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Some Invariant Solutions of Two Dimensional Heat Equation

Mersaid Aripov [a], Otabek Narmanov[b]

[a]*Department of Applied mathematics and computer analysis, National University of Uzbekistan, Tashkent, 100174, Tashkent, Uzbekistan*
[b]*Department of Mathematical Modelling and algorithms,Tashkent university of information technologies, Tashkent, 100200, Tashkent, Uzbekistan*

Abstract

The symmetry group of a differential equation is the group of transformations which transform solutions of the differential equation to solutions. For systems of partial differential equations, the symmetry group can be used to explicitly find particular types of solutions that are themselves invariant with respect to some subgroup of the full group of symmetries of the system. Group analysis methods are widely used to study partial differential equations and to integrate ordinary differential equations.

The heat equation is a certain partial differential equation and the most widely studied equation in pure mathematics, and its analysis is regarded as fundamental to the broader field of partial differential equations.

In the paper we find solutions of the two-dimensional heat equation which are invariant with respect some symmetry groups and show that some solutions can be found using the well-known Bessel functions.

Keywords: Heat equation, Symmetry group, Invariant solution, Lie group

2010 MSC: 35K05, 35K57, 35K10

1. Introduction

Numerous studies [1] - [14] have been devoted to finding symmetry groups of differential equations and their applications for research. Study of the group-theoretical properties of differential equations associated with the name of the outstanding Norwegian mathematician Sophus Lie (1842-1899). He created group theory of continuous transformations. This theory arose at the junction three major mathematical disciplines - the theory of differential equations, algebra and differential geometry [6], [7]. Further development of the theory of groups of continuous transformations is associated with the name of L.V. Ovsyannikov. Note that the term group analysis was also introduced by L.V. Ovsyannikov. Many researches are devoted to the study of symmetry groups, such as [6], [7], [11], [12], [13] in which the foundations of this theory are laid and modern methods are developed. In the paper [12] it is discussed the group classification of differential equations and their solutions. In the paper [11] it is developed a computational method that explicitly defines the complete symmetry group of an arbitrary partial differential equation.

The solutions of quasilinear equations of parabolic type that describe the processes of heat conduction and combustion in continuous nonlinear media, which grow infinitely over a finite time, are considered in classical monograph [13]. Also in this book special methods for studying nonlinear parabolic equations are presented.

This paper is a continuation of papers [9], [10] and it is devoted to finding invariant solutions of the two-dimensional heat equation, which are invariant with respect some symmetry groups of the equation. Some invariant

†*Article ID:* MTJPAM-D-20-00060
Email addresses: mirsaidaripov@mail.ru (Mersaid Aripov), otabek.narmanov@mail.ru (Otabek Narmanov)
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⋆*Corresponding Author:* Mersaid Aripov

solutions the two-dimensional heat equation were found in the papers [9], [10], where considered the case when the thermal conductivity coefficients are power functions of temperature.

In this paper the two-dimensional heat equation is considered with the thermal conductivity coefficients which are exponential functions of temperature and there is a heat source in the equation. In this case arise symmetry groups which are different from the case considered in papers [9], [10] and respectively we get different class of invariant solutions. The class of solutions that are invariant with respect to the group of solutions includes exact solutions that have immediate mathematical or physical values. It is found solutions with using well known Bessel functions.

Let us be given a differential equation of order m

$$\Delta(x, u^{(m)}) = 0 \tag{1.1}$$

from n independent $x = (x^1, x^2, ..., x^n)$ and q dependent variables $u = (u^1, u^2, ..., u^q)$, containing derivatives of u with respect to x to order m.

Definition 1.1. A group G of transformations acting on the set M of the space of independent and dependent variables of a differential equation is called the symmetry group of the equation (1.1) if for each solution $u = f(x)$ of the equation (1.1) and for $g \in G$ such that $g \circ f$, is defined, then the function $\tilde{u} = g \circ f$ is also a solution of the equation (1.1).

One of the advantages of knowing the symmetry group of differential equations is that if we know the solution $u = f(x)$, then according to the definition of symmetry group the function $\tilde{u} = g \circ f$ is also a solution for any element g of the group G, so that we have the opportunity to build a whole family of solutions, exposing the known solution to the action of all possible elements of the group.

