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Cyclic and Noncyclic Geraghty Type Condensing Operators and Optimal Solutions of Nonlocal Integro-Differential Equations

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Abstract

The present work considers a family of cyclic (non-cyclic) relatively Geraghty type condensing functions and primarily aims to study the existence of (coupled) points and pairs of best proximity in Banach spaces. The occurrence of optimal solutions for a system of non-local integro-differential equations is demonstrated as an application. As numerical illustrations, we present the optimal solution of integro-differential systems (type cell growth), which can be optimized by using fractal entropy (the measurement of complexity). The fractal power is playing an important role to state the stability and maximization of solutions.

Keywords: oupled point (pair) of best proximity, Cyclic (non-cyclic) relatively Geraghty condensing operator, Optimal solution, Non-local integro-differential equation, Fractal, Entropy, Fractional calculus

2010 MSC: 53C22, 47H09, 34A12

1. Introduction

It is palpable fact that the fixed point theory has a considerable number of applications in several branches of mathematics, computer science and economics [24, 28]. One of the interesting topic in this branch of study is to exhibit the actuality of solutions of the operator equation $T\omega = \omega$, for the function $T : \Omega \to \mathcal{S}$, where ω is a non-null subset in a metric space $(\mathcal{S}, d)$. A point ω_0 satisfying this equation is called fixed point of T. It is remarkable that the equation $T\omega = \omega$ may not possesses any solution. As in the absence of fixed point, here the problem changes the occurs to exhibit the existence of approximate outcomes of the equation taking the formula $T\omega = \omega$. A point $\omega^\dagger \in \Omega$ is called an approximate solution of the equation $T\omega = \omega$, if $\omega^\dagger$ is in the close proximity of $T\omega^\dagger$. Ky Fan (1969) was first to prove a existence theorem for approximate solution which is stated as follows.

Theorem 1.1. ([15]) *Let $\mathcal{S}$ be a locally convex, Hausdorff topological vector space and Ω be a nonempty, convex and compact subset of $\mathcal{S}$. Let d be a semi-metric induced by a continuous semi-norm defined on $\mathcal{S}$ and $T : \Omega \to \mathcal{S}$ be a continuous function. Then a point $\omega_0 \in \Omega$ occurs such that*

$$d(\omega_0, T\omega_0) = \operatorname{dist}(\{T\omega_0\}, \Omega).$$

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This result served as an motivation for the introduction of points of best proximity. Let $T : \Omega \to \Omega^\dagger$ be a non-self function, where Ω and $\Omega^\dagger$ be two nonempty disjoint subsets of a metric space $(\mathcal{S}, d)$. The approximate solution of $T\omega = \omega$ is a point $\upsilon \in \Omega$ satisfying $d(\upsilon, T\upsilon) = \text{dist}(\Omega, \Omega^\dagger)$. This point υ is then called a *point of best proximity* for the function T.

The present paper, primarily focuses on the investigation of points of best proximity for a class of functions $T : \Omega \cup \Omega^\dagger \to \Omega \cup \Omega^\dagger$. If $T(\Omega) \subseteq \Omega^\dagger$ and $T(\Omega^\dagger) \subseteq \Omega$ then T is called a *cyclic function*. Besides, if $T(\Omega) \subseteq \Omega$ and $T(\Omega^\dagger) \subseteq \Omega^\dagger$, then T is called as a *non-cyclic function*. In case of non-cyclic functions, a pair $(\omega, \upsilon) \in \Omega \times \Omega^\dagger$ is called a *pair of best proximity* for the function T if ω and υ are fixed points of T and estimate the distance between the sets Ω and $\Omega^\dagger$, that is, $\omega = T\omega$, $\upsilon = T\upsilon$ and $d(\omega, \upsilon) = \text{dist}(\Omega, \Omega^\dagger)$.

The study of existence of points and pairs of best proximity for different classes of non-self functions is of utmost interest in functional analysis for nonlinear case so that it has attracted the attention of fixed point theorists ([1, 4, 5, 7, 11]. Eldred et al. [8]) were first to establish the results for occurrence of points (pairs) of best proximity for cyclic (non-cyclic) relatively non-expansive functions. We inform that the function $T : \Omega \cup \Omega^\dagger \to \Omega \cup \Omega^\dagger$ is called relatively non-expansive, if $d(T\omega, T\upsilon) \leq d(\omega, \upsilon)$ for all $(\omega, \upsilon) \in \Omega \times \Omega^\dagger$. Later several works on points of best proximity have appeared in literature, we refer reader [12, 13, 14, 7] and references therein.

Recently Gabeleh and Markin [19] proved the results for the occurrence of the point (pair) of best proximity. They considered a set of *cyclic (as well as non-cyclic) condensing functions* with the help of a concept called measure of non-compactness (MNC) in Banach spaces. Some other works on proving existence of points of best proximity using MNC includes [26] and references therein.

The present work considers a new category of cyclic and non-cyclic condensing functions and focuses on proving the results concerning the points and pairs of best proximity in Banach spaces. In addition, we prove a theorem for coupled point of best proximity with the application of our conclusion. The existence of an optimal solution for a non-local integro-differential system is studied as an application of our main result. As numerical illustrations, we present the optimal outcome of integro-differential systems, which can be optimized by using fractal entropy.

2. Preliminaries

Throughout the paper, we consider the following notation. Let $\mathbb{N}$ and $\mathbb{R}$ denote usual set of all natural and real numbers respectively, whereas $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $\mathbb{R}^+ = [0, +\infty)$. Let $(\mathcal{S}, \|.\|)$ be a real Banach space with zero element $\bar{0}$. The closed ball centered at υ with radius r is denoted by $\mathcal{B}(\upsilon, r)$. While $\overline{\mathcal{Z}}$ and $\overline{conv}(\mathcal{Z})$ denote the closure and the convex closure of a nonempty subset $\mathcal{Z}$ of $\mathcal{S}$. Furthermore, $diam(\mathcal{Z})$ represents diameter of the set $\mathcal{Z}$. Moreover, $\mathfrak{M}_{\mathcal{S}}$ is the family of nonempty bounded subsets of $\mathcal{S}$. Let nonempty bounded closed and convex (*NBCC*) set means nonempty, bounded, closed and convex set.

2.1. Measure of non-compactness

The concept of measure of non-compactness (MNC) authorizes us to choose a category of functions that are more general than that of compact functions. We will use MNC to present our new class of condensing functions.

