a+n+5/2, 6a, 2/3-2a\\ a+n+7/2, 2a+5/6\end{matrix}\middle| z\right]$$
$$+ 256z^4\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(2a+7+2n)}z^n\,{}_3F_2\left[\begin{matrix}a+n+7/2, 6a, 2/3-2a\\ a+n+9/2, 2a+5/6\end{matrix}\middle| z\right].$$

Proof. We start with the Goursat's transformation [7, Eq.135, p. 142]

$${}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle| z\right] = (1+30z-96z^2+64z^3)^{-2a}\,{}_2F_1\left[\begin{matrix}a, a+1/2\\ 2a+5/6\end{matrix}\middle| \frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right]. \tag{3.10}$$

Putting $g(z) = \frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}$ and $h(z) = z$ in (3.4), we have $g'(z) = \frac{(1-16z+16z^2)^2}{(1+30z-96z^2+64z^3)^3}$ and

$${}_{\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}}O^{\alpha}_{\alpha+1} f(z) = \frac{(1-z)^{-\alpha}}{(1+30z-96z^2+64z^3)^{-2\alpha}}\,{}_zO^{\alpha}_{\alpha+1}\left\{\frac{(1-z)^{\alpha-1}(1-16z+16z^2)^2}{(1+30z-96z^2+64z^3)^{2\alpha-1}}f(z)\right\} \tag{3.11}$$

Using (3.10), we have

$${}_{\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}}O^{\alpha}_{\alpha+1}\,{}_2F_1\left[\begin{matrix}a, a+1/2\\ 2a+5/6\end{matrix}\middle| \frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] = \frac{(1-z)^{-\alpha}}{(1+30z-96z^2+64z^3)^{-2\alpha}} \tag{3.12}$$
$${}_zO^{\alpha}_{\alpha+1}\left\{\frac{(1-z)^{\alpha-1}(1-16z+16z^2)^2}{(1+30z-96z^2+64z^3)^{2\alpha+2a-1}}\,{}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle| z\right]\right\}.$$

Choosing $\alpha = a - \frac{1}{2}$, developing in series $(1-z)^{a-\frac{3}{2}}$ and using the linear property (2.4), we obtain

$$\frac{(1-z)^{\alpha}}{(1+30z-96z^2+64z^3)^{2\alpha}}\,{}_{\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}}O^{a-\frac{1}{2}}_{a+\frac{1}{2}}\,{}_2F_1\left[\begin{matrix}a, a+1/2\\ 2a+5/6\end{matrix}\middle| \frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] \tag{3.13}$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{3}{2}-a)_n}{n!} \, {}_zO_{a+\frac{1}{2}}^{a-\frac{1}{2}}\left\{z^n(1-32z+288z^2-512z^3+256z^4)\, {}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle| z\right]\right\}$$
$$= \sum_{n=0}^{\infty} \frac{(\frac{3}{2}-a)_n}{n!} \, {}_zO_{a+\frac{1}{2}}^{a-\frac{1}{2}}\left\{z^n \, {}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle| z\right]\right\} - 32 \sum_{n=0}^{\infty} \frac{(\frac{3}{2}-a)_n}{n!} \, {}_zO_{a+\frac{1}{2}}^{a-\frac{1}{2}}\left\{z^{n+1} {}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle| z\right]\right\}$$
$$+ 288 \sum_{n=0}^{\infty} \frac{(\frac{3}{2}-a)_n}{n!} {}_zO_{a+\frac{1}{2}}^{a-\frac{1}{2}}\left\{z^{n+2} {}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle| z\right]\right\} - 512 \sum_{n=0}^{\infty} \frac{(\frac{3}{2}-a)_n}{n!} \, {}_zO_{a+\frac{1}{2}}^{a-\frac{1}{2}}\left\{z^{n+3} {}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle| z\right]\right\}$$
$$+ 256 \sum_{n=0}^{\infty} \frac{(\frac{3}{2}-a)_n}{n!} \, {}_zO_{a+\frac{1}{2}}^{a-\frac{1}{2}}\left\{z^{n+4} {}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle| z\right]\right\}.$$

Now, using the property (2.5), we have

$$\frac{(1-z)^{\alpha}}{(1+30z-96z^2+64z^3)^{2\alpha}} \, {}_2F_1\left[\begin{matrix}a, a-1/2\\ 2a+5/6\end{matrix}\middle| \frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] \tag{3.14}$$
$$= \sum_{n=0}^{\infty} \frac{(\frac{3}{2}-a)_n (a-\frac{1}{2})_n}{(a+\frac{1}{2})_n n!} z^n \, {}_zO_{a+\frac{1}{2}+n}^{a-\frac{1}{2}+n} {}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle| z\right]$$
$$- 32 \sum_{n=0}^{\infty} \frac{(\frac{3}{2}-a)_n (a-\frac{1}{2})_{n+1}}{(a+\frac{1}{2})_{n+1} n!} z^{n+1} \, {}_zO_{a+\frac{3}{2}+n}^{a+\frac{1}{2}+n} {}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle| z\right]$$
$$+ 288 \sum_{n=0}^{\infty} \frac{(\frac{3}{2}-a)_n (a-\frac{1}{2})_{n+2}}{(a+\frac{1}{2})_{n+2} n!} z^{n+2} \, {}_zO_{a+\frac{5}{2}+n}^{a+\frac{3}{2}+n} {}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle| z\right]$$
$$- 512 \sum_{n=0}^{\infty} \frac{(\frac{3}{2}-a)_n (a-\frac{1}{2})_{n+3}}{(a+\frac{1}{2})_{n+3} n!} z^{n+3} \, {}_zO_{a+\frac{7}{2}+n}^{a+\frac{5}{2}+n} {}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle| z\right]$$
$$+ 256 \sum_{n=0}^{\infty} \frac{(\frac{3}{2}-a)_n (a-\frac{1}{2})_{n+4}}{(a+\frac{1}{2})_{n+4} n!} z^{n+4} \, {}_zO_{a+\frac{9}{2}+n}^{a+\frac{7}{2}+n} {}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle| z\right].$$

After simplifications, we obtain (3.9). □

Corollary 3.5. *If* $\left|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right| < 1$ *, we have*

$$\frac{(1-z)}{2(1+30z-96z^2+64z^3)^2} \, {}_2F_1\left[\begin{matrix}3/2, 1\\ 23/6\end{matrix}\middle| \frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] \tag{3.15}$$
$$= \frac{1}{2} \, {}_3F_2\left[\begin{matrix}1, 9, -7/3\\ 2, 23/6\end{matrix}\middle| z\right] - 8z \, {}_3F_2\left[\begin{matrix}2, 9, -7/3\\ 3, 23/6\end{matrix}\middle| z\right] + 48z^2 \, {}_3F_2\left[\begin{matrix}3, 9, -7/3\\ 4, 23/6\end{matrix}\middle| z\right]$$
$$- 64z^3 \, {}_3F_2\left[\begin{matrix}4, 9, -7/3\\ 5, 23/6\end{matrix}\middle| z\right] + \frac{128}{5} z^4 \, {}_3F_2\left[\begin{matrix}5, 9, -7/3\\ 6, 23/6\end{matrix}\middle| z\right],$$

$$\frac{(1-z)^{-1/6}}{(1+30z-96z^2+64z^3)^{-1/3}} \, {}_2F_1\left[\begin{matrix}1/3, -1/6\\ 3/2\end{matrix}\middle| \frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] \tag{3.16}$$
$$= {}_2F_1\left[\begin{matrix}7/6, -1/6\\ 5/6\end{matrix}\middle| z\right] + \frac{32}{5} z \, {}_2F_1\left[\begin{matrix}7/6, 5/6\\ 11/6\end{matrix}\middle| z\right] - \frac{288}{11} z^2 \, {}_2F_1\left[\begin{matrix}7/6, 11/6\\ 17/6\end{matrix}\middle| z\right]$$
$$+ \frac{512}{17} z^3 \, {}_2F_1\left[\begin{matrix}7/6, 17/6\\ 23/6\end{matrix}\middle| z\right] - \frac{256}{23} z^4 \, {}_2F_1\left[\begin{matrix}7/6, 23/6\\ 29/6\end{matrix}\middle| z\right]$$

and

$$\frac{(1+30z-96z^2+64z^3)}{\sqrt{1-z}} = \frac{30z+1}{\sqrt{1-z}} + 288z^2(\frac{\sqrt{1-z}}{z(z-1)} + \frac{\arcsin(\sqrt{z})}{z^{3/2}}) + 512z^3(\frac{1}{2}\frac{(z-3)\sqrt{1-z}}{z^2(z-1)} - \frac{3}{2}\frac{\arcsin(\sqrt{z})}{z^{5/2}}) - 256z^4(\frac{1}{56}\frac{(14z^2+35z-105)\sqrt{1-z}}{(z-1)z^3} - \frac{15}{8}\frac{\arcsin(\sqrt{z})}{z^{7/2}}). \tag{3.17}$$

Proof. Putting respectively $a = \frac{3}{2}$, $a = \frac{1}{3}$ and $a = 0$ in (3.9), we obtain (3.15), (3.16) and (3.17). □

Theorem 3.6. *If* $\left|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right| < 1$ *and* $a \neq 1/2, 3/2$ *and* $-5/12$ *then*

$$\frac{(1-z)^{a-1/2}}{(1+30z-96z^2+64z^3)^{2a-3}(2a-1)(2a-3)}\,{}_2F_1\left[\begin{matrix}a, a-3/2\\ 2a+5/6\end{matrix}\middle|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] = \sum_{i=0}^{9} Q_i z^i \sum_{n=0}^{\infty} \frac{(5/2-a)_n}{n!(2a-3+2i+2n)(2a-1+2i+2n)} z^n\,{}_3F_2\left[\begin{matrix}a-3/2+i+n, 6a, 2/3-2a\\ a+1/2+i+n, 2a+5/6\end{matrix}\middle| z\right] \tag{3.18}$$

where coefficient $Q_i(a,z)$ are given in Table 1.

i	$Q_i(a,z)$
0	$(1-z)$
1	$(-4096z^5+12288z^4-13056z^3+5632z^2-736z-33)$
2	$(135168z^5-409600z^4+443136z^3-198912z^2+29920z+320)$
3	$(-1310720z^5+4067328z^4-4587520z^3+2245376z^2-413952z-800)$
4	$(3276800z^5-11141120z^4+14512128z^3-9093120z^2+2445056z+768)$
5	$(-3145728z^5+12713984z^4-21168128z^3+18837504z^2-7237632z-256)$
6	$-65536z(1-z)(16z^3-80z^2+165z-180)$
7	$-65536z(1-z)(16z^2-80z+165)$
8	$1048576z(1-z)(5-z)$
9	$-1048576z(1-z)$