In this paper, we find invariant, with respect to symmetry groups, solutions of the two-dimensional heat equation using the Bessel functions of the first and second kind. Bessel functions satisfy the Bessel equation. The Bessel equation arises in many problems of mathematical physics in cylindrical and spherical coordinates. Therefore, Bessel functions are used in solving many problems of wave propagation, static potentials. Bessel functions, first defined by the famous mathematician Daniel Bernoulli and then generalized by Friedrich Bessel, are canonical solutions of Bessel's differential equation.

2. Invariant solutions of two dimensional heat equation

Let us consider two dimensional heat equation

$$u_t = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(k_i(u)\frac{\partial u}{\partial x_i}) + Q(u) \tag{2.1}$$

where $u = u(x_1, x_2, t) \geq 0$ is the temperature function, $k_i(u) \geq 0$, $Q(u)$ are functions on temperature u. Function $Q(u)$ describes the heat release process if $Q(u) > 0$ and the heat absorption process if $Q(u) < 0$.

Consider the case when the thermal conductivity coefficients $k_1(u), k_2(u)$ in equation (2.1) are exponential functions of temperature i.e. they have the form $k_1(u) = k_2(u) = exp(u)$.

Suppose that $Q(u) = \pm exp(\alpha u)$, where α is a real number.

In this case, equation (2.1) has the following form:

$$u_t = exp(u)\Delta u + exp(u)(\nabla u)^2 \pm exp(\alpha u) \tag{2.2}$$

where $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$ — Laplace operator,$\nabla u = \{\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\}$ —gradient of the function u.

Suppose that $\alpha \neq 0$. It was shown in [2] that the following vector field is infinitesimal generator of the symmetry group of equation (2.2):

$$X = 2\alpha t\frac{\partial}{\partial t} - (\alpha - 1)x_1\frac{\partial}{\partial x_1} - (\alpha - 1)x_2\frac{\partial}{\partial x_2} - 2\frac{\partial}{\partial u}.$$

This means that the flow of this vector field X generates a group of transformations of the space of variables (t, x_1, x_2, u), whose elements translate the solutions of equation (2.2) into its solutions. The flow of the vector field X generates the following group of transformations

$$(t, x_i, u) \to (te^{2\alpha s}, x_i e^{(\alpha-1)s}, u - 2s), s \in R. \tag{2.3}$$

We find solutions of equation (2.2) that are invariant with respect to transformation groups (2.3). To do this, we first find the invariant functions of these transformations.

It is known that [11] the smooth function $f : M \to R$ is an invariant function of the transformation group G, acting on the manifold M if and only if $Xf = 0$ for each infinitesimal generator X groups G.

Using this criterion, we find that the functions

$$I = e^{\frac{u}{2}} t^{\frac{1}{2\alpha}}, \xi = \frac{x_1 - x_2}{t^{\beta}},$$

where $\beta = \frac{\alpha-1}{2\alpha}$, are invariant functions of the transformation group (2.3), which follows from the following equalities $X_1(I) = 0$, $X_1(\xi) = 0$.

We will seek the solution of equation (2.2) in the form

$$u = \ln \frac{V(\xi)}{t^{\frac{1}{\alpha}}}. \quad (2.4)$$

Substituting function (2.4) into equation (2.2), we obtain the following ordinary differential equation for the function V :

$$2V'' + \beta\xi \frac{V'}{V} \pm V^{\alpha} + \frac{1}{\alpha} = 0. \quad (2.5)$$

In the general case, equation (2.5) equation does not integrate. We need to investigate by numerical methods. For example if $\alpha = -1$, then $\beta = 1$, and equation (2.5) has followng form

$$2\xi V \frac{d^2V}{d\xi^2} + \xi \frac{dV}{d\xi} - V \pm 1 = 0. \quad (2.6)$$

As shown by numerical studies of the equation (2.6) by the program Maple, the function $V(\xi)$ increases unlimitedly when $\xi \to \infty$ (Fig-1).

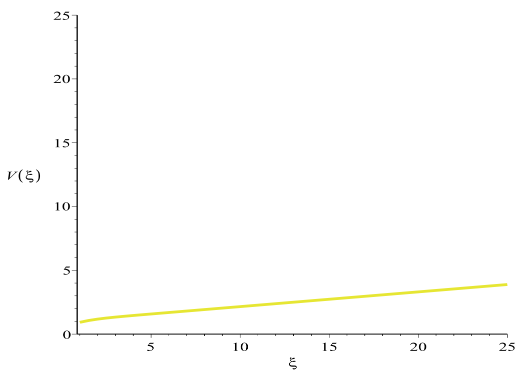

Figure 1. Solution of equation (2.6)

As follows from the formula (2.4) of solution of equation (2.2) temperature function $u = u(x_1, x_2, t)$ increases unlimitedly at $t \to \infty$.