Definition 2.1 ([6, 3])**.** Let $(\mathcal{S}, d)$ be a complete metric space. A function $\mu : \mathfrak{M}_{\mathcal{S}} \to \mathbb{R}^+$ is said to be a measure of non-compactness (MNC) if the next conditions hold:
(i) $\mu(\Omega) = 0 \Leftrightarrow \Omega$ is relatively compact,
(ii) $\mu(\Omega) = \mu(\overline{\Omega})$ for all $\Omega \in \mathfrak{M}_{\mathcal{S}}$,
(iii) $\mu(\Omega \cup \Omega^\dagger) = \max\{\mu(\Omega), \mu(\Omega^\dagger)\}$ for all $\Omega, \Omega^\dagger \in \mathfrak{M}_{\mathcal{S}}$.

We have trailing properties for an MNC μ on $\mathfrak{M}_{\mathcal{S}}$.
(1) If $\Omega \subseteq \Omega^\dagger \Rightarrow \mu(\Omega) \leq \mu(\Omega^\dagger)$.
(2) $\mu(\Omega \cap \Omega^\dagger) \leq \min\{\mu(\Omega), \mu(\Omega^\dagger)\}$ for each $\Omega, \Omega \in \mathfrak{M}_{\mathcal{S}}$.
(3) For a set Ω, which is finite, $\mu(\Omega) = 0$.
(4) Let $\{\Omega_n\}$ be a descending sequence of *NBCC* subsets of $\mathcal{S}$ such that $\lim_{n\to\infty} \mu(\Omega_n) = 0 \Rightarrow, \emptyset \neq \Omega_\infty = \cap_{n\geq 1}\Omega_n$ is compact.
We say that μ is *invariant based on convex hull* if

$$\mu(\text{con}(\Omega)) = \mu(\Omega), \quad \forall \Omega \in \mathfrak{M}_{\mathcal{S}}.$$

It is to be remembered while reading the present article that, the MNC under consideration is invariant w.r.t. convex hulls.

We recall two famous examples of MNCs ($\mathfrak{K}$ introduced by Kuratowski whereas $\mathfrak{H}$ is introduced by Hausdorff) in the next proposition.

Proposition 2.2. ([6]) *Let $\Omega \in \mathfrak{M}_{\mathcal{S}}$ where $(\mathcal{S}, d)$ is complete metric space. Define*

$$\mathfrak{K}(\Omega) = \inf\{k > 0 \mid \Omega \subseteq \cup_{i=1}^{n} \Omega_i : \ \Omega_i \in \mathfrak{M}_{\mathcal{S}}, \ \mathrm{diam}(\Omega_i) \leq k, \ \forall \ 1 \leq i \leq n < \infty\},$$

$$\mathfrak{H}(\Omega) = \inf\{k > 0 \mid \Omega \subseteq \cup_{i=1}^{n} \mathcal{B}(\omega_i; r_i) : \ \omega_i \in \mathcal{S}, \ r_i \leq k, \ \forall \ 1 \leq i \leq n < \infty\}.$$

Then $\mathfrak{K}$ and $\mathfrak{H}$ are MNC. *Moreover, the following inequality holds for these two MNC's*

$$\mathfrak{H}(\Omega) \leq \mathfrak{K}(\Omega) \leq 2\mathfrak{H}(\Omega), \quad \forall \Omega \in \mathfrak{M}_{\mathcal{S}}.$$

We use the following terminologies from the literature: Let us take two nonempty subsets Ω and $\Omega^{\dagger}$ of a set $\mathcal{S}$. It is assumed that a pair $(\Omega, \Omega^{\dagger})$ has a certain nature, if Ω and $\Omega^{\dagger}$ individually have that nature. For instant, we call a pair $(\Omega, \Omega^{\dagger})$ is closed if and only if Ω and $\Omega^{\dagger}$ both are closed. For the pair $(\Omega, \Omega^{\dagger})$, we formulate

$$\Omega_0 = \{p \in \Omega : \ \exists \, q' \in \Omega^{\dagger} \mid ||p - q'|| = dist(\Omega, \Omega^{\dagger})\},$$

$$\Omega_0^{\dagger} = \{q \in \Omega^{\dagger} : \ \exists \, p' \in \Omega \mid ||p' - q|| = dist(\Omega, \Omega^{\dagger})\}.$$

In Banach space $\mathcal{S}$, the pair $(\Omega_0, \Omega_0^{\dagger})$ is convex and weakly compact if $(\Omega_0, \Omega_0^{\dagger})$ is convex and weakly compact. If $\Omega_0 = \Omega$ and $\Omega_0^{\dagger} = \Omega^{\dagger}$ then the pair $(\Omega_0, \Omega_0^{\dagger})$ in $\mathcal{S}$ is called proximinal.

Eldred et al. in [8] obtained the point (pair) of best proximity results for cyclic (non-cyclic) relatively non-expansive functions in Banach spaces using the concept of proximal normal structure (PNS). In 2017, Gabeleh reviewed PNS and concluded that every compact and convex (nonempty) pair in a Banach space has PNS [17]. With this fact in mind he obtained the following result. Recall that $T : \Omega \cup \Omega^{\dagger} \to \Omega \cup \Omega^{\dagger}$ is compact means $(\overline{T(\Omega)}, \overline{T(\Omega^{\dagger})})$ is compact.

Theorem 2.3. *[17] Let $\mathcal{S}$ be a Banach space and $T : \Omega \cup \Omega^{\dagger} \to \Omega \cup \Omega^{\dagger}$ be a relatively non-expansive cyclic function, then T has a best proximity point provided T is compact and $\Omega_0 \neq \emptyset$.*

Let us revisit the concept of strict convexity of Banach spaces in order to state the result for non-cyclic functions. A Banach space $\mathcal{S}$ is said to have strict convexity if for $\omega, \upsilon, x \in \mathcal{S}$ and $\Lambda > 0$, [30]

$$[||\omega - x|| \leq \Lambda, \ ||\upsilon - x|| \leq \Lambda, \ \omega \neq \upsilon] \Rightarrow \left\| \frac{\omega + \upsilon}{2} - x \right\| < \Lambda$$

holds.

Example 2.4. The Hilbert spaces and L^p spaces $(1 < p < \infty)$ are strictly convex Banach spaces.

Theorem 2.5. *[17] Let $\mathcal{S}$ be a Banach space with strict convexity and $T : \Omega \cup \Omega^{\dagger} \to \Omega \cup \Omega^{\dagger}$ be a relatively non-expansive non-cyclic function, then T has a pair of best proximity, where $(\Omega_0, \Omega_0^{\dagger})$ is a pair of* NBCC *subset in $\mathcal{S}$.*

3. Best proximity point (pair) outcomes

We consider the following class of functions presented in [21] to present a novel category of cyclic (non-cyclic) condensing functions.