Table 1. Coefficients $Q_i(a,z)$ of (3.18)

Proof. With the same choices of $g(z)$ et $h(z)$ then Theorem 2 in (3.5), we have

$$\frac{g(z)-g(w)}{z-w} = \frac{(z+w-1)H(w,z)}{(1+30z-96z^2+64z^3)^2(1+30w-96w^2+64w^3)^2} \tag{3.19}$$

where

$$\begin{aligned} H(w,z) &= 4096w^4z^2+4096w^2z^4-4096w^4z-8192w^3z^2-8192w^2z^3-8192w^2z^3-4096wz^4 \\ &+8192w^3z+8960w^2z^2+8192wz^3-4864w^2z-4864wz^2+768wz-1 \\ &= 4096w(-1+w)+4096w(-1+w)(w-3)z+256w(-1+w)(16w^2-48w+51)z^2 \\ &+256w(-1+w)(16w^3-48w^2+51w-22)z^3+(-4096w^5+12288w^4-13056w^3+5632w^2-768w-1) \end{aligned} \tag{3.20}$$

and (3.5) becomes

$$\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2} O^{\alpha}_{\alpha+2} f(z) = \frac{(1-z)^{-\alpha-1}}{(1+30z-96z^2+64z^3)^{-2\alpha-2}}\, {}_zO^{\alpha}_{\alpha+2}\left\{\frac{(1-z)^{\alpha-1}(1-16z+16z^2)^2}{(1+30z-96z^2+64z^3)^{2\alpha+1}}\right. \tag{3.21}$$

$$\left.\left(\frac{H(w,z)}{(1+30z-96z^2+64z^3)^2(1+30w-96w^2+64w^3)^2}\right)f(z)\right\}\Bigg|_{w=z}.$$

Using (3.10), we have

$$_{\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}}O^{\alpha}_{\alpha+2}\ {}_2F_1\left[\begin{matrix}a, a+1/2\\ 2a+5/6\end{matrix}\middle|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] = \frac{(1-z)^{-\alpha-1}}{(1+30z-96z^2+64z^3)^{-2\alpha-2}} \tag{3.22}$$

$$_zO^{\alpha}_{\alpha+2}\left\{\frac{(1-z)^{\alpha-1}(1-16z+16z^2)^2}{(1+30z-96z^2+64z^3)^{2\alpha-2a+1}}\left(\frac{H(w,z)}{(1+30z-96z^2+64z^3)^2(1+30w-96w^2+64w^3)^2}\right){}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle|z\right]\right\}\Bigg|_{w=z}$$

$$= \frac{(1-z)^{-\alpha-1}}{(1+30z-96z^2+64z^3)^{-2\alpha}}\,{}_zO^{\alpha}_{\alpha+2}\left\{\frac{(1-z)^{\alpha-1}(1-16z+16z^2)^2H(w,z)}{(1+30z-96z^2+64z^3)^{2\alpha-2a+3}}\,{}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle|z\right]\right\}\Bigg|_{w=z}$$

or

$$\frac{(1-z)^{\alpha+1}}{(1+30z-96z^2+64z^3)^{2\alpha}}\,{}_{\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}}O^{\alpha}_{\alpha+2}\,{}_2F_1\left[\begin{matrix}a, a+1/2\\ 2a+5/6\end{matrix}\middle|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] \tag{3.23}$$

$$= {}_zO^{\alpha}_{\alpha+2}\left\{\frac{(1-z)^{\alpha-1}(1-16z+16z^2)^2H(w,z)}{(1+30z-96z^2+64z^3)^{2\alpha-2a+3}}\,{}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle|z\right]\right\}\Bigg|_{w=z}.$$

But

$$\begin{aligned}(1-16z+16z^2)^2H(w,z) &= (1-w)+(-4096w^5+12288w^4-13056w^3+5632w^2-736w-33)z \\ &\quad +(135168w^5-409600w^4+442136w^3-198912w^2+29920w+320)z^2 \\ &\quad +(-1310720w^5+4067328w^4-4587520w^3+224537w^2-413952w-800)z^3 \\ &\quad +(3276800w^5-11141120w^4+14512128w^3-9093120w^2+2445056w+768)z^4 \\ &\quad +(-3145728w^5+1271398w^4-21168128w^3+1883750w^2-7237632w-256)z^5 \\ &\quad -65536w(1-w)(16w^3-80w^2+165w-189)z^6 \\ &\quad -65536w(1-w)(16w^2-80w+165)z^7+1048576w(1-w)(5-w)z^8-1048576w(1-w)z^9 \\ &= \sum_{i=0}^{9} Q_i(w)z^i.\end{aligned} \tag{3.24}$$

Putting $\alpha = a-\frac{3}{2}$ in (3.23), we get

$$\begin{aligned}&\frac{(1-z)^{a-\frac{1}{2}}}{(1+30z-96z^2+64z^3)^{2a-3}}\,{}_2F_1\left[\begin{matrix}a, a-3/2\\ 2a+5/6\end{matrix}\middle|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] \\ &= {}_zO^{a-\frac{3}{2}}_{a+\frac{1}{2}}\left\{(1-z)^{a-\frac{5}{2}}(1-16z+16z^2)^2H(w,z)\,{}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle|z\right]\right\}\Bigg|_{w=z} \\ &= \sum_{i=0}^{9} Q_i(z)\,{}_zO^{a-\frac{3}{2}}_{a+\frac{1}{2}}\left\{(1-z)^{a-\frac{5}{2}}z^i\,{}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle|z\right]\right\} \\ &= \sum_{i=0}^{9} Q_i(z)\sum_{n=0}^{\infty}\frac{(\frac{5}{2}-a)_n}{n!}\,{}_zO^{a-\frac{3}{2}}_{a+\frac{1}{2}}\left\{z^{i+n}\,{}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle|z\right]\right\} \\ &= \sum_{i=0}^{9} Q_i(z)\sum_{n=0}^{\infty}\frac{(\frac{5}{2}-a)_n(a-\frac{3}{2})_{i+n}}{(a+\frac{1}{2})_{i+n}n!}z^{i+n}\,{}_zO^{a-\frac{3}{2}+i+n}_{a+\frac{1}{2}+i+n}\,{}_2F_1\left[\begin{matrix}6a, 2/3-2a\\ 2a+5/6\end{matrix}\middle|z\right] \\ &= \sum_{i=0}^{9} Q_i(z)\frac{(a-\frac{3}{2})_i}{(a+\frac{1}{2})_i}z^i\sum_{n=0}^{\infty}\frac{(\frac{5}{2}-a)_n(a-\frac{3}{2}+i)_n}{(a+\frac{1}{2}+i)_nn!}z^n\,{}_3F_2\left[\begin{matrix}a-\frac{3}{2}+i+n, 6a, 2/3-2a\\ a+\frac{1}{2}+i+n, 2a+5/6\end{matrix}\middle|z\right].\end{aligned} \tag{3.25}$$

After simplifications, we obtain (3.18). ☐

Corollary 3.7. *If* $\left|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right| < 1$ *, we have*

$$\frac{(1-z)^2}{8(1+30z-96z^2+64z^3)^2}\,{}_2F_1\left[\begin{matrix}5/2,1\\35/6\end{matrix}\middle|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] = \frac{1}{8}(1-z)\,{}_3F_2\left[\begin{matrix}1,15,-13/3\\3,35/6\end{matrix}\middle|z\right] \tag{3.26}$$
$$+(-\frac{512}{3}z^5+512z^4-544z^3+\frac{704}{3}z^2-\frac{92}{3}z-\frac{11}{8})z\,{}_3F_2\left[\begin{matrix}2,15,-13/3\\4,35/6\end{matrix}\middle|z\right]$$
$$+(2816z^5-\frac{25600}{3}z^4+9232z^3-4144z^2+\frac{1870}{3}z+\frac{20}{3})z^2\,{}_3F_2\left[\begin{matrix}3,15,-13/3\\5,35/6\end{matrix}\middle|z\right]$$
$$+(-16384z^5+\frac{254208}{5}z^4-57344z^3+\frac{140336}{5}z^2-\frac{25872}{5}z-10)z^3\,{}_3F_2\left[\begin{matrix}4,15,-13/3\\6,35/6\end{matrix}\middle|z\right]$$
$$+(\frac{81920}{3}z^5-\frac{278528}{3}z^4+\frac{604672}{5}z^3-75776z^2+\frac{305632}{15}z+\frac{32}{5})z^4\,{}_3F_2\left[\begin{matrix}5,15,-13/3\\7,35/6\end{matrix}\middle|z\right]$$
$$+(-\frac{131072}{7}z^5+\frac{1589248}{21}z^4-\frac{2646016}{21}z^3+112128z^2-\frac{301568}{7}z-\frac{32}{21})z^5\,{}_3F_2\left[\begin{matrix}6,15,-13/3\\8,35/6\end{matrix}\middle|z\right]$$
$$-\frac{2048}{7}z^7(1-z)(16z^3-80z^2+165z-180)\,{}_3F_2\left[\begin{matrix}7,15,-13/3\\9,35/6\end{matrix}\middle|z\right]$$
$$-\frac{2048}{9}z^8(1-z)(16z^2-80z+165)\,{}_3F_2\left[\begin{matrix}8,15,-13/3\\10,35/6\end{matrix}\middle|z\right]$$
$$+\frac{131072}{45}z^9(1-z)(5-z)\,{}_3F_2\left[\begin{matrix}9,15,-13/3\\11,35/6\end{matrix}\middle|z\right]-\frac{131072}{55}z^{10}(1-z)\,{}_3F_2\left[\begin{matrix}10,15,-13/3\\12,35/6\end{matrix}\middle|z\right],$$