In the case when $\alpha = 1$ equation (2.5) has following form

$$2\frac{d^2V}{d\xi^2} \pm V + 1 = 0. \quad (2.7)$$

In the case when $Q(u) = -exp(\alpha u)$, making the change $p(V) = \frac{dV}{d\xi}$ in the equation (2.7) we obtain linearly a first order equation

$$2p\frac{dp}{dV} - V + 1 = 0.$$

Solving this equation, we find that

$$p = \frac{1}{\sqrt{2}}\sqrt{V^2 - 2V + C_1}.$$

Now from the equation

$$\frac{dV}{d\xi} = \frac{1}{\sqrt{2}}\sqrt{V^2 - 2V + C_1}$$

find that

$$V - 1 + \sqrt{V^2 - 2V + C_1} = C_2 e^{\frac{\xi}{\sqrt{2}}},$$

where C_1, C_2 are arbitrary constants. If $C_1 = 1$, then the function can be written in explicit form:

$$V = C_2 e^{\frac{\xi}{\sqrt{2}}} + 1$$

Thus, in the case when $\alpha = 1$ the invariant solution of equation (2.2) has the following form

$$u = \ln \frac{V(\xi)}{t} \tag{2.8}$$

Since there is a heat absorption source in the equation, it follows from the formula of solution (2.8) that for $0 < t \leq 1$ the temperature at each point on the plane increases. Starting from $t \geq 1$ the temperature decreases and tends to $u = \ln V(\xi)$ at $t \to \infty$.

In the case when $Q(u) = exp(\alpha u)$, making the change $p(V) = \frac{dV}{d\xi}$ in the equation (2.7) we obtain linearly a first order equation

$$2p\frac{dp}{dV} + V + 1 = 0.$$

Solving this equation, we find that

$$p = \frac{1}{\sqrt{2}}\sqrt{C_1 - V^2 - 2V}.$$

Now from the equation

$$\frac{dV}{d\xi} = \frac{1}{\sqrt{2}}\sqrt{C_1 - V^2 - 2V}$$

find that

$$V + 1 - \sqrt{V^2 - 2V + C_1} = C_2 e^{\frac{\xi}{\sqrt{2}}},$$

where C_1, C_2 are arbitrary constants.

The paper [10] shows that if $Q(u) = exp(u)$, a solution of the equation can be found using Bessel functions.

Recall a solution of a differential equation

$$x^2 y''(x) + xy'(x) + (x^2 - v^2)y(x) = 0$$

is called the Bessel function of the index v.

A large number of the most diverse problems associated with almost all the most important sections of mathematical physics, and those that are designed to answer topical technical questions, are associated with the use of Bessel functions. Bessel functions are widely used in solving problems of acoustics, radiophysics, hydrodynamics, problems of atomic and nuclear physics. There are many applications of Bessel functions to the theory of thermal conductivity.

For integer v, the solution of the Bessel equation is given as a linear combination of Bessel functions of the first and second kind.

Bessel function of the first kind is equal to

$$J(v, x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+v+1)}\{\frac{x}{2}\}^{2k+v}$$

Here $\Gamma(s)$ is so called gamma function which is given for complex number s by the formula

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

for $Re(s) > 0$. From the above equation, for any positive integer n, we note that $\Gamma(n) = (n-1)!$ [5].

Bessel function of the second kind is given by the formula

$$Y(\nu, x) = \coth(\pi\nu) J(\nu, x) + \frac{1}{\sin(\pi\nu)} J(-\nu, x).$$

In the case $\alpha = 1$, the equality $\beta = 0$ holds, therefore the function ξ does not depend on t. It can be any function from x_1, x_2. In this case function $\xi = \frac{x_1^2+x_2^2}{4}$ also is a invariant function for the group of transformations (2.3). If we look for a solution in the form (2.4), where $\xi = \frac{x_1^2+x_2^2}{4}$, then for the function $V(\xi)$ we obtain the following differential equation for the $Q(u) = exp(\alpha u)$:

$$\xi \frac{d^2V}{d\xi^2} + \frac{dV}{d\xi} + V + 1 = 0. \tag{2.9}$$

This case was considered in the paper [9]. It was shown in the paper [9] the general solution of equation (2.9) is given by the function

$$V(\xi) = C_1 J(0, 2\sqrt{\xi}) + C_2 Y(0, \sqrt{\xi}) + 1,$$

where C_1, C_2 are arbitrary constants. Here J, Y are the Bessel functions of the first and second type, respectively.