Let Υ be the collection of all functions $\beta : \mathbb{R}^+ \to [0, 1)$ satisfying the condition $\beta(t_n) \to 1$ implies $t_n \to 0$.

Now we redefine the concept of ξ-admissibility which is introduced by Rehman et al. in [27], with slight modification as follows.

Definition 3.1. Let $\xi : 2^{\mathcal{S}} \to [0, +\infty)$. A function $T : \mathcal{S} \to \mathcal{S}$ is said to be ξ-admissible if for every $M_1, M_2 \in 2^{\mathcal{S}}$, we have $\xi(M_1 \cup M_2) \geq 1 \implies \xi(\overline{con}(T(M_1) \cup T(M_2))) \geq 1$.

3.1. Results for ξ_μ-condensing function

We enunciate with the introduction of a new class of cyclic (noncyclic) condensing functions in the following definition. In this section, we consider $(\Omega, \Omega^\dagger)$ as a nonempty and convex pair in a Banach space $\mathcal{S}$ and μ an MNC on $\mathcal{S}$.

Definition 3.2. A function $T : \Omega \cup \Omega^\dagger \to \Omega \cup \Omega^\dagger$ is called a cyclic (non-cyclic) relatively ξ_μ-condensing function if for any *NBCC*, T-invariant and proximinal pair $(M_1, M_2) \subseteq (\Omega, \Omega^\dagger)$ with $\mathrm{dist}(M_1, M_2) = \mathrm{dist}(\Omega, \Omega^\dagger)$ and there exists $\xi : 2^X \to [0, +\infty)$ such that

$$\xi(M_1 \cup M_2)\mu(T(M_1) \cup T(M_2)) \le \beta(\mu(M_1 \cup M_2))\mu(M_1 \cup M_2)$$

where $\beta \in \Upsilon$.

We now present our first result.

Theorem 3.3. *Let $(\Omega, \Omega^\dagger)$ be a* NBCC *pair in a Banach space $\mathcal{S}$ such that $\Omega_0 \neq \emptyset$ and $T : \Omega \cup \Omega^\dagger \to \Omega \cup \Omega^\dagger$ be a relatively non-expansive cyclic ξ-admissible, ξ_μ-condensing function. Then T has a best proximity point provided there exist* NBCC *sets $X_0, Y_0 \subseteq \mathcal{S}$ such that $\xi(X_0 \cup Y_0) \ge 1$.*

Proof. From $\Omega_0 \neq \emptyset$, $(\Omega_0, \Omega_0^\dagger) \neq \emptyset$. It is also easy to show that $(\Omega_0, \Omega_0^\dagger)$ is convex, closed, T-invariant and proximinal pair considering conditions on T. For $\omega \in \Omega_0$, there is a $\upsilon \in \Omega_0^\dagger$ such that $\|\omega - \upsilon\| = dist(\Omega_0, \Omega_0^\dagger)$. Since T is relatively non-expansive

$$\|T\omega - T\upsilon\| \le \|\omega - \upsilon\| = dist(\Omega_0, \Omega_0^\dagger),$$

which gives $T\omega \in \Omega_0^\dagger$, that is, $T(\Omega_0) \subseteq \Omega_0^\dagger$. Similarly, $T(\Omega_0^\dagger) \subseteq \Omega_0$ and so T is cyclic on $\Omega_0 \cup \Omega_0^\dagger$.

We start with assumption $\mathcal{U}_0 = \Omega_0$ and $\mathcal{V}_0 = \Omega_0^\dagger$ and define a sequence pair $\{(\mathcal{U}_n, \mathcal{V}_n)\}$ as $\mathcal{U}_n = \overline{conv}(T(\mathcal{U}_{n-1}))$ and $\mathcal{V}_n = \overline{conv}(T(\mathcal{V}_{n-1}))$, $n \ge 1$. We claim that $\mathcal{U}_{n+1} \subseteq \mathcal{V}_n$ and $\mathcal{V}_n \subseteq \mathcal{U}_{n-1}$ for all $n \in \mathbb{N}$. We have $\mathcal{V}_1 = \overline{conv}(T(\mathcal{V}_0)) = \overline{conv}(T\Omega_0^\dagger)) = \overline{conv}(\Omega_0) \subseteq \Omega_0 = \mathcal{U}_0$. Therefore, $T(\mathcal{V}_1) \subseteq T(\mathcal{U}_0)$. So $\mathcal{V}_2 = \overline{conv}(T(\mathcal{V}_1)) \subseteq \overline{conv}(T(\mathcal{U}_0)) = \mathcal{U}_1$. Continuing this pattern, we get $\mathcal{V}_n \subseteq \mathcal{U}_{n-1}$. Similarly, $\mathcal{U}_{n+1} \subseteq \mathcal{V}_n$ for all $n \in \mathbb{N}^*$. Thus $\mathcal{U}_{n+2} \subseteq \mathcal{V}_{n+1} \subseteq \mathcal{U}_n \subseteq \mathcal{V}_{n-1}$ for all $n \in \mathbb{N}^*$. Hence, we get a descending sequence $\{(\mathcal{U}_{2n}, \mathcal{V}_{2n})\}$ of nonempty, closed and convex pairs in $\Omega_0 \times \Omega_0^\dagger$. Moreover, $T(\mathcal{V}_{2n}) \subseteq T(\mathcal{U}_{2n-1}) \subseteq \overline{conv}(T(\mathcal{U}_{2n-1})) = \mathcal{U}_{2n}$ and $T(\mathcal{U}_{2n}) \subseteq T(\mathcal{V}_{2n-1}) \subseteq \overline{conv}(T(\mathcal{V}_{2n-1})) = \mathcal{V}_{2n}$. Therefore for all $n \in \mathbb{N}$, the pair $(\mathcal{U}_{2n}, \mathcal{V}_{2n})$ is T-invariant.