$$\frac{(1-z)^{-1/6}}{7(1+30z-96z^2+64z^3)^{-7/3}}\,{}_2F_1\left[\begin{matrix}1/3,-7/6\\3/2\end{matrix}\middle|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] = \frac{1}{7}(1-z)\,{}_2F_1\left[\begin{matrix}-7/6,13/6\\5/6\end{matrix}\middle|z\right] \tag{3.27}$$
$$-\frac{1}{5}(-4096z^5+12288z^4-13056z^3+5632z^2-736z-33)z\,{}_2F_1\left[\begin{matrix}-1/6,11/6\\5/6\end{matrix}\middle|z\right]$$
$$+\frac{1}{55}(135168z^5-409600z^4+443136z^3-198912z^2+29920z+320)z^2\,{}_2F_1\left[\begin{matrix}5/6,13/6\\17/6\end{matrix}\middle|z\right]$$
$$+\frac{1}{187}(-1310720z^5+4067328z^4-4587520z^3+2245376z^2-413952z-800)z^3\,{}_2F_1\left[\begin{matrix}11/6,13/6\\23/6\end{matrix}\middle|z\right]$$
$$+\frac{1}{391}(3276800z^5-11141120z^4+14512128z^3-9093120z^2+2445056z+768)z^4\,{}_2F_1\left[\begin{matrix}17/6,13/6\\29/6\end{matrix}\middle|z\right]$$
$$+\frac{1}{667}(-3145728z^5+12713984z^4-21168128z^3+18837504z^2-7237632z-256)z^5\,{}_2F_1\left[\begin{matrix}23/6,13/6\\35/6\end{matrix}\middle|z\right]$$
$$-\frac{65536}{1015}(1-z)(16z^3-80z^2+165z-180)z^7\,{}_2F_1\left[\begin{matrix}29/6,13/6\\41/6\end{matrix}\middle|z\right]$$
$$-\frac{65536}{1435}(1-z)(16z^2-80z+165)z^7\,{}_2F_1\left[\begin{matrix}35/6,13/6\\47/6\end{matrix}\middle|z\right]$$
$$+\frac{1048576}{1927}(1-z)(5-z)z^9\,{}_2F_1\left[\begin{matrix}41/6,13/6\\53/6\end{matrix}\middle|z\right]-\frac{1048576}{2491}(1-z)z^{10}\,{}_2F_1\left[\begin{matrix}47/6,13/6\\59/6\end{matrix}\middle|z\right]$$

and

$$\frac{1}{3}\frac{(1+30z-96z^2+64z^3)^3}{\sqrt{1-z}} = \frac{1}{(2a-3)(2a-1)}(1-z)\,{}_2F_1\left[\begin{matrix}5/2,a-3/2\\a+1/2\end{matrix}\middle|z\right] \tag{3.28}$$

$$+\frac{1}{(2a-1)(2a+1)}z(-4096z^5+12288z^4-13056z^3+5632z^2-736z-33)\,{}_2F_1\left[\begin{matrix}5/2, a-1/2\\ a+3/2\end{matrix}\middle| z\right]$$
$$+\frac{1}{(2a+1)(2a+3)}z^2(135168z^5-409600z^4+443136z^3-198912z^2+29920z+320)\,{}_2F_1\left[\begin{matrix}5/2, a+1/2\\ a+5/2\end{matrix}\middle| z\right]$$
$$+\frac{1}{(2a+3)(2a+5)}z^3(-1310720z^5+4067328z^4-4587520z^3+2245376z^2-413952z-800)_2F_1\left[\begin{matrix}5/2, a+3/2\\ a+7/2\end{matrix}\middle| z\right]$$
$$+\frac{1}{(2a+5)(2a+7)}z^4(3276800z^5-11141120z^4+14512128z^3-9093120z^2+2445056z+768)_2F_1\left[\begin{matrix}5/2, a+5/2\\ a+9/2\end{matrix}\middle| z\right]$$
$$+\frac{1}{(2a+7)(2a+9)}z^5(-3145728z^5+12713984z^4-21168128z^3+18837504z^2-7237632z-256)_2F_1\left[\begin{matrix}5/2, a+7/2\\ a+11/2\end{matrix}\middle| z\right]$$
$$+\frac{1}{(2a+9)(2a+11)}z^7(1-z)(16z^3-80z^2+165z-180)_2F_1\left[\begin{matrix}5/2, a+9/2\\ a+13/2\end{matrix}\middle| z\right]$$
$$-65536\frac{1}{(2a+11)(2a+13)}z^8(1-z)(16z^2-80z+165)_2F_1\left[\begin{matrix}5/2, a+11/2\\ a+15/2\end{matrix}\middle| z\right]$$
$$+1048576\frac{1}{(2a+13)(2a+15)}z^9(1-z)(5-z)_2F_1\left[\begin{matrix}5/2, a+13/2\\ a+17/2\end{matrix}\middle| z\right]$$
$$-1048576\frac{1}{(2a+15)(2a+17)}z^{10}(1-z)_2F_1\left[\begin{matrix}5/2, a+15/2\\ a+19/2\end{matrix}\middle| z\right].$$

Proof. Putting respectively $a=\frac{5}{2}$, $a=\frac{1}{3}$ and $a=0$ in (3.18), we obtain (3.26), (3.27) and (3.28). □

Theorem 3.8. *If* $\left|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right|<1$ *and* $a\neq 1/2,\ 3/2$ *and* $-5/12$ *then*

$$64^{2a}\frac{4z^3(1-z)^{5a-13/2}}{(64-96z+30z^2+z^3)^{2a-3}(2a-1)(2a-3)}\,{}_2F_1\left[\begin{matrix}a, a-3/2\\ 2a+5/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] \tag{3.29}$$
$$=\sum_{i=0}^{9}C_i(z)\sum_{n=0}^{\infty}\frac{(5/2-a)_n}{n!(4a+i+n-5)(4a+i+n-6)}(-\frac{z}{1-z})^{i+n}{}_3F_2\left[\begin{matrix}4a+i+n-6, 6a, 4a+1/6\\ 4a+i+n-4, 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$

where coefficient $C_i(z)$ *are given in Tables 2*

i	$C_i(z)$
0	$-1048576z^3(z-1)$
1	$1048576z^2(z-1)(4z+1)$
2	$-65536z(z-1)(101z^2+48z+16)$
3	$65536(z-1)(79z^3+53z^2+32z+16)$
4	$-256z^5-2060288z^4+159744z^3+1769472z^2-3014656z+3145728$
5	$512z^5+385280z^4+364544z^3-2715648z^2+5242880z-3276800$
6	$-288z^5-29952z^4-220416z^3+1425408z^2-2486272z+1310720$
7	$32z^5+1056z^4+24576z^3-156416z^2+266249z-135168$
8	$-z^5-32z^4-768z^3+4864z^2-8192z+4096$
9	z^4

Table 2. Coefficients $C_i(z)$ of (3.29).

Proof. We start with the Goursat's transformation [7, Eq.137, p. 142]

$$ {}_2F_1\left[\begin{matrix}6a, 4a+1/6\\ 8a+1/3\end{matrix}\middle| z\right] = \left(\frac{64-96z+30z^2+z^3}{64}\right)^{-2a} {}_2F_1\left[\begin{matrix}a, a+1/2\\ 2a+5/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right]. \tag{3.30}$$

Putting $g(z) = \dfrac{z^4(1-z)}{(64-96z+30z^2+z^3)^2}$ and $h(z) = \dfrac{z}{1-z}$ in (3.5), we have $g'(z) = \dfrac{z^3(1-16z+16z^2)^2}{(64-96z+30z^2+z^3)^3}$, $h'(z) = (1-z)^{-2}$ and we have

$$\frac{g(z)-g(w)}{h(z)-h(w)} = \frac{(1-w)(1-z)(wz-w-z)H(w,z)}{(64-96z+30z^2+z^3)^2(64-96w+30w^2+w^3)^2} \tag{3.31}$$

where

$$\begin{aligned} H(w,z) = &-4096(1-w)w^2 + 8192(1-w)w^2z - 256w(1-w)(19w^2-16w+16)z^2 \\ &+ 256w(1-w)(3w-4)(w-4)z^3 + w^4z^4 \end{aligned} \tag{3.32}$$

and (3.5) becomes

$$\begin{aligned} &\frac{z^{3\alpha+3}(1-z)^{2\alpha+1}}{(64-96z+30z^2+z^3)^{2\alpha}} {}_{\frac{z^4(1-z)}{(64-96z+30z^2+z^3)^2}}O^{\alpha}_{\alpha+2} f(z) \\ &= {}_{\frac{z}{1-z}}O^{\alpha}_{\alpha+2}\left\{\frac{z^{3\alpha}(1-z)^{2\alpha+1}(1-16z+16z^2)^2}{(64-96z+30z^2+z^3)^{2\alpha+3}}(wz-w-z)H(w,z)f(z)\right\}\Bigg|_{w=z}. \end{aligned} \tag{3.33}$$

From the Euler transformation

$$ {}_2F_1\left[\begin{matrix}6a, 4a+1/6\\ 8a+1/3\end{matrix}\middle| z\right] = (1-z)^{-6a} {}_2F_1\left[\begin{matrix}6a, 4a+1/6\\ 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right] \tag{3.34}$$

and using (3.30), we have

$$\begin{aligned} &{}_{\frac{z^4(1-z)}{(64-96z+30z^2+z^3)^2}}O^{\alpha}_{\alpha+2}\ {}_2F_1\left[\begin{matrix}a, a+1/2\\ 2a+5/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] \\ &= \frac{64^{-2a}z^{-3\alpha-3}(1-z)^{-2\alpha-2}}{(64-96z+30z^2+z^3)^{-2\alpha-2}} {}_{\frac{z}{1-z}}O^{\alpha}_{\alpha+2}\left\{\frac{z^{3\alpha}(1-z)^{2\alpha-6a}(z^2-16z+16)^2}{(64-96z+30z^2+z^3)^{2\alpha-2a+1}}\right. \\ &\left.\frac{(1-w)(1-z)(wz-w-z)}{(64-96z+30z^2+z^3)^2(64-96w+30w^2+w^3)^2}H(w,z) {}_2F_1\left[\begin{matrix}6a, 4a+1/6\\ 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right]\right\}\Bigg|_{w=z} \end{aligned} \tag{3.35}$$

or

$$\begin{aligned} &\frac{64^{2a}z^{3\alpha+3}(1-z)^{2\alpha+1}}{(64-96z+30z^2+z^3)^{2\alpha}} {}_{\frac{z^4(1-z)}{(64-96z+30z^2+z^3)^2}}O^{\alpha}_{\alpha+2}\ {}_2F_1\left[\begin{matrix}a, a+1/2\\ 2a+5/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] \\ &= {}_{\frac{z}{1-z}}O^{\alpha}_{\alpha+2}\left\{\frac{z^{3\alpha}(1-z)^{2\alpha-6a+1}(z^2-16z+16)^2(wz-w-z)H(w,z)}{(64-96z+30z^2+z^3)^{2\alpha-2a+3}}\ {}_2F_1\left[\begin{matrix}6a, 4a+1/6\\ 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right]\right\}\Bigg|_{w=z} \end{aligned} \tag{3.36}$$

or

$$\begin{aligned} &\frac{64^{2a}(\frac{z}{1-z})^{3\alpha+3}(1-z)^{5\alpha+4}}{(64-96z+30z^2+z^3)^{2\alpha}} {}_{\frac{z^4(1-z)}{(64-96z+30z^2+z^3)^2}}O^{\alpha}_{\alpha+2} {}_2F_1\left[\begin{matrix}a, a+1/2\\ 2a+5/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] \\ &= {}_{\frac{z}{1-z}}O^{\alpha}_{\alpha+2}\left\{\frac{(\frac{z}{1-z})^{3\alpha}(1-z)^{5\alpha-6a+1}(z^2-16z+16)^2(wz-w-z)H(w,z)}{(64-96z+30z^2+z^3)^{2\alpha-2a+3}}\ {}_2F_1\left[\begin{matrix}6a, 4a+1/6\\ 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right]\right\}\Bigg|_{w=z}. \end{aligned} \tag{3.37}$$