Let us consider the zero-index Bessel equation i.e. $\nu = 0$:

$$x^2 y''(x) + xy'(x) + x^2 y(x) = 0.$$

If in this equation we make the replacement of an independent variable by the formula $x = 2\sqrt{\xi}$, then we get the following equation

$$\xi y''(\xi) + y'(\xi) + y(\xi) = 0. \tag{2.10}$$

If in equation (2.9) we make the replacement of the dependent variable by the formula $y = V + 1$,, we obtain equation (2.10).Therefore general solution of the equation can be obtained by using Bessel functions.

In the case of $Q(u) = -exp(\alpha u)$ for $\alpha = 1$, if we look for solutions (2.2) in the form (2.4) where $\xi = \frac{x_1^2+x_2^2}{4}$, then we get the following second-order differential equation

$$\xi \frac{d^2V}{d\xi^2} + \frac{dV}{d\xi} - V + 1 = 0. \tag{2.11}$$

The general solution of equation (2.11) is given by modified Bessel functions. The modified Bessel functions satisfy the modified Bessel equation:

$$x^2 y''(x) + xy'(x) - (x^2 + \nu^2) y(x) = 0.$$

Consider the modified Bessel equation of zero index, i.e. for $\nu = 0$:

$$x^2 y''(x) + xy'(x) - x^2 y(x) = 0.$$

If in this equation we make the replacement of an independent variable by the formula $x = 2\sqrt{\xi}$, then we get the following equation

$$\xi y''(\xi) + y'(\xi) - y(\xi) = 0. \tag{2.12}$$

If in equation (2.11) we make the replacement of the dependent variable by the formula $y = V - 1$, then we get equation (2.12).

Therefore, the general solution of equation (2.11) is given as a linear combination of modified Bessel functions:

$$V(\xi) = C_1 I(0, 2\sqrt{\xi}) + C_2 K(0, \sqrt{\xi}) + 1,$$

where C_1, C_2 are arbitrary constants. Here I, K are modified Bessel functions of the first and second kind, respectively.

The modified Bessel function of the first kind is determined by the formula

$$I(v,x) = e^{\frac{iv\pi}{2}} J(v, xe^{\frac{i\pi}{2}})$$

The modified Bessel function of the second kind is determined by the formula

$$K(v,x) = \frac{\pi}{2sin\pi v}[I(-v,x) - I(v,x)]$$

We show one more family of invariant solutions of equation (2.2) for $\alpha = 1$. For this, we seek a solution in the form (2.4), where

$$\xi = \sqrt{x_1^2 + x_2^2}.$$

In this case, substituting function (2.4) in equation (2.2) we obtain the following ordinary differential equation second order for the function $V(\xi)$:

$$\frac{d^2V}{d\xi^2} + \frac{1}{\xi}\frac{dV}{d\xi} - V + 1 = 0. \tag{2.13}$$

Multiplying this equation by ξ^2 we get the following equation

$$\xi^2\frac{d^2V}{d\xi^2} + \xi\frac{dV}{d\xi} - \xi^2 V = -\xi^2. \tag{2.14}$$

Recall that the equation

$$x^2y''(x) + xy'(x) + (x^2 - v^2)y(x) = f(x)$$

called the inhomogeneous Bessel equation. Equation (2.14) is a modified inhomogeneous Bessel equation with $v = 0, f(\xi) = -\xi^2$. It is known that the solution of equation (2.14) has the following form [5]

$$V(\xi) = \frac{\pi}{2}[I(0,\xi)\int_0^\xi K(0,t)dt - K(0,\xi)\int_0^\xi I(0,t)dt]$$

Thus, the invariant solution of equation (2.2) is expressed through the solution of the inhomogeneous Bessel equation.