Now if $(\nu, \vartheta) \in \Omega_0 \times \Omega_0^\dagger$ is a proximinal pair then

$$dist(\mathcal{U}_{2n}, \mathcal{V}_{2n}) \le \|T^{2n}\nu - T^{2n}\vartheta\| \le \|\nu - \vartheta\| = dist(\Omega_0, \Omega_0^\dagger).$$

Next, with the help of mathematical induction, we show that the pair $(\mathcal{U}_n, \mathcal{V}_n)$ is proximinal. Obviously for $n = 0$, the pair $(\mathcal{U}_0, \mathcal{V}_0)$ is proximinal. Suppose that $(\mathcal{U}_k, \mathcal{V}_k)$ is proximinal. We show that $(\mathcal{U}_{k+1}, \mathcal{V}_{k+1})$ is also proximinal. Let ω be an arbitrary member in $\mathcal{U}_{k+1} = \overline{conv}(T(\mathcal{U}_k))$. Then it is represented as $\omega = \sum_{l=1}^m \lambda_l T(\omega_l)$ with $\omega_l \in \mathcal{U}_k$, $m \in \mathbb{N}$, $\lambda_l \ge 0$ and $\sum_{l=1}^m \lambda_l = 1$. Due to proximinality of the pair $(\mathcal{U}_k, \mathcal{V}_k)$, there occurs $\upsilon_l \in \mathcal{V}_k$ for $1 \le l \le m$ such that $\|\omega_l - \upsilon_l\| = dist(\mathcal{U}_k, \mathcal{V}_k) = dist(\Omega_0, \Omega_0^\dagger)$. Take $\upsilon = \sum_{l=1}^m \lambda_l T(\upsilon_l)$. Then $\upsilon \in \overline{conv}(T(\mathcal{V}_k)) = \mathcal{V}_{k+1}$ and

$$\|\omega - \upsilon\| = \|\sum_{l=1}^m \lambda_l T(\omega_l) - \sum_{l=1}^m \lambda_l T(\upsilon_l)\| \le \sum_{l=1}^m \lambda_l \|\omega_l - \upsilon_l\| = dist(\Omega_0, \Omega_0^\dagger).$$

This means that the pair $(\mathcal{U}_{k+1}, \mathcal{V}_{k+1})$ is proximinal and hence by induction $(\mathcal{U}_n, \mathcal{V}_n)$ is proximinal for all $n \in \mathbb{N}^*$.
We observe that if for some $j \in \mathbb{N}$ we have $\max\{\mu(\mathcal{U}_{2j}), \mu(\mathcal{V}_{2j})\} = 0$, then $T : \mathcal{U}_{2j} \cup \mathcal{V}_{2j} \to \mathcal{U}_{2j} \cup \mathcal{V}_{2j}$ is a compact cyclic relatively non-expansive function. Hence, the conclusion of theorem follows from Theorem 2.3.
Let us suppose that $\max\{\mu(\mathcal{U}_{2n}), \mu(\mathcal{V}_{2n})\} > 0$ for all $n \in \mathbb{N}$. Since $\xi(\mathcal{U}_0 \cup \mathcal{V}_0) \ge 1$, ξ-admissibility of T implies that $\xi(\mathcal{U}_1 \cup \mathcal{V}_1) \ge 1$. Repeating this process $2n$ times, we get $\beta(\mathcal{U}_{2n} \cup \mathcal{U}_{2n}) \ge 1$, for every $n \ge 0$. Since T is a ξ_μ-condensing function,

$$\begin{aligned}
\mu(\mathcal{U}_{2n+1} \cup \mathcal{V}_{2n+1}) &\le \xi(\mathcal{U}_{2n} \cup \mathcal{V}_{2n})\mu(\mathcal{U}_{2n+1} \cup \mathcal{V}_{2n+1}) \\
&= \xi(\mathcal{U}_{2n} \cup \mathcal{V}_{2n})\mu(\overline{\mathrm{con}}(T(\mathcal{U}_{2n})) \cup \overline{\mathrm{con}}(T(\mathcal{V}_{2n}))) \\
&\le \xi(\mathcal{U}_{2n} \cup \mathcal{V}_{2n})\mu(T(\mathcal{U}_{2n}) \cup T(\mathcal{V}_{2n})) \\
&\le \beta(\mu(\mathcal{U}_{2n} \cup \mathcal{V}_{2n}))\mu(\mathcal{U}_{2n} \cup \mathcal{V}_{2n}). \qquad (3.1)
\end{aligned}$$

From the definition of β, we get
$$\mu(\mathcal{U}_{2n+1} \cup \mathcal{V}_{2n+1}) < \mu(\mathcal{U}_{2n} \cup \mathcal{V}_{2n}).$$
Therefore, $\{\mu(\mathcal{U}_{2n} \cup \mathcal{V}_{2n})\}$ is a descending sequence in $\mathbb{R}^+$.
Also from (3.1), we have
$$\frac{\mu(\mathcal{U}_{2n+1} \cup \mathcal{V}_{2n+1})}{\mu(\mathcal{U}_{2n} \cup \mathcal{V}_{2n})} \leq \beta(\mu(\mathcal{U}_{2n} \cup \mathcal{V}_{2n})) < 1,$$
which yields us $\beta(\mu(\mathcal{U}_{2n} \cup \mathcal{V}_{2n})) \to 1$ as $n \to \infty$. By definition of β we get $\lim_{n\to\infty} \mu(\mathcal{U}_{2n} \cup \mathcal{V}_{2n}) = 0$. This means,
$$\lim_{n\to\infty} \max\{\mu(\mathcal{U}_{2n}), \mu(\mathcal{V}_{2n})\} = 0.$$
Let
$$\mathcal{U}_\infty = \bigcap_{n=0}^{\infty} \mathcal{U}_{2n}, \quad \& \quad \mathcal{V}_\infty = \bigcap_{n=0}^{\infty} \mathcal{V}_{2n}.$$
By the property (4) of MNC, the pair $(\mathcal{U}_\infty, \mathcal{V}_\infty) \subseteq (\Omega, \Omega^\dagger)$ is nonempty, compact and convex with $\text{dist}(\mathcal{U}_\infty, \mathcal{V}_\infty) = \text{dist}(\Omega, \Omega^\dagger)$. Besides, $T : \mathcal{U}_\infty \cup \mathcal{V}_\infty \to \mathcal{U}_\infty \cup \mathcal{V}_\infty$ is a relatively non-expansive and cyclic function. Thus, Theorem 2.3 guarantees that T has a best proximity point. □

Our next result investigates existence of pair of best proximity for non-cyclic case of Theorem 3.3.

Theorem 3.4. *Let $\mathcal{S}$ be a Banach space with strict convexity and Ω and $\Omega^\dagger$ be* NBCC *subsets in $\mathcal{S}$ such that Ω_0 is nonempty. Let $T : \Omega \cup \Omega^\dagger \to \Omega \cup \Omega^\dagger$ be a relatively non-expansive non-cyclic ξ-admissible and ξ_μ-condensing function. Then T have a pair of best proximity if there exist nonempty, closed, bounded, and convex $X_0, Y_0 \subseteq \mathcal{S}$ such that $\xi(X_0 \cup Y_0) \geq 1$.*

Proof. Consider the sequence of pairs $(\mathcal{U}_n, \mathcal{V}_n) \subseteq (\Omega, \Omega^\dagger)$ for $n \in \mathbb{N}^*$ as in Theorem 3.3. Thus (from the proof of Theorem 3.3) $\{(\mathcal{U}_n, \mathcal{V}_n)\}$ is a descending sequence of *NBCC* and proximinal pairs with $\text{dist}(\mathcal{U}_n, \mathcal{V}_n) = \text{dist}(\Omega, \Omega^\dagger)$ which are T-invariant.