Using the property (2.5), we have

$$\frac{64^{2a}z^3(1-z)^{5\alpha+1}}{(64-96z+30z^2+z^3)^{2\alpha}}\ {}_{\frac{z^4(1-z)}{(64-96z+30z^2+z^3)^2}}O^{\alpha}_{\alpha+2}\,{}_2F_1\left[\begin{matrix}a, a+1/2\\ 2a+5/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] \tag{3.38}$$
$$= \frac{\alpha+1}{4(4\alpha+1)}\ {}_{\frac{z}{1-z}}O^{4\alpha}_{4\alpha+2}\left\{\frac{(1-z)^{5\alpha-6a+1}(z^2-16z+16)^2(wz-w-z)H(w,z)}{(64-96z+30z^2+z^3)^{2\alpha-2a+3}}\ {}_2F_1\left[\begin{matrix}6a, 4a+1/6\\ 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right]\right\}\Bigg|_{w=z}.$$

From (3.32), it is easy to prove that

$$S(w,z) = (z^2-16z+16)^2(wz-w-z)H(w,z) \tag{3.39}$$
$$= (1-z)^9\sum_{i=0}^{9} C_i(w)\left(-\frac{z}{1-z}\right)^i.$$

Putting $\alpha = a - \frac{3}{2}$ and and expanding in series $(1-z)^{\frac{5}{2}-a} = (1+\frac{z}{1-z})^{a-\frac{5}{2}}$, we obtain

$$\frac{64^{2a}z^3(1-z)^{5a-\frac{13}{2}}}{(64-96z+30z^2+z^3)^{2a-3}}\,{}_2F_1\left[\begin{matrix}a, a-3/2\\ 2a+5/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] = \frac{2a-1}{8(4a+5)}\sum_{i=0}^{9} C_i(z) \tag{3.40}$$
$$\sum_{n=0}^{\infty}\frac{(\frac{5}{2}-a)_n}{n!}\ {}_{\frac{z}{1-z}}O^{4a-6}_{4a-4}\left(-\frac{z}{1-z}\right)^{i+n}{}_2F_1\left[\begin{matrix}6a, 4a+1/6\\ 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$

Using properties (2.3) and (2.5), we have after simplifications

$$\frac{64^{2a}z^3(1-z)^{5a-\frac{13}{2}}}{(64-96z+30z^2+z^3)^{2a-3}}\,{}_2F_1\left[\begin{matrix}a, a-3/2\\ 2a+5/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] \tag{3.41}$$
$$= \frac{(2a-1)(2a-3)}{4}\sum_{i=0}^{9} C_i(z)\sum_{n=0}^{\infty}\frac{(\frac{5}{2}-a)_n}{n!(4a-6+i+n)(4a-5+i+n)}\left(-\frac{z}{1-z}\right)^{i+n}$$
$${}_3F_2\left[\begin{matrix}4a+6+i+n, 6a, 4a+1/6\\ 4a-4+i+n, 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right].$$

□

Corollary 3.9. *If* $\left|\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right| < 1$ *, we have*

$$53687092\frac{z^3(1-z)^6}{(64-96z+30z^2+z^3)^2}\,{}_2F_1\left[\begin{matrix}1, 5/2\\ 35/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] \tag{3.42}$$
$$= \frac{262144}{5}(1-z)z^3\,{}_3F_2\left[\begin{matrix}4, 15, 61/6\\ 6, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right] + \frac{524288}{15}z^3(4z+1)\,{}_3F_2\left[\begin{matrix}5, 15, 61/6\\ 7, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$+\frac{32768}{21}\frac{z^3(101z^2+48z+16)}{1-z}\,{}_3F_2\left[\begin{matrix}6, 15, 61/6\\ 8, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$+\frac{8192}{7}\frac{(79z^3+53z^2+32z+16)}{(1-z)^2}z^3\,{}_3F_2\left[\begin{matrix}7, 15, 61/6\\ 9, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$-\frac{32}{9}(z^5+8048z^4-624z^3-6912z^2+11776z-12288)\left(\frac{z}{1-z}\right)^4{}_3F_2\left[\begin{matrix}8, 15, 61/6\\ 10, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$-\frac{128}{45}(2z^5+1505z^4+1424z^3-10608z^2+20480z-12800)\left(\frac{z}{1-z}\right)^5{}_3F_2\left[\begin{matrix}9, 15, 61/6\\ 11, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$

$$- \frac{16}{55}(9z^5 + 936z^4 + 6888z^3 - 44544z^2 + 77696z - 40960)(\frac{z}{1-z})^6 {}_3F_2\left[\begin{matrix}10, 15, 61/6\\ 12, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$- \frac{8}{33}(z^5 + 33z^4 + 768z^3 - 4888z^2 + 8320z - 4224)(\frac{z}{1-z})^7 {}_3F_2\left[\begin{matrix}11, 15, 61/6\\ 13, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$- \frac{1}{156}(z^5 + 32z^4 + 768z^3 - 4864z^2 + 8192z - 4096)(\frac{z}{1-z})^8 {}_3F_2\left[\begin{matrix}12, 15, 61/6\\ 14, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$- \frac{1}{182}\frac{z^{13}}{(1-z)^9} {}_2F_1\left[\begin{matrix}13, 61/6\\ 61/3\end{matrix}\middle| -\frac{z}{1-z}\right].$$

Proof. Putting $a = \frac{5}{2}$ in (3.29), we obtain (3.42). □

4. Special cases

We now give a list of nine transformation formulas with different choices for the arguments $g(z)$ and $h(z)$ in formulas (3.4) and (3.5) in order to demonstrate the efficiency of the method.

Case 1. *([7, Eq.135, p. 142]; (3.4); $g(z) = z(1-z)/(1+30z-96z^2+64z^3)^2$; $h(z) = z/(1-z)$)*

$$\frac{(1-z)^{2a-1}}{(1+30z-96z^2+64z^3)^{2a-1}(2a-1)} {}_2F_1\left[\begin{matrix}a, a-1/2\\ 2a+5/6\end{matrix}\middle| \frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] \tag{4.1}$$
$$= \sum_{n=0}^{\infty} \frac{(3-4a)_n}{n!(2a-1+2n)}(-\frac{z}{1-z})^n {}_3F_2\left[\begin{matrix}a+n-1/2, 6a, 4a+1/6\\ a+n+1/2, 2a+5/6\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$- 28(\frac{z}{1-z}) \sum_{n=0}^{\infty} \frac{(3-4a)_n}{n!(2a+1+2n)}(-\frac{z}{1-z})^n {}_3F_2\left[\begin{matrix}a+n+1/2, 6a, 4a+1/6\\ a+n+3/2, 2a+5/6\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$+ 198(\frac{z}{1-z})^2 \sum_{n=0}^{\infty} \frac{(3-4a)_n}{n!(2a+3+2n)}(-\frac{z}{1-z})^n {}_3F_2\left[\begin{matrix}a+n+3/2, 6a, 4a+1/6\\ a+n+5/2, 2a+5/6\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$- 28(\frac{z}{1-z})^3 \sum_{n=0}^{\infty} \frac{(3-4a)_n}{n!(2a+5+2n)}(-\frac{z}{1-z})^n {}_3F_2\left[\begin{matrix}a+n+5/2, 6a, 4a+1/6\\ a+n+7/2, 2a+5/6\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$+ (\frac{z}{1-z})^4 \sum_{n=0}^{\infty} \frac{(3-4a)_n}{n!(2a+7+2n)}(-\frac{z}{1-z})^n {}_3F_2\left[\begin{matrix}a+n+7/2, 6a, 4a+1/6\\ a+n+9/2, 2a+5/6\end{matrix}\middle| -\frac{z}{1-z}\right].$$

If $\frac{-z}{1-z} \to z$ in (4.1), putting $a = 5/4$, we get

$$\frac{(1-z)}{\sqrt{(1+z)}\sqrt{(z^2-34z+1)}} {}_2F_1\left[\begin{matrix}3/4, 1/4\\ 7/3\end{matrix}\middle| -\frac{108z(1-z)^4}{(1+z)^2(1-34z+z^2)^2}\right] = {}_3F_2\left[\begin{matrix}1/4, 9/2, 19/6\\ 5/4, 7/3\end{matrix}\middle| z\right] \tag{4.2}$$
$$+ \frac{28}{5}z\, {}_3F_2\left[\begin{matrix}5/4, 9/2, 19/6\\ 9/4, 7/3\end{matrix}\middle| z\right] + 22z^2\, {}_3F_2\left[\begin{matrix}9/4, 9/2, 19/6\\ 13/4, 7/3\end{matrix}\middle| z\right] + \frac{28}{13}z^3\, {}_3F_2\left[\begin{matrix}13/4, 9/2, 19/6\\ 17/4, 7/3\end{matrix}\middle| z\right]$$
$$+ \frac{1}{17}z^4\, {}_3F_2\left[\begin{matrix}17/4, 9/2, 19/6\\ 21/4, 7/3\end{matrix}\middle| z\right].$$