Now consider the general case of $\alpha \neq 0$. For the transformation group (2.3), the functions

$$I = e^{\frac{u}{2}} t^{\frac{1}{2\alpha}}, \xi = \frac{\sqrt{x_1^2 + x_2^2}}{t^\beta},$$

where $\beta = \frac{\alpha-1}{2\alpha}$, are invariant functions which follows from the following equalities $X(I) = 0$, $X(\xi) = 0$. In this case, we seek the solution of equation (2.2) in the form

$$u = \ln\frac{V(\xi)}{t^{\frac{1}{\alpha}}}, \tag{2.15}$$

where $\xi = \frac{\sqrt{x_1^2+x_2^2}}{t^\beta}$.

Substituting function (2.15) in equation (2.2) we obtain the following ordinary differential equation second order for the function $V(\xi)$:

$$\frac{d^2V}{d\xi^2} + \beta\xi\frac{dV}{d\xi} + \frac{1}{\xi}\frac{dV}{d\xi} \pm V^\alpha + \frac{1}{\alpha} = 0. \tag{2.16}$$

In the general case, equation (2.16) equation does not integrate. Need to be investigated by numerical methods.

For example if $\alpha = -1$, then $\beta = 1$, and equation (2.16) has followng form

$$\xi V\frac{d^2V}{d\xi^2} + \xi^2 V\frac{dV}{d\xi} + V\frac{dV}{d\xi} - \xi V \pm \xi = 0. \tag{2.17}$$

As shown by numerical studies of the equation (2.17) by the program Maple, the function $V(\xi)$ remains limited when $\xi \to \infty$ (Fig-2).

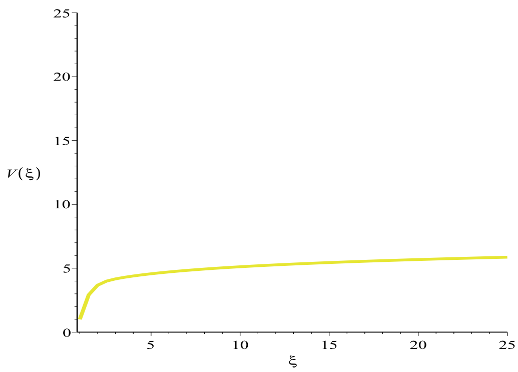

Figure 2. Solution of equation (2.17)

As follows from the formula (2.15) of solution of equation (2.2) temperature function $u = u(x_1, x_2, t)$ increases unlimitedly at $t \to \infty$.

3. The equation has constant source of heat generation

In this section we consider the case when $\alpha = 0$. In this case, equation (2.2) takes the following form:

$$u_t = exp(u)\Delta u + exp(u)(\nabla u)^2 \pm 1. \tag{3.1}$$

Firtsly we consider one of equations (3.1):

$$u_t = exp(u)\Delta u + exp(u)(\nabla u)^2 + 1. \tag{3.2}$$

It was shown in [2] that the following vector field is infinitesimal generator of the symmetry group of equation (3.2):

$$X = \exp(-t)\frac{\partial}{\partial t} + \exp(-t)\frac{\partial}{\partial u}.$$

This means that the flow of this vector field generates a group of transformations of the space of variables (t, x_1, x_2, u), whose elements translate the solutions of equation (3.2) into its solutions.

The flow of the vector field X generates the following group of transformations

$$(t, x_i, u) \to (\ln(e^t + s), x_i, u + \ln(e^t + s) - t). \tag{3.3}$$

We find solutions of equation (3.2) that are invariant with respect to transformation groups (3.3). To do this, we first find the invariant functions of these transformations. For the group of transformations (3.3), the function

$$I = e^{\frac{-u+t}{2}} V(\xi) \tag{3.4}$$

where $\xi = \sqrt{x_1^2 + x_2^2}$, is an invariant function, which follows from the following equality $X(I) = 0$.

The form of the invariant function allows us to search for a solution to equation (3.2) in the form

$$u = \ln(e^t V(\xi)), \tag{3.5}$$

where $\xi = \sqrt{x_1^2 + x_2^2}$.

In this case, substituting function (3.5) in equation (3.2) we obtain the following ordinary differential equation second order for the function $V(\xi)$:

$$\frac{d^2V}{d\xi^2} + \frac{1}{\xi}\frac{dV}{d\xi} = 0. \tag{3.6}$$

Solving this equation, we find that

$$V = C_1 \ln \sqrt{x_1^2 + x_2^2} + C_2, \tag{3.7}$$

where C_1, C_2 are arbitrary constants.

Since in equation (3.2) there is a constant source of heat generation, it follows from the form of solution (3.5) that for $0 < t \leq 1$ the temperature at each point on the plane decreases. Starting from $t \geq 1$ the temperature rises unlimitedly at $t \to \infty$.