Observe that if for some $k \in \mathbb{N}$, $\max\{\mu(\mathcal{U}_k), \mu(\mathcal{V}_k)\} = 0$, then using Theorem 2.5 the result holds.

Now by ξ-admissibility of T and $\xi(\mathcal{U}_0 \cup \mathcal{V}_0) \geq 1$, we get $\beta(\mathcal{U}_n, \mathcal{V}_n) \geq 1$. Suppose that $\max\{\mu(\mathcal{U}_n), \mu(\mathcal{V}_n)\} > 0$ for all $n \in \mathbb{N}^*$. Thus, as discussion follows in the proof of Theorem 3.3, we can prove that $\lim_{n\to\infty} \max\{\mu(\mathcal{U}_n), \mu(\mathcal{V}_n)\} = 0$. Setting
$$\mathcal{U}_\infty = \bigcap_{n=0}^{\infty} \mathcal{U}_n, \quad \& \quad \mathcal{V}_\infty = \bigcap_{n=0}^{\infty} \mathcal{V}_n,$$
we get $(\mathcal{U}_\infty, \mathcal{V}_\infty)$ a nonempty, compact and convex pair with $\text{dist}(\mathcal{U}_\infty, \mathcal{V}_\infty) = \text{dist}(\Omega_0, \Omega_0^\dagger)$. Hence, the existence of a pair of best proximity is ensured by Theorem 2.5 for the function T. □

4. Coupled point of best proximity

First we revisit the concept of *coupled points of best proximity* from [18]. Let $(\Omega, \Omega^\dagger)$ a nonempty pair in a metric space $(\mathcal{S}, d)$. A function $\mathfrak{F} : (\Omega\times\Omega) \cup (\Omega^\dagger\times\Omega^\dagger) \to \Omega \cup \Omega^\dagger$ is a cyclic function if $\mathfrak{F}(\Omega^\dagger\times\Omega^\dagger) \subseteq \Omega$ and $\mathfrak{F}(\Omega\times\Omega) \subseteq \Omega^\dagger$. A point $(\omega, \upsilon) \in (\Omega \times \Omega) \cup (\Omega^\dagger \times \Omega^\dagger)$ is said to be a coupled best proximity point of $\mathfrak{F}$ if
$$d(\omega, \mathfrak{F}(\omega, \upsilon)) = d(\upsilon, \mathfrak{F}(\upsilon, \omega)) = \text{dist}(\Omega_0, \Omega_0^\dagger).$$

The following results will be useful in proof of our results.

Lemma 4.1. ([3]) *Let $\mu_1, \mu_2, \cdots, \mu_n$ are MNCs on the metric spaces $\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_n$, respectively. Let $\kappa : \mathbb{R}^{+^n} \to \mathbb{R}^+$ is a convex function with $\kappa(a_1, a_2, \cdots, a_n) = 0$ if and only if $a_j = 0$ for all $j = 1, 2, \cdots, n$. Then μ defined as*
$$\mu(\mathcal{S}) = \kappa(\mu_1(\mathcal{S}_1), \mu_2(\mathcal{S}_2), \cdots, \mu_n(\mathcal{S}_n)),$$
is a MNC on $\mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$, where $\mathcal{S}_j$ denotes the natural projection of E into $\mathcal{S}_j$ for $j = 1, 2, \cdots, n$.

Lemma 4.2. ([20]) *Let $\mathcal{S} \times \mathcal{S}$ be the product space with the metric*

$$d_\infty((\omega_1, \upsilon_1), (\omega_2, \upsilon_2)) = \max\{d(\omega_1, \omega_2), d(\upsilon_1, \upsilon_2)\}, \quad \forall(\omega_1, \upsilon_1), (\omega_2, \upsilon_2) \in \mathcal{S} \times \mathcal{S}.$$

Then the pair $(\Omega_0, \Omega_0^\dagger)$ is proximinal in $\mathcal{S}$ if and only if $(\Omega \times \Omega, \Omega^\dagger \times \Omega^\dagger)$ is proximinal in $\mathcal{S} \times \mathcal{S}$.

We now state our first result for coupled point of best proximity.

Theorem 4.3. *Let $(\Omega_0, \Omega_0^\dagger)$ be a* NBCC *pair in a Banach space $\mathcal{S}$ such that Ω_0 is nonempty. Let $\mathfrak{F} : (\Omega \times \Omega) \cup (\Omega^\dagger \times \Omega^\dagger) \to \Omega \cup \Omega^\dagger$ is a cyclic function satisfying following conditions.*

(i) For all NBCC*, proximinal and $\mathfrak{F}$-invariant pairs $(\mathcal{S}_1, \mathcal{S}_2) \subseteq (\Omega, \Omega^\dagger)$ and $(\mathcal{S}_1', \mathcal{S}_2') \subseteq (\Omega, \Omega^\dagger)$ with* $\operatorname{dist}(\mathcal{S}_1, \mathcal{S}_2) = \operatorname{dist}(\Omega, \Omega^\dagger) = \operatorname{dist}(\mathcal{S}_1', \mathcal{S}_2')$ *and $\gamma : 2^{\mathcal{S}\times\mathcal{S}} \to [0, +\infty)$ such that*

$$\gamma((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2'))\mu((\mathfrak{F}(\mathcal{S}_1 \times \mathcal{S}_1') \cup \mathfrak{F}(\mathcal{S}_2 \times \mathcal{S}_2'))) \leq \frac{1}{2}\beta(\max\{\mu(\mathcal{S}_1 \cup \mathcal{S}_1'), \mu(\mathcal{S}_2 \cup \mathcal{S}_2')\})\mu((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2')),$$

where $\beta \in \Upsilon$.

(ii) For any $(X, Y) \in \mathcal{S} \times \mathcal{S}$ and $\gamma((X \times Y) \cup (Y \times X)) \geq 1$ we have

$$\gamma(\overline{con}(\mathfrak{F}(X \cup Y) \times \mathfrak{F}(Y \times U))) \geq 1.$$

(iii) Moreover, $d(\mathfrak{F}(\omega_1, \omega_2), \mathfrak{F}(\upsilon_1, \upsilon_2)) \leq d_\infty((\omega_1, \upsilon_1), (\omega_2, \upsilon_2)), \quad \forall(\omega_1, \omega_2) \in \Omega \times \Omega,\ \forall(\upsilon_1, \upsilon_2) \in \Omega^\dagger \times \Omega^\dagger$,

(iv) Furthermore, there occurs NBCC *$X_0, Y_0 \subseteq \mathcal{S}$ such that $\gamma((X_0 \times Y_0))) \geq 1$ and $\gamma(Y_0 \times X_0) \geq 1$.*

Then $\mathfrak{F}$ admits a coupled point of best proximity.