Putting $a = -1/24$, we obtain

$$\frac{(1+z)^{13/12}(1-34z+z^2)^{13/12}}{(1-z)^{13/6}} {}_2F_1\left[\begin{matrix}-13/24, -1/24\\ 3/4\end{matrix}\middle| -\frac{108z(1-z)^4}{(1+z)^2(1-34z+z^2)^2}\right] = {}_2F_1\left[\begin{matrix}-13/24, 19/6\\ 11/24\end{matrix}\middle| z\right] \tag{4.3}$$
$$- \frac{364}{11}z\, {}_2F_1\left[\begin{matrix}11/24, 19/6\\ 35/24\end{matrix}\middle| z\right] - \frac{2574}{35}z^2\, {}_2F_1\left[\begin{matrix}35/24, 19/6\\ 59/24\end{matrix}\middle| z\right] - \frac{364}{59}z^3\, {}_2F_1\left[\begin{matrix}59/24, 19/6\\ 83/24\end{matrix}\middle| z\right]$$

$$- \frac{13}{83} z^4 \;{}_2F_1\left[\begin{matrix}83/24, 19/6\\ 107/24\end{matrix}\middle| z\right].$$

Case 2. *([7, Eq.135, p. 142]; (3.5); $g(z) = z(1-z)/(1+30z-96z^2+64z^3)^2$; $h(z) = z/(1-z)$)*

$$\frac{(1-z)^{2a-2}}{(1+30z-96z^2+64z^3)^{2a-3}(2a-1)(2a-3)}\;{}_2F_1\left[\begin{matrix}a, a-3/2\\ 2a+5/6\end{matrix}\middle| \frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] \tag{4.4}$$
$$= \sum_{i=0}^{9} r_i(z)\left(-\frac{z}{1-z}\right)^i \left\{ \sum_{n=0}^{\infty} \frac{(7-4a)_n}{n!(2a-1+2n+2i)(2a-3+2n+2i)}\left(-\frac{z}{1-z}\right)^n {}_3F_2\left[\begin{matrix}a+n+i-3/2, 6a, 4a+1/6\\ a+n+i+1/2, 2a+5/6\end{matrix}\middle| -\frac{z}{1-z}\right]\right\}$$

where coefficient $r_i(z)$ are given in Table 3.

i	$r_i(z)$
0	$(1-z)$
1	$4096z^5 - 12288z^4 + 13056z^3 - 5632z^2 + 745z + 24$
2	$102400z^5 - 311296z^4 + 338688z^3 - 153856z^2 + 23996z + 92$
3	$479232z^5 - 1544192z^4 + 1851136z^3 - 1010688z^2 + 225204z - 600$
4	$-1978368z^5 + 5349376z^4 - 4438272z^3 + 517376z^2 + 548322z + 966$
5	$1978368z^5 - 4542464z^4 + 2824448z^3 - 484864z^2 + 226078z - 600$
6	$-479232z^5 + 851968z^4 - 466688z^3 + 69888z^2 + 23372z + 92$
7	$-102400z^5 + 200704z^4 - 117504z^3 + 18432z^2 + 836z + 24$
8	$-4096z^5 + 8192z^4 - 4864z^3 + 768z^2 + 23z + 1$
9	z

Table 3. Coefficients $r_i(z)$ of (4.4)

If $\frac{-z}{1-z} \to z$ in (4.4),we obtain the equivalent formula

$$\frac{(1-z)^{4a-7}(1-33z-33z^2+z^3)^{3-2a}}{(2a-1)(2a-3)}\;{}_2F_1\left[\begin{matrix}a, a-3/2\\ 2a+5/6\end{matrix}\middle| \frac{-108z(1-z)^4}{(1-33z-33z^2+z^3)^2}\right] \tag{4.5}$$
$$= \sum_{i=0}^{9} P_i(z) z^i \left\{ \sum_{n=0}^{\infty} \frac{(7-4a)_n}{n!(2a-1+2n+2i)(2a-3+2n+2i)} z^n \;{}_3F_2\left[\begin{matrix}a+n+i-3/2, 6a, 4a+1/6\\ a+n+i+1/2, 2a+5/6\end{matrix}\middle| z\right]\right\}$$

where coefficient $r_i(z)$ are given in Table 4.

i	$P_i(z)$
0	$(1-z)^4$
1	$-z^5 + 28z^4 - 870z^3 - 2412z^2 - 865z + 24$
2	$-24z^5 + 956z^4 - 22016z^3 - 56952z^2 - 24456z + 92$
3	$-92z^5 + 23832z^4 - 164296z^3 - 115872z^2 - 222204z - 600$
4	$600z^5 + 223078z^4 - 413448z^3 + 2720324z^2 - 553152z + 966$
5	$-966z^5 + 553152z^4 - 2720324z^3 + 413448z^2 - 223078z - 600$
6	$600z^5 + 222204z^4 + 115872z^3 + 164296z^2 - 23832z + 92$
7	$-92z^5 + 24456z^4 + 56952z^3 + 22016z^2 - 956z + 24$
8	$-24z^5 + 865z^4 + 2412z^3 + 870z^2 - 28z + 1$
9	$-z(1-z)^4$

Table 4. Coefficients $P_i(z)$ of (4.5)

If $a = 7/4$ in (4.5), we obtain

$$\frac{1}{5}\frac{(1-z)^5}{\sqrt{(1+z)(1-34z+z^2)}}\,{}_2F_1\left[\begin{matrix}7/4,1/4\\13/3\end{matrix}\middle|-\frac{108z(1-z)^4}{(1+z)^2(1-34z+z^2)^2}\right] = \frac{1}{5}(1-z)^4{}_3F_2\left[\begin{matrix}1/4,21/2,43/6\\9/4,13/3\end{matrix}\middle|z\right] \tag{4.6}$$
$$-\frac{1}{45}z(z^5-28z^4+870z^3+2412z^2+865z-24)\,{}_3F_2\left[\begin{matrix}5/4,21/2,43/6\\13/4,13/3\end{matrix}\middle|z\right]$$
$$-\frac{4}{117}z^2(6z^5-239z^4+5504z^3+14238z^2+6114z-23)\,{}_3F_2\left[\begin{matrix}9/4,21/2,43/6\\17/4,13/3\end{matrix}\middle|z\right]$$
$$-\frac{4}{221}z^3(23z^5-5958z^4+41074z^3+28968z^2+55551z+150)\,{}_3F_2\left[\begin{matrix}13/4,21/2,43/6\\21/4,13/3\end{matrix}\middle|z\right]$$
$$+\frac{2}{357}z^4(300z^5+111539z^4-206724z^3+1360162z^2-276576z+483)\,{}_3F_2\left[\begin{matrix}17/4,21/2,43/6\\25/4,13/3\end{matrix}\middle|z\right]$$
$$-\frac{2}{525}z^5(483z^5-276576z^4+1360162z^3-206724z^2+111539z+300)\,{}_3F_2\left[\begin{matrix}21/4,21/2,43/6\\29/4,13/3\end{matrix}\middle|z\right]$$
$$+\frac{4}{725}z^6(150z^5+55551z^4+28968z^3+41074z^2-5958z+23)\,{}_3F_2\left[\begin{matrix}25/4,21/2,43/6\\33/4,13/3\end{matrix}\middle|z\right]$$
$$-\frac{4}{957}z^7(23z^5-6114z^4-14238z^3-5504z^2+239z-6)\,{}_3F_2\left[\begin{matrix}29/4,21/2,43/6\\37/4,13/3\end{matrix}\middle|z\right]$$
$$-\frac{1}{1221}z^8(24z^5-865z^4-2412z^3-870z^2+28z-1)\,{}_3F_2\left[\begin{matrix}33/4,21/2,43/6\\41/4,13/3\end{matrix}\middle|z\right]$$
$$-\frac{1}{1517}z^{10}(1-z)^4{}_3F_2\left[\begin{matrix}37/4,21/2,43/6\\45/4,13/3\end{matrix}\middle|z\right]$$

and putting $a = -1/24$, we get

$$\frac{(1+z)^{37/12}(1-34z+z^2)^{37/12}}{(1-z)^{13/6}}\,{}_2F_1\left[\begin{matrix}-37/24,-1/24\\3/4\end{matrix}\middle|-\frac{108z(1-z)^4}{(1+z)^2(1-34z+z^2)^2}\right] = (1-z)^4{}_2F_1\left[\begin{matrix}-37/24,43/6\\11/24\end{matrix}\middle|z\right] \tag{4.7}$$
$$+\frac{37}{11}(z^5-28z^4+870z^3+2412z^2+865z-24)z\;{}_2F_1\left[\begin{matrix}-13/24,43/6\\35/24\end{matrix}\middle|z\right]$$
$$-\frac{1924}{385}(6z^5-239z^4+5504z^3+14238z^2+6114z-23)z^2\;{}_2F_1\left[\begin{matrix}11/24,43/6\\59/24\end{matrix}\middle|z\right]$$
$$-\frac{1924}{2065}(23z^5-5958z^4+41074z^3+28968z^2+55551z+159)z^3\;{}_2F_1\left[\begin{matrix}35/24,43/6\\83/24\end{matrix}\middle|z\right]$$
$$+\frac{962}{4897}(300z^5+111539z^4-206724z^3+1360162z^2-276576z+383)z^4\;{}_2F_1\left[\begin{matrix}59/24,43/6\\107/24\end{matrix}\middle|z\right]$$
$$-\frac{962}{8881}(483z^5-276576z^4+1360162z^3-206724z^2+111539z+300)z^5\;{}_2F_1\left[\begin{matrix}83/24,43/6\\131/24\end{matrix}\middle|z\right]$$
$$+\frac{1924}{14017}(150z^6+55551z^4+28968z^3+41074z^2-5958z+23)z^6\;{}_2F_1\left[\begin{matrix}107/24,43/6\\155/24\end{matrix}\middle|z\right]$$
$$+\frac{1924}{20305}(23z^7-6114z^4-14238z^3-5504z^2+239z-6)\;z^7{}_2F_1\left[\begin{matrix}131/24,43/6\\179/24\end{matrix}\middle|z\right]$$
$$-\frac{481}{27745}(24z^8-865z^4-2412z^3-870z^2+28z-1)\;z^8{}_2F_1\left[\begin{matrix}155/24,43/6\\203/24\end{matrix}\middle|z\right]$$
$$-\frac{481}{36337}(1-z)^4z^{10}{}_2F_1\left[\begin{matrix}179/24,43/6\\227/24\end{matrix}\middle|z\right].$$