Now consider the second equation from equations (3.1):

$$u_t = exp(u)\Delta u + exp(u)(\nabla u)^2 - 1. \tag{3.8}$$

In this case, the vector field

$$Y = \exp(t)\frac{\partial}{\partial t} - \exp(t)\frac{\partial}{\partial u}$$

is infinitesimal generator of the symmetry group of equation (3.8).

The flow of the vector field Y generates the following group of transformations

$$(t, x_i, u) \to (-\ln(e^{-t} - s), x_i, u + \ln(e^{-t} - s) + t). \tag{3.9}$$

For the group of transformations (3.9), the function

$$I = e^{u-t}V(\xi) \tag{3.10}$$

where $\xi = \sqrt{x_1^2 + x_2^2}$, is an invariant function which follows from the following equalities $Y(I) = 0$.

The form of the invariant function allows us to search for a solution of the equation (3.8) in the form

$$u = \ln(e^{-t}V(\xi)), \tag{3.11}$$

where $\xi = \sqrt{x_1^2 + x_2^2}$.

In this case, substituting function (3.11) in equation (3.8), we obtain the ordinary differential equation (3.6) for the function $V(\xi)$. Thus, in this case, the function $V(\xi)$ is also defined by formula:

$$V = C_1 \ln \sqrt{x_1^2 + x_2^2} + C_2, \tag{3.12}$$

where C_1, C_2 are arbitrary constants.

Since in equation (3.8) there is a constant source of heat absorption, it follows from the form of solution (3.11) that for $0 < t \leq 1$ the temperature at each point on the plane increases. Starting from $t \geq 1$ the temperature decreases with $t \to \infty$.

4. The equation has not source of heat generation

In conclusion, we consider the case when $Q = 0$. In this case, equation (2.2) takes the following form:

$$u_t = exp(u)\Delta u + exp(u)(\nabla u)^2. \tag{4.1}$$

It was shown in [2] that the following vector fields are infinitesimal generators of the symmetry group of equation (4.1):

$$X = 2t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2},$$

$$Y = t\frac{\partial}{\partial t} - \frac{\partial}{\partial u}.$$

Solutions of equation (4.1) invariant with respect to the group generated by vector field Y are found in the paper[9].

We will find solutions of equation (4.1) invariant with respect to the group generated by vector field X.

For the group of transformations, generated by vector field X, the function

$$I = e^{-u}V(\xi), \tag{4.2}$$

where $\xi = \frac{x_1^2+x_2^2}{t}$, is an invariant function which follows from the following equalities $X(I) = 0$.

This fact allows us to seek solution of the equation (4.1) in the following form

$$u = \ln V(\xi). \tag{4.3}$$

In this case, substituting function (4.3) in equation (4.1), we obtain following ordinary differential equation for the function $V(\xi)$.

$$4\xi V\frac{d^2V}{d\xi^2} + 4V\frac{dV}{d\xi} + \xi\frac{dV}{d\xi} = 0. \tag{4.4}$$

By putting $V = U(\eta)$,where $\eta = \ln\xi$, we have following equation for U

$$4\xi\frac{d^2U}{d\xi^2} + \frac{1}{U}\frac{dU}{d\xi} = 0. \tag{4.5}$$

Integrating this equation gives first order equation

$$4\frac{dU}{d\xi} - 4U - \ln U = 0. \tag{4.6}$$

Numerical studies of this equation show that the solution of the equation (4.6) (the function U), and therefore the solution of the equation (4.3)(the function $V(\xi)$), are limited at $\xi \to \infty$.

As the function $\xi = \frac{x_1^2+x_2^2}{t} \to 0$ at $t \to \infty$, it follows from the formula (4.3) of solution of equation (4.1) temperature function $u = u(x_1, x_2, t)$ tends to zero at $t \to \infty$. This is explained by the fact that in the process described by equation (4.1) there is no heat source.

5. Conclusion

We considered the two-dimensional heat equation in the case when the heat conductivity coefficients $k_1(u), k_2(u)$ are exponential functions of temperature and the heat source is given by the function $Q(u) = \pm exp(\alpha u)$, where α is a real number. It is shown in the paper that in the case $\alpha = 1$ the invariant solutions are defined by the well-studied Bessel functions. In the case $\alpha = 0$ invariant solutions are found in explicit form.

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