Proof. Let X_i be the natural projection of X into X_i for $i = 1, 2$ and $\widetilde{\mu}(X) := \max\{\mu(X_1), \mu(X_2)\}$. Therefore by Lemma 4.1 $\widetilde{\mu}$ is an MNC on $\mathcal{S} \times \mathcal{S}$. Let $T : (\Omega \times \Omega) \cup (\Omega^\dagger \times \Omega^\dagger) \to (\Omega \times \Omega) \cup (\Omega^\dagger \times \Omega^\dagger)$ defined by

$$T(\upsilon, \omega) = (\mathfrak{F}(\upsilon, \omega), \mathfrak{F}(\omega, \upsilon)), \quad \forall(\upsilon, \omega) \in (\Omega \times \Omega) \cup (\Omega^\dagger \times \Omega^\dagger).$$

Observe that if $(\omega, \upsilon) \in \Omega \times \Omega$, then since $\mathfrak{F}$ is cyclic, $(\mathfrak{F}(\omega, \upsilon), \mathfrak{F}(\upsilon, \omega)) \in \Omega^\dagger \times \Omega^\dagger$, this means, $T(\Omega \times \Omega) \subseteq \Omega^\dagger \times \Omega^\dagger$. Similarly, $T(\Omega^\dagger \times \Omega^\dagger) \subseteq \Omega \times \Omega$. Therefore, T is cyclic on $(\Omega \times \Omega) \cup (\Omega^\dagger \times \Omega^\dagger)$.

Besides, for any $((\omega_1, \omega_2), (\upsilon_1, \upsilon_2)) \in (\Omega \times \Omega) \times (\Omega^\dagger \times \Omega^\dagger)$ we have

$$\begin{aligned}
d_\infty\Big(T(\omega_1, \omega_2), T(\upsilon_1, \upsilon_2)\Big) &= d_\infty\Big((\mathfrak{F}(\omega_1, \omega_2), \mathfrak{F}(\omega_2, \omega_1)), (\mathfrak{F}(\upsilon_1, \upsilon_2), \mathfrak{F}(\upsilon_2, \upsilon_1))\Big) \\
&= \max\{d(\mathfrak{F}(\omega_1, \omega_2), \mathfrak{F}(\upsilon_1, \upsilon_2)), d(\mathfrak{F}(\omega_2, \omega_1), \mathfrak{F}(\upsilon_2, \upsilon_1))\} \\
&\leq \max\{d_\infty((\omega_1, \upsilon_1), (\omega_2, \upsilon_2)), d_\infty((\omega_2, \upsilon_2), d(\omega_1, \upsilon_1))\} \\
&= d_\infty((\omega_1, \omega_2), (\upsilon_1, \upsilon_2)).
\end{aligned}$$

Hence, T is relatively non-expansive.
Let us now define $\xi : 2^{\mathcal{S}\times\mathcal{S}} \to [0, +\infty)$ as follows:

$$\xi((S_1 \times S_1') \cup (S_2 \times S_2')) = \min\{\gamma((S_1 \times S_1') \cup (S_2 \times S_2')), \gamma((S_1' \times S_1) \cup (S_2' \times S_2))\}.$$

By our hypothesis (*ii*), it is clear that whenever $\xi((S_1 \times S_1') \cup (S_2 \times S_2')) \geq 1$, we have $\xi(\overline{\text{con}}(T(S_1 \times S_1') \cup (E_2 \times E_2'))) \geq 1$, which shows that T is ξ-admissible. Also by hypothesis (*iv*) it is clear that there exist $X_0, Y_0 \subseteq \mathcal{S}$ such that $\xi(X_0 \times Y_0) \geq 1$.

Moreover, we have

$$
\begin{aligned}
&\xi((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2'))\widetilde{\mu}(T(\mathcal{S}_1 \times \mathcal{S}_1') \cup T(\mathcal{S}_2 \times \mathcal{S}_2'))\\
&\quad = \xi((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2')) \max\{\widetilde{\mu}(T(\mathcal{S}_1 \times \mathcal{S}_1')), \widetilde{\mu}(T(\mathcal{S}_2 \times \mathcal{S}_2'))\}\\
&\quad = \xi((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2')) \max\{\widetilde{\mu}(\mathfrak{F}(\mathcal{S}_1 \times \mathcal{S}_1') \times \mathfrak{F}(\mathcal{S}_1' \times \mathcal{S}_1)), \widetilde{\mu}(\mathfrak{F}(\mathcal{S}_2 \times \mathcal{S}_2') \times \mathfrak{F}(\mathcal{S}_2' \times \mathcal{S}_2))\}\\
&\quad = \xi((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2')) \max\{\max\{\mu(\mathfrak{F}(\mathcal{S}_1 \times \mathcal{S}_1')), \mu(\mathfrak{F}(\mathcal{S}_1' \times \mathcal{S}_1))\}, \max\{\mu(\mathfrak{F}(\mathcal{S}_2 \times \mathcal{S}_2')), \mu(\mathfrak{F}(\mathcal{S}_2' \times \mathcal{S}_2))\}\}\\
&\quad = \xi((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2')) \max\{\max\{\mu(\mathfrak{F}(\mathcal{S}_1 \times \mathcal{S}_1')), \mu(\mathfrak{F}(\mathcal{S}_2 \times \mathcal{S}_2'))\}, \max\{\mu(\mathfrak{F}(\mathcal{S}_1' \times \mathcal{S}_1)), \mu(\mathfrak{F}(\mathcal{S}_2' \times \mathcal{S}_2))\}\}\\
&\quad = \xi((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2')) \max\{\mu(\mathfrak{F}(\mathcal{S}_1 \times \mathcal{S}_1') \cup \mathfrak{F}(\mathcal{S}_2 \times \mathcal{S}_2')), \mu(\mathfrak{F}(\mathcal{S}_1' \times \mathcal{S}_1) \cup \mathfrak{F}(\mathcal{S}_2' \times \mathcal{S}_2))\}\\
&\quad \le \xi((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2'))[\mu(\mathfrak{F}(\mathcal{S}_1 \times \mathcal{S}_1') \cup \mathfrak{F}(\mathcal{S}_2 \times \mathcal{S}_2')) + \mu(\mathfrak{F}(\mathcal{S}_1' \times \mathcal{S}_1) \cup \mathfrak{F}(\mathcal{S}_2' \times \mathcal{S}_2))]\\
&\quad \le \frac{1}{2}\beta(\max\{\mu(\mathcal{S}_1 \cup \mathcal{S}_1'), \mu(\mathcal{S}_2 \cup \mathcal{S}_2')\})\mu((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2'))\\
&\qquad + \frac{1}{2}\beta(\max\{\mu(\mathcal{S}_1' \cup \mathcal{S}_1), \mu(\mathcal{S}_2' \cup \mathcal{S}_2)\})\mu((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2')) \quad \text{by hypothesis}(i)\\
&\quad = \beta(\max\{\mu(\mathcal{S}_1 \cup \mathcal{S}_1'), \mu(\mathcal{S}_2 \cup \mathcal{S}_2')\})\mu((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2'))\\
&\quad = \beta\left(\max\left\{\max\{\mu(\mathcal{S}_1), \mu(\mathcal{S}_1')\}, \max\{\mu(\mathcal{S}_2), \mu(\mathcal{S}_2')\}\right\}\right)\mu((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2'))\\
&\quad = \beta\left(\max\left\{\widetilde{\mu}(\mathcal{S}_1 \times \mathcal{S}_1'), \widetilde{\mu}(\mathcal{S}_2 \times \mathcal{S}_2')\right\}\right)\mu((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2'))\\
&\quad = \beta(\widetilde{\mu}((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2')))\mu((\mathcal{S}_1 \times \mathcal{S}_1') \cup (\mathcal{S}_2 \times \mathcal{S}_2')).
\end{aligned}
$$