Case 3. *([7, Eq.137, p. 142]; (3.4);* $g(z) = -108z^4(1-z)/(64-96z+30z^2+z^3)^2$; $h(z) = z$*)*

$$64^{2a}\frac{(1-z)^{a-1/2}}{(64-96z+30z^2+z^3)^{2a-1}(2a-1)}\,{}_2F_1\left[\begin{matrix}a, a-1/2\\ 2a+5/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] \tag{4.8}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(4a-1+2n)}z^{n+4}{}_3F_2\left[\begin{matrix}4a+n+2, 6a, 4a+1/6\\ 4a+n+3, 8a+1/3\end{matrix}\middle| z\right]$$
$$-16\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(4a+n+1)}z^{n+3}{}_3F_2\left[\begin{matrix}4a+n+1, 6a, 4a+1/6\\ 4a+n+2, 8a+1/3\end{matrix}\middle| z\right]$$
$$+144\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(4a+n)}z^{n+2}{}_3F_2\left[\begin{matrix}4a+n, 6a, 4a+1/6\\ 4a+n+1, 8a+1/3\end{matrix}\middle| z\right]$$
$$-256\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(4a+n-1)}z^{n+1}{}_3F_2\left[\begin{matrix}4a+n-1, 6a, 4a+1/6\\ 4a+n, 8a+1/3\end{matrix}\middle| z\right]$$
$$+128\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(4a+n-2)}z^{n}{}_3F_2\left[\begin{matrix}4a+n-2, 6a, 4a+1/6\\ 4a+n-1, 8a+1/3\end{matrix}\middle| z\right].$$

If $a = 3/2$ *in (4.8), we obtain*

$$131072\frac{(1-z)}{(64-96z+30z^2+z^3)^2}\,{}_2F_1\left[\begin{matrix}1, 3/2\\ 23/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] = \frac{1}{16}z^4{}_2F_1\left[\begin{matrix}8, 37/6\\ 37/3\end{matrix}\middle| z\right] \tag{4.9}$$
$$-\frac{16}{7}z^3{}_3F_2\left[\begin{matrix}7, 9, 37/6\\ 8, 37/3\end{matrix}\middle| z\right] + 24z^2{}_3F_2\left[\begin{matrix}6, 9, 37/6\\ 7, 37/3\end{matrix}\middle| z\right] - \frac{256}{5}z{}_3F_2\left[\begin{matrix}5, 9, 37/6\\ 6, 37/3\end{matrix}\middle| z\right] + 32{}_3F_2\left[\begin{matrix}4, 9, 37/6\\ 5, 37/3\end{matrix}\middle| z\right].$$

Case 4. *([7, Eq.137, p. 142]; (3.4);* $g(z) = -108z^4(1-z)/(64-96z+30z^2+z^3)^2$; $h(z) = \frac{z}{1-z}$*)*

$$64^{2a}\frac{(1-z)^{5a-5/2}}{(64-96z+30z^2+z^3)^{2a-1}(2a-1)}\,{}_2F_1\left[\begin{matrix}a, a-1/2\\ 2a+5/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] \tag{4.10}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(4a-1+2n)}(-1)^n\left(\frac{z}{1-z}\right)^{n+4}{}_3F_2\left[\begin{matrix}4a+n+2, 6a, 4a+1/6\\ 4a+n+3, 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$+16\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(4a+n+1)}(-1)^n\left(\frac{z}{1-z}\right)^{n+3}{}_3F_2\left[\begin{matrix}4a+n+1, 6a, 4a+1/6\\ 4a+n+2, 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$+144\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(4a+n)}(-1)^n\left(\frac{z}{1-z}\right)^{n+2}{}_3F_2\left[\begin{matrix}4a+n, 6a, 4a+1/6\\ 4a+n+1, 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$+256\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(4a+n-1)}(-1)^n\left(\frac{z}{1-z}\right)^{n+1}{}_3F_2\left[\begin{matrix}4a+n-1, 6a, 4a+1/6\\ 4a+n, 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$+128\sum_{n=0}^{\infty}\frac{(3/2-a)_n}{n!(4a+n-2)}(-1)^n\left(\frac{z}{1-z}\right)^{n}{}_3F_2\left[\begin{matrix}4a+n-2, 6a, 4a+1/6\\ 4a+n-1, 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right].$$

If $a = 3/2$ *in (4.10), we obtain*

$$262144\frac{(1-z)^5}{(64-96z+30z^2+z^3)^2}\,{}_2F_1\left[\begin{matrix}1, 3/2\\ 23/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] = \frac{1}{8}z^4{}_2F_1\left[\begin{matrix}8, 37/6\\ 37/3\end{matrix}\middle| -\frac{z}{1-z}\right] \tag{4.11}$$
$$+\frac{32}{7}z^3{}_3F_2\left[\begin{matrix}7, 9, 37/6\\ 8, 37/3\end{matrix}\middle| -\frac{z}{1-z}\right] + 48z^2{}_3F_2\left[\begin{matrix}6, 9, 37/6\\ 7, 37/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$

$$+\frac{512}{5}z\,{}_3F_2\left[\begin{matrix}5,9,37/6\\6,37/3\end{matrix}\middle|-\frac{z}{1-z}\right]+64\,{}_3F_2\left[\begin{matrix}4,9,37/6\\5,37/3\end{matrix}\middle|z\right].$$

Case 5. *([7, Eq.137, p. 142]; (3.5); $g(z)=-108z^4(1-z)/(64-96z+30z^2+z^3)^2$; $h(z)=z$)*

$$64^{2a}\frac{4z^3(1-z)^{a-1/2}}{(64-96z+30z^2+z^3)^{2a-3}(2a-1)(2a-3)}\,{}_2F_1\left[\begin{matrix}a,a-3/2\\2a+5/6\end{matrix}\middle|-\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] \tag{4.12}$$
$$=\sum_{s=0}^{5}c_s(z)\sum_{j=0}^{4}f_j(z)\sum_{n=0}^{\infty}\frac{(5/2-a)_n}{n!(4a-6+s+n+j)(4a-5+s+n+j)}$$
$$z^{s+n+j}{}_3F_2\left[\begin{matrix}4a+s+n+j-6,6a,4a+1/6\\4a+s+n+j-4,8a+1/3\end{matrix}\middle|z\right]$$

where coefficients $c_i(z)$ and $f_i(z)$ are given in Tables 5 and 6.

i	$c_i(z)$
0	$-256z$
1	$768z-256$
2	$-800z+512$
3	$320z-288$
4	$-33z+32$
5	$z-1$

Table 5. Coefficients $c_i(z)$ of (4.12)

i	$f_i(z)$
0	$4096z^2(z-1)$
1	$-8192z^2(z-1)$
2	$256(z-1)(19z^2-16z+16)$
3	$-256(z-1)(3z-4)(z-4)$
4	z^4

Table 6. Coefficients $f_i(z)$ of (4.12)

Explicitly, we have

$$64^{2a}\frac{4z^3(1-z)^{a-1/2}}{(z^3+30z^2-96z+64)^{2a-3}}\,{}_2F_1\left[\begin{matrix}a,a-1/2\\2a+5/6\end{matrix}\middle|-\frac{108z^4(1-z)}{(z^3+30z^2-96z+64)^2}\right] \tag{4.13}$$
$$=1048576z^3(1-z)\sum_{n=0}^{\infty}\frac{(5/2-a)_n}{(4a-5+n)(4a-6+n)n!}z^n{}_3F_2\left[\begin{matrix}4a-6+n,6a,4a+1/6\\4a-4+n,8a+1/3,\end{matrix}\middle|z\right]$$
$$-1048576z^3(5z-1)(1-z)\sum_{n=0}^{\infty}\frac{(5/2-a)_n}{(4a-4+n)(4a-5+n)n!}z^n{}_3F_2\left[\begin{matrix}4a-5+n,6a,4a+1/6\\4a-3+n,8a+1/3,\end{matrix}\middle|z\right]$$
$$+65536z^3(1-z)(165z^2-80z+16)\sum_{n=0}^{\infty}\frac{(5/2-a)_n}{(4a-3+n)(4a-4+n)n!}z^n{}_3F_2\left[\begin{matrix}4a-4+n,6a,4a+1/6\\4a-2+n,8a+1/3,\end{matrix}\middle|z\right]$$
$$-z^3(11796480z^3-10813440z^2+5242880z-1048576)(1-z)$$

$$\sum_{n=0}^{\infty}\frac{(5/2-a)_n}{(4a-2+n)(4a-3+n)n!}z^n{}_3F_2\left[\begin{matrix}4a-3+n,6a,4a+1/6\\4a-1+n,8a+1/3,\end{matrix}\middle|z\right]$$
$$-z^4(256z^5+7237632z^4-18837504z^3+21168128z^2-12713984z+3145728)$$
$$\sum_{n=0}^{\infty}\frac{(5/2-a)_n}{(4a-1+n)(4a-2+n)n!}z^n{}_3F_2\left[\begin{matrix}4a-2+n,6a,4a+1/6\\4a+n,8a+1/3,\end{matrix}\middle|z\right]$$
$$+z^5(768z^5+2445056z^4-9093120z^3+14512128z^2-11141120z+3276800)$$
$$\sum_{n=0}^{\infty}\frac{(5/2-a)_n}{(4a+n)(4a-1+n)n!}z^n{}_3F_2\left[\begin{matrix}4a-1+n,6a,4a+1/6\\4a+n+1,8a+1/3,\end{matrix}\middle|z\right]$$
$$-z^6(800z^5+413952z^4-2245376z^3+4587520z^2-4067328z+1310720)$$
$$\sum_{n=0}^{\infty}\frac{(5/2-a)_n}{(4a+n+1)(4a+n)n!}z^n{}_3F_2\left[\begin{matrix}4a+n,6a,4a+1/6\\4a+n+2,8a+1/3,\end{matrix}\middle|z\right]$$
$$+z^7(320z^5+29920z^4-198912z^3+443136z^2-409600z+131168)$$
$$\sum_{n=0}^{\infty}\frac{(5/2-a)_n}{(4a+n+2)(4a+n+1)n!}z^n{}_3F_2\left[\begin{matrix}4a+n+1,6a,4a+1/6\\4a+n+3,8a+1/3,\end{matrix}\middle|z\right]$$
$$-z^8(33z^5+736z^4-5632z^3+13056z^2-12288z+4096)$$
$$\sum_{n=0}^{\infty}\frac{(5/2-a)_n}{(4a+n+3)(4a+n+2)n!}z^n{}_3F_2\left[\begin{matrix}4a+n+2,6a,4a+1/6\\4a+n+4,8a+1/3,\end{matrix}\middle|z\right]$$
$$-z^{13}(1-z)\sum_{n=0}^{\infty}\frac{(5/2-a)_n}{(4a+n+4)(4a+n+3)n!}z^n{}_3F_2\left[\begin{matrix}4a+n+3,6a,4a+1/6\\4a+n+5,8a+1/3,\end{matrix}\middle|z\right].$$