Therefore, T is a ξ_μ-condensing function. Thus by Theorem 3.3, T has a point of best proximity, called $(p, q) \in (\Omega \times \Omega) \cup (\Omega^\dagger \times \Omega^\dagger)$. Thus $\mathfrak{F}$ has a coupled point of best proximity namely (p, q). □

5. Application to system of non-local integro-differential equations

In this section, we are concerned to prove a result which shows the existence of optimal solutions for a system of nonlocal integro-differential equations. Such integro-differential equations are studied in [9, 10] for existence of their solutions.

We consider the system of non-local problem for the integro-differential equation:

$$
\begin{cases} u'(s) = g_1(s, u(s), \int_0^s f_1(r, u(r))dr), \text{a.e. } s \in (0, 1) \\ \text{with } \sum_{k=1}^m \alpha_k u(\tau_k) = u_1, \end{cases} \tag{5.1}
$$

$$
\begin{cases} v'(s) = g_2(s, v(s), \int_0^t f_2(r, v(r))dr), \text{a.e. } s \in (0, 1) \\ \text{with } \sum_{k=1}^m \alpha_k v(\tau_k) = v_1, \end{cases} \tag{5.2}
$$

where $\alpha_k \geq 0$ and $\tau_k \in (0, 1)$.

The nonlocal problem (5.1) can be represented in integral form as follows:

Lemma 5.1. *[10] The solution (if it exists) of the problem* (5.1) *has the following representation in the form of integral equation*

$$
u(s) = \frac{1}{\Lambda}[u_1 - \sum_{k=1}^m \alpha_k \int_0^{\tau_k} g_1(r, u(r), \int_0^r f_1(\sigma, u(\sigma))d\sigma)dr] + \int_0^s g_1(r, u(r), \int_0^r f_1(\sigma, u(\sigma))d\sigma)dr,
$$

where $\Lambda = \sum_{k=1}^m \alpha_k \neq 0$.

Let $b, \rho \in \mathbb{R}^+$ and $\mathcal{S}$ be a Banach space. Let $C(I, \mathcal{S})$ be the Banach space of all continuous functions from $I = [0, 1]$ into $\mathcal{S}$, equipped with the supremum norm. Let $B_1 = B(u_1; b)$ and $B_2 = B(u_2; b)$ be closed balls in $\mathcal{S}$, where $u_1, u_2 \in \mathcal{S}$. Let $J \subseteq I$ and define

$$
\mathcal{K}_1 = \{u : J \to B_1 : u \in C(J, \mathcal{S}), \sum_{k=1}^m \alpha_k u(\tau_k) = u_1\}
$$

and

$$\mathcal{K}_2 = \{v : J \to B_2 : v \in C(J, \mathcal{S}), \sum_{k=1}^{m} \alpha_k v(\tau_k) = v_1\}.$$

Assume that $f_i : I \times B_i \to \mathcal{S}$, $g_i : I \times B_i \times B_i \to \mathcal{S}$, be two continuous functions and f_i is f_i-invariant for $i = 1, 2$. Then clearly $(\mathcal{K}_1, \mathcal{K}_2)$ is a closed, bounded and convex pair in $C(J, \mathcal{S})$. Moreover, for $(u, v) \in \mathcal{K}_1 \times \mathcal{K}_2$, we have $\|u_1 - v_1\| \leq \sup_{t\in J}\|u(t) - v(t)\| = \|u - v\|$, and so, $\text{dist}(\mathcal{K}_1, \mathcal{K}_2) = \|u_1 - v_1\|$.

Now, let $T : \mathcal{K}_1 \cup \mathcal{K}_2 \to C(J, \mathcal{S})$ be the function defined as

$$Tu(t) = \begin{cases} \frac{1}{\Lambda}[v_1 - \sum_{k=1}^{m} \alpha_k \int_0^{\tau_k} g_1(s, u(s), \int_0^s f_1(r, u(r))dr)ds] \\ + \int_0^t g_1(s, u(s), \int_0^s f_1(r, u(r))dr)ds, & u \in \mathcal{K}_1, \\ \\ \frac{1}{\Lambda}[u_1 - \sum_{k=1}^{m} \alpha_k \int_0^{\tau_k} g_2(s, u(s), \int_0^s f_2(r, u(r))dr)ds] \\ + \int_0^t g_2(s, u(s), \int_0^s f_2(r, u(r))dr)ds, & u \in \mathcal{K}_2. \end{cases} \tag{5.3}$$

We call $w \in \mathcal{K}_1 \cup \mathcal{K}_2$ is an optimal solution of the system (5.1)-(5.2) if $\|w - Tw\| = \text{dist}(\mathcal{K}_1 \cup \mathcal{K}_2)$ holds for T. That is, w is the point of best proximity of the function T. Before proving the existence result for optimal solutions of system (5.1)-(5.2), we recall an extended version of the *Mean-Value Theorem*.

Theorem 5.2. ([19]) *Consider I, J, B_i, g_i and f_i from above discussion with $i \in \{1, 2\}$ and $s \in J$. Then*

$$u_j + \int_0^s g_i(\eta, u(\eta), \int_0^\eta f_i(r, u(r))dr)d\eta \in u_j + s\,\overline{con}(\{g_i(\eta, u(\eta), \int_0^\eta k_i(r, u(r))dr) : \eta \in [0, t]\}),$$

for $(i, j) \in \{(1, 2), (2, 1)\}$.

5.1. Existence of optimal solution

Consider the following assumptions to prove the existence of optimal solutions. Let μ be any MNC on $C(J, \mathcal{S})$.

(A_1) $\|g_1(t, u(t), \int_0^t f_1(s, u(s))ds) - g_2(t, v(t), \int_0^t f_2(s, v(s))ds)\| \leq \frac{1}{2\rho}(\|u(t) - v(t)\| - \frac{1}{\Lambda}\|v_1 - u_1\|), \quad \forall (u, v) \in \mathcal{K}_1 \times \mathcal{K}_2.$

(A_2) For any bounded pair $(N_1, N_2) \subseteq (\mathcal{K}_1, \mathcal{K}_2)$, there is a function $\beta \in \Upsilon$ such that

$$\mu(g_1(\mathcal{J} \times N_1 \times N_1) \cup g_2(\mathcal{J} \times N_2 \times N_2)) < \frac{\beta(\mu(N_1 \cup N_2))\mu(N_1 \cup N_2)}{2\rho}.$$

(A_3) $\rho \leq \dfrac{b}{2\max_{i\in\{1,2\}} M_i}$ where $M_i = \sup\{\|g_i(t, u(t), \int_0^t f_i(s, u(s))ds)\| : (t, u) \in I \times B_i\}$, $i = 1, 2$.

The following result establishes the existence of an optimal solution of System (5.1)-(5.2).

Theorem 5.3. *The non-local problem* (5.1)-(5.2) *has atleast one optimal solution if the assumptions* (A_1)-(A_3) *hold.*

Proof. In order to prove the existence of optimal solution of nolocal problem (5.1)-(5.2), we show that the function defined in (5.3) has a point of best proximity.

Step 1: To Show that T is a cyclic function. Indeed for $u \in \mathcal{K}_1$, we have

$$\begin{aligned} \|Tu(t) - \Lambda^{-1}v_1\| &= \Big\|\frac{-1}{\Lambda}\sum_{k=1}^{m}\alpha_k \int_0^{\tau_k} g_1(s, u(s), \int_0^s f_1(\eta, u(\eta))d\eta)ds + \int_0^t g_1(s, u(s), \int_0^s f_1(\eta, u(\eta))d\eta)ds\Big\| \\ &\leq \left|\frac{1}{\Lambda}\sum_{k=1}^{m}\alpha_k \int_0^{\tau_k} \|g_1(s, u(s), \int_0^s f_1(\eta, u(\eta))d\eta)\|ds + \int_0^t \|g_1(s, u(s), \int_0^s f_1(\eta, u(\eta))d\eta)\|ds\right| \\ &\leq \left|\frac{1}{\Lambda}\sum_{k=1}^{m}\alpha_k \int_0^{\tau_k} M_1 ds + \int_0^t M_1 ds\right| \\ &\leq 2M_1\rho = b. \end{aligned}$$

Thus we get $\|Tu(t) - \Lambda^{-1}v_1\| \leq b$ for all $t \in J$ and $u \in \mathcal{K}_1$ which yields $Tu \in \mathcal{K}_2$. The same argument shows that $u \in \mathcal{K}_2$ implies $Tu \in \mathcal{K}_1$.

Step 2: Because T is a cyclic function, we conclude that $T(\mathcal{K}_1)$ is a bounded subset of $\mathcal{K}_2$. We show that $T(\mathcal{K}_1)$ is also an equicontinuous subset of $\mathcal{K}_2$. Suppose $t_1, t_2 \in J$ and $u \in \mathcal{K}_1$ such that $|t_2 - t_1| < \delta$. We observe that

$$\begin{aligned}\|Tu(t_2) - Tu(t_1)\| &= \left\| \int_0^{t_2} g_1\left(\tau, u(\tau), \int_0^{\tau} f_1(s, u(s))ds\right)d\tau - \int_{t_0}^{t_1} g_1\left(\tau, u(\tau), \int_0^{\tau} f_1(s, u(s))ds\right)d\tau \right\| \\ &\leq \left| \int_{t_1}^{t_2} \|g_1(\tau, u(\tau), \int_0^{\tau} f_1(s, u(s))ds)\| d\tau \right| \\ &\leq M_1|t_2 - t_1|.\end{aligned}$$

This means that $T(\mathcal{K}_1)$ is equicontinuous. With the same argument, one can show that $T(\mathcal{K}_2)$ is equicontinuous. Now, by applying Arzela-Ascoli theorem, it follows that the pair $(\mathcal{K}_1, \mathcal{K}_2)$ is relatively compact.

Step 3: Now, we aim to prove T as a relatively non-expansive function. For all $(u, v) \in \mathcal{K}_1 \times \mathcal{K}_2$ with the help of assumption (A_1), we have

$$\begin{aligned}\|Tu(t) - Tv(t)\| = & \left\| \frac{1}{\Lambda}[v_1 - \sum_{k=1}^{m} \alpha_k \int_0^{\tau_k} g_1(s, u(s), \int_0^s f_1(\eta, u(\eta))d\eta)ds] + \int_0^t g_1(s, u(s), \int_0^s f_1(\eta, u(\eta))d\eta)ds \right. \\ & \left. - \left\{ \frac{1}{\Lambda}[u_1 - \sum_{k=1}^{m} \alpha_k \int_0^{\tau_k} g_2(s, v(s), \int_0^s f_2(\eta, v(\eta))d\eta)ds] + \int_0^t g_2(s, v(s), \int_0^s f_2(\eta, v(\eta))d\eta)ds \right\} \right\| \\ & \leq \frac{1}{\Lambda}\|v_1 - u_1\| + \frac{1}{\Lambda}\left| \sum_{k=1}^{m} \alpha_k \int_0^{\tau_k} \|g_1(s, u(s), \int_0^s f_1(\eta, u(\eta))d\eta) - g_2(s, v(s), \int_0^s f_2(\eta, v(\eta))d\eta)\| ds \right| \\ & + \left| \int_0^t \|g_1(s, v(s), \int_0^s f_1(\eta, u(\eta))d\eta) - g_2(s, v(s), \int_0^s f_2(\eta, v(\eta))d\eta)\| ds \right| \\ & \leq \frac{1}{\Lambda}\|v_1 - u_