If $a=5/2$ in (4.13), we obtain

$$536870912\frac{z^3(1-z)^2}{(64-96z+30z^2+z^3)^2}\ {}_2F_1\left[\begin{matrix}1,5/2\\35/6\end{matrix}\middle|-\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] \tag{4.14}$$
$$=-\frac{262144}{5}z^3(z-1)\ {}_3F_2\left[\begin{matrix}4,15,61/6\\6,61/3\end{matrix}\middle|z\right]+\frac{524288}{15}\ z^3(5z-1)(z-1)\ {}_3F_2\left[\begin{matrix}5,15,61/6\\7,61/3\end{matrix}\middle|z\right]$$
$$-\frac{32768}{21}z^3(z-1)(165z^2-80z+16)\ {}_3F_2\left[\begin{matrix}6,15,61/6\\8,61/3\end{matrix}\middle|z\right]$$
$$+\frac{8192}{7}z^3(z-1)(180z^3-165z^2+80z-16)\ {}_3F_2\left[\begin{matrix}7,15,61/6\\9,61/3\end{matrix}\middle|z\right]$$
$$-\frac{32}{9}(z^5+28272z^4-73584z^3+82688z^2-49664z+12288)z^4\ {}_3F_2\left[\begin{matrix}8,15,61/6\\10,61/3\end{matrix}\middle|z\right]$$
$$+\frac{128}{45}(3z^5+9551z^4-35520z^3+56688z^2-43520z+12800)z^5\ {}_3F_2\left[\begin{matrix}9,15,61/6\\11,61/3\end{matrix}\middle|z\right]$$
$$-\frac{16}{55}(25z^5+12936z^4-70168z^3+143360z^2-127104z+40960)z^6\ {}_3F_2\left[\begin{matrix}10,15,61/6\\12,61/3\end{matrix}\middle|z\right]$$
$$+\frac{8}{33}(10z^5+935z^4-6216z^3+13848z^2-12800z+4224)z^7\ {}_3F_2\left[\begin{matrix}11,15,61/6\\13,61/3\end{matrix}\middle|z\right]$$
$$-\frac{1}{156}(33z^5+736z^4-5632z^3+13056z^2-12288z+4096)z^8\ {}_3F_2\left[\begin{matrix}12,15,61/6\\14,61/3\end{matrix}\middle|z\right]$$
$$+\frac{1}{182}z^{13}(z-1)\ {}_3F_2\left[\begin{matrix}13,61/6\\61/3\end{matrix}\middle|z\right].$$

Case 6. *([7, Eq.137, p. 142]; (3.5);* $g(z) = -108z^4(1-z)/(64-96z+30z^2+z^3)^2$; $h(z) = \frac{z}{1-z}$*)*

$$64^{2a}\frac{4z^3(1-z)^{5a-13/2}}{(64-96z+30z^2+z^3)^{2a-3}(2a-1)(2a-3)}\,{}_2F_1\left[\begin{matrix}a, a-3/2\\ 2a+5/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] \tag{4.15}$$
$$= \sum_{i=0}^{9} F_i(z)\sum_{n=0}^{\infty}\frac{(5/2-a)_n}{n!(4a-6+n+j)(4a-5+n+j)}(-1)^n(-\frac{z}{1-z})^{n+i}{}_3F_2\left[\begin{matrix}4a+n+i-6, 6a, 4a+1/6\\ 4a+s+n+j-4, 8a+1/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$

where coefficient $F_i(z)$ *are given in Tables 7.*

i	$c_i(z)$
0	$1048576z^3(1-z)$
1	$1048576z^2(1-z)(1+4z)$
2	$65536z(1-z)(101z^2+48z+16)$
3	$65536z(1-z)(79z^3+53z^2+32z+16)$
4	$-256z^5-2060288z^4+159744z^3+1769472z^2-3014656z+3145728$
5	$-512z^5-385280z^4-364544z^3+2715648z^2-5242880z+3276800$
6	$-288z^5-29952z^4-220416z^3+1425408z^2-2486272z+1310720$
7	$-32z^5-1056z^4-24576z^3+156416z^2-266240z+135168$
8	$-z^5-32z^4-768z^3+4864z^2-8192z+4096$
9	$-z^4$

Table 7. Coefficients $F_i(z)$ of (4.12)

If $a = 5/2$ *in (4.15), we obtain*

$$536870912\frac{z^3(1-z)^6}{(64-96z+30z^2+z^3)^2}\,{}_2F_1\left[\begin{matrix}1, 5/2\\ 35/6\end{matrix}\middle| -\frac{108z^4(1-z)}{(64-96z+30z^2+z^3)^2}\right] \tag{4.16}$$
$$= -\frac{262144}{5}z^3(z-1)\,{}_3F_2\left[\begin{matrix}4, 15, 61/6\\ 6, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$+\frac{524288}{15}\,z^3(4z+1)\,{}_3F_2\left[\begin{matrix}5, 15/2, 61/6\\ 7, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$+\frac{32768}{21}\frac{z^3(101z^2+48z+16)}{1-z}\,{}_3F_2\left[\begin{matrix}6, 15/2, 61/6\\ 8, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$-\frac{8192}{7}\frac{z^3(79z^3+53z^2+32z-16)}{(1-z)^3}\,{}_3F_2\left[\begin{matrix}7, 15/2, 61/6\\ 9, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$+(-\frac{32}{9}z^5-\frac{257536}{9}z^4+\frac{6656}{3}z^3+24576z^2-\frac{376832}{9}z+\frac{131072}{3})$$
$$(\frac{z}{1-z})^4\,{}_3F_2\left[\begin{matrix}8, 15/2, 61/6\\ 10, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$+(-\frac{256}{45}z^5-\frac{38528}{9}z^4-\frac{182272}{45}z^3+\frac{452608}{9}z^2-\frac{524288}{9}z+\frac{327680}{9})$$
$$(\frac{z}{1-z})^5\,{}_3F_2\left[\begin{matrix}9, 15/2, 61/6\\ 11, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$
$$+(-\frac{144}{55}z^5-\frac{14976}{55}z^4-\frac{110208}{55}z^3+\frac{712704}{55}z^2-\frac{1243136}{55}z+\frac{131072}{11})$$
$$(\frac{z}{1-z})^6\,{}_3F_2\left[\begin{matrix}10, 15/2, 61/6\\ 12, 61/3\end{matrix}\middle| -\frac{z}{1-z}\right]$$

$$+(-\frac{8}{33}z^5-8z^4-\frac{2048}{11}z^3+\frac{39104}{33}z^2-\frac{66560}{33}z+1024)(\frac{z}{1-z})^7\ {}_3F_2\left[\begin{matrix}11,15/2,61/6\\13,61/3\end{matrix}\middle|-\frac{z}{1-z}\right]$$
$$+(-\frac{1}{156}z^5-\frac{8}{39}z^4-\frac{64}{13}z^3+\frac{1216}{39}z^2-\frac{2048}{39}z+\frac{1024}{39})(\frac{z}{1-z})^8\ {}_3F_2\left[\begin{matrix}12,15/2,61/6\\14,61/3\end{matrix}\middle|-\frac{z}{1-z}\right]$$
$$-\frac{1}{182}\frac{z^{13}}{(1-z)^9}\ {}_2F_1\left[\begin{matrix}13,61/6\\61/3\end{matrix}\middle|-\frac{z}{1-z}\right].$$

Case 7. *([7, Eq.135, p. 142]; (3.5); $g(z)=z(1-z)/(1+30z-96z^2+64z^3)^2$, $h(z)=z(1-z)$)*

$$(1+30z-96z^2+64z^3)^{4-2a}{}_3F_2\left[\begin{matrix}a-2,a+1/2\\2a+5/6\end{matrix}\middle|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right]={}_3F_2\left[\begin{matrix}3a,1/3-a,a-2\\a,2a+5/6\end{matrix}\middle|4z(1-z)\right] \tag{4.17}$$
$$-256\frac{(a-2)}{a}z^2(1-z)^2(4z-3)(4z-1){}_3F_2\left[\begin{matrix}3a,1/3-a,a-1\\a+1,2a+5/6\end{matrix}\middle|4z(1-z)\right]$$
$$-256\frac{(a-1)(a-2)}{a(a+1)}z^2(1-z)^2{}_3F_2\left[\begin{matrix}3a,1/3-a,a\\a+2,2a+5/6\end{matrix}\middle|4z(1-z)\right]$$
$$+(-3145728z^4+6291456z^3-3735552z^2+589824z+8192)$$
$$\frac{(a-1)(a-2)}{(a+2)(a+1)}z^3(1-z)^3{}_3F_2\left[\begin{matrix}3a,1/3-a,a+1\\a+3,2a+5/6\end{matrix}\middle|4z(1-z)\right]-1048576(32z^2-32z+9)$$
$$\frac{(a-1)(a-2)}{(a+3)(a+2)}z^5(1-z)^5{}_3F_2\left[\begin{matrix}3a,1/3-a,a+2\\a+4,2a+5/6\end{matrix}\middle|4z(1-z)\right]$$
$$+33554432\frac{(a-1)(a-2)}{(a+4)(a+3)}z^6(1-z)^6{}_3F_2\left[\begin{matrix}3a,1/3-a,a+3\\a+5,2a+5/6\end{matrix}\middle|4z(1-z)\right].$$

If $a=1/2\pm\sqrt{3}/4$ in (4.17), we have

$$(27/4)^{2-a}\frac{\Gamma(2a+5/6)\Gamma(7/3)}{\Gamma(a+17/6)\Gamma(a+1/3)}={}_3F_2\left[\begin{matrix}3a,1/3-a,a-2\\a,2a+5/6\end{matrix}\middle|\frac{1}{4}\right]-2\frac{(a-2)}{a}{}_3F_2\left[\begin{matrix}3a,1/3-a,a-1\\a+1,2a+5/6\end{matrix}\middle|\frac{1}{4}\right] \tag{4.18}$$
$$-2\frac{(a-1)(a-2)}{a(a+1)}{}_3F_2\left[\begin{matrix}3a,1/3-a,a\\a+2,2a+5/6\end{matrix}\middle|\frac{1}{4}\right]+8\frac{(a-1)(a-2)}{(a+2)(a+1)}{}_3F_2\left[\begin{matrix}3a,1/3-a,a+1\\a+3,2a+5/6\end{matrix}\middle|\frac{1}{4}\right]$$
$$-7\frac{(a-1)(a-2)}{(a+3)(a+2)}{}_3F_2\left[\begin{matrix}3a,1/3-a,a+2\\a+4,2a+5/6\end{matrix}\middle|\frac{1}{4}\right]+2\frac{(a-1)(a-2)}{(a+4)(a+3)}{}_3F_2\left[\begin{matrix}3a,1/3-a,a+3\\a+5,2a+5/6\end{matrix}\middle|\frac{1}{4}\right].$$

Case 8. *([7, Eq.135, p. 142]; (3.4); $g(z)=z(1-z)/(1+30z-96z^2+64z^3)^2$, $h(z)=z(1-z)$)*

$$(1+30z-96z^2+64z^3)^{2-2a}{}_3F_2\left[\begin{matrix}a-1,a+1/2\\2a+5/6\end{matrix}\middle|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right]={}_3F_2\left[\begin{matrix}3a,1/3-a,a-1\\a,2a+5/6\end{matrix}\middle|4z(1-z)\right] \tag{4.19}$$
$$-768\frac{(a-1)}{(a+1)}z^2(1-z)^2{}_3F_2\left[\begin{matrix}3a,1/3-a,a+1\\a+2,2a+5/6\end{matrix}\middle|4z(1-z)\right]$$
$$+8192\frac{(a-1)}{(a+2)}z^3(1-z)^3{}_3F_2\left[\begin{matrix}3a,1/3-a,a+2\\a+3,2a+5/6\end{matrix}\middle|4z(1-z)\right].$$

If $a=1/3$ in (4.21), we obtain

$$(1+30z-96z^2+64z^3)^{2-2a}{}_2F_1\left[\begin{matrix}-2/3,5/6\\3/2\end{matrix}\middle|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] \tag{4.20}$$
$$=1+384z^2-\frac{21760}{7}z^3+\frac{51840}{7}z^4-\frac{49152}{7}z^5+\frac{16384}{7}z^6.$$

If $a = 1/2 \pm \sqrt{3}/4$ in (4.21), we have

$$3^{\frac{5}{2}-3a}2^{4-\frac{1}{3}}\frac{\Gamma(\pi)\Gamma(2a+\frac{5}{6})}{(6a+5)\Gamma(2a+\frac{2}{3}\Gamma(\frac{2}{3}))} = {}_3F_2\left[\begin{matrix}3a, 1/3-a, a-1\\ a, 2a+5/6\end{matrix}\middle|\frac{1}{4}\right] - 48\frac{(a-1)}{(a+1)}{}_3F_2\left[\begin{matrix}3a, 1/3-a, a+1\\ a+2, 2a+5/6\end{matrix}\middle|\frac{1}{4}\right]$$
$$+ 128\frac{(a-1)}{(a+2)}{}_3F_2\left[\begin{matrix}3a, 1/3-a, a+2\\ a+3, 2a+5/6\end{matrix}\middle|\frac{1}{4}\right].$$

Putting $u = z(1-z)$ in (4.20) and applying the operator ${}_uO_\beta^\alpha$ on the both sides, we obtain

$$\sum_{n=0}^{\infty}\frac{(-2/3)_n(5/6)_n(\alpha)_n}{(3/2)_n(\beta)_n}108^n\frac{u^n}{n!}F_1(\alpha+n; -2/3+n, -4/3+2n; \beta+n; 4u, -32u) \tag{4.21}$$
$$= 1 + 384\frac{\alpha(\alpha+1)}{\beta(\beta+1)}u^2 - 4096\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}u^3$$

where $F_1(\alpha, \alpha', \beta, \beta'; \gamma; x, y)$ represent the first Appell function of two complex variables ([3], Vol. 1, Eq. 8, p. 224).

Case 9. *([7, Eq.135, p. 142]; (3.5); $g(z) = z(1-z)/(1+30z-96z^2+64z^3)^2$, $h(z) = z(1-z)$)*

$$(1+30z-96z^2+64z^3)^{4-2a}{}_3F_2\left[\begin{matrix}a-2, a+1/2\\ 2a+5/6\end{matrix}\middle|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] = {}_3F_2\left[\begin{matrix}3a, 1/3-a, a-2\\ a, 2a+5/6\end{matrix}\middle|4z(1-z)\right] \tag{4.22}$$
$$-256\frac{(a-2)}{a}z^2(1-z)^2(4z-3)(4z-1){}_3F_2\left[\begin{matrix}3a, 1/3-a, a-1\\ a+1, 2a+5/6\end{matrix}\middle|4z(1-z)\right]$$
$$-256\frac{(a-1)(a-2)}{a(a+1)}z^2(1-z)^2{}_3F_2\left[\begin{matrix}3a, 1/3-a, a\\ a+2, 2a+5/6\end{matrix}\middle|4z(1-z)\right]$$
$$+(-3145728z^4+6291456z^3-3735552z^2+589824z+8192)$$
$$\frac{(a-1)(a-2)}{(a+2)(a+1)}z^3(1-z)^3{}_3F_2\left[\begin{matrix}3a, 1/3-a, a+1\\ a+3, 2a+5/6\end{matrix}\middle|4z(1-z)\right]$$
$$-1048576(32z^2-32z+9)\frac{(a-1)(a-2)}{(a+3)(a+2)}z^5(1-z)^5{}_3F_2\left[\begin{matrix}3a, 1/3-a, a+2\\ a+4, 2a+5/6\end{matrix}\middle|4z(1-z)\right]$$
$$+33554432\frac{(a-1)(a-2)}{(a+4)(a+3)}z^6(1-z)^6{}_3F_2\left[\begin{matrix}3a, 1/3-a, a+3\\ a+5, 2a+5/6\end{matrix}\middle|4z(1-z)\right].$$

If $a = 1/2 \pm \sqrt{3}/4$ in (4.22), we have

$$(27/4)^{2-a}\frac{\Gamma(2a+5/6)\Gamma(7/3)}{\Gamma(a+17/6)\Gamma(a+1/3)} = {}_3F_2\left[\begin{matrix}3a, 1/3-a, a-2\\ a, 2a+5/6\end{matrix}\middle|\frac{1}{4}\right] - 2\frac{(a-2)}{a}{}_3F_2\left[\begin{matrix}3a, 1/3-a, a-1\\ a+1, 2a+5/6\end{matrix}\middle|\frac{1}{4}\right] \tag{4.23}$$
$$-2\frac{(a-1)(a-2)}{a(a+1)}{}_3F_2\left[\begin{matrix}3a, 1/3-a, a\\ a+2, 2a+5/6\end{matrix}\middle|\frac{1}{4}\right] + 8\frac{(a-1)(a-2)}{(a+2)(a+1)}{}_3F_2\left[\begin{matrix}3a, 1/3-a, a+1\\ a+3, 2a+5/6\end{matrix}\middle|\frac{1}{4}\right]$$
$$-7\frac{(a-1)(a-2)}{(a+3)(a+2)}{}_3F_2\left[\begin{matrix}3a, 1/3-a, a+2\\ a+4, 2a+5/6\end{matrix}\middle|\frac{1}{4}\right] + 2\frac{(a-1)(a-2)}{(a+4)(a+3)}{}_3F_2\left[\begin{matrix}3a, 1/3-a, a+3\\ a+5, 2a+5/6\end{matrix}\middle|\frac{1}{4}\right].$$

If $a = 1/3$ in (4.22), we obtain

$$(1+30z-96z^2+64z^3)^{10/3}{}_2F_1\left[\begin{matrix}-5/3, 5/6\\ 3/2\end{matrix}\middle|\frac{108z(1-z)}{(1+30z-96z^2+64z^3)^2}\right] \tag{4.24}$$
$$= 1 + 1920z^2 - \frac{78080}{7}z^3 + \frac{1641600}{7}z^4 - 3336192z^5 + \frac{1911367680}{91}z^6$$
$$-\frac{6352404480}{91}z^7 + \frac{1762099200}{13}z^8 - \frac{14546370560}{91}z^9$$

$$+\frac{10291249152}{91})z^{10}-\frac{4026531840}{91}z^{11}+\frac{671088640}{91}z^{12}.$$

Putting $u = z(1-z)$ in (4.20) and applying the operator ${}_uO^{\alpha}_{\beta}$ on the both sides, we obtain

$$\sum_{n=0}^{\infty}\frac{(-5/3)_n(5/6)_n(\alpha)_n}{(3/2)_n(\beta)_n}108^n\frac{u^n}{n!}F_1(\alpha+n;-5/3+n,-10/3+2n;\beta+n;4u,-32u) \tag{4.25}$$
$$=1+1920\frac{\alpha(\alpha+1)}{\beta(\beta+1)}u^2-\frac{51200}{7}\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}u^3$$
$$+\frac{1474560}{7}\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{\beta(\beta+1)(\beta+2)(\beta+3)}u^4$$
$$-\frac{17301504}{7}\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}{\beta(\beta+1)(\beta+2)(\beta+3)(\beta+4)}u^5$$
$$+\frac{671088640}{91}\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)}{\beta(\beta+1)(\beta+2)(\beta+3)(\beta+4)(\beta+5)}u^6$$

where $F_1(\alpha,\alpha',\beta,\beta';\gamma;x,y)$ is the first Appell function of two variables x and y ([3], Vol. 1, p. 224, Eq(6)).

5. Conclusion

In this article we give a list of twelve new transformations involving the Gaussian hypergeometric function with higher level rational arguments. They are deduced from two known transformation formulas of Gaussian hypergeometric function chosen in Goursat's thesis cite Goursat1881. These formulas are added to those of two recent articles [25, 24]. The discovery of these transformation formulas was possible by applying a systematic method developed in [25] involving the well-posed operator ${}_{g(z)}O^{\alpha}_{\beta}$ (introduced by the author [23]) defined with the fractional derivative and its representation using the Pochhammer contour integral.

Applications in section 2 clearly demonstrates the utility and efficiency of the operator ${}_zO^{\alpha}_{\beta}$. It is certainly a powerful tool for systematically obtain new formulas involving hypergeometric functions as well as the special functions of mathematical physics. In future work we will explore other applications of this fractional operator, in particular the search for additional summation theorems for hypergeometric functions.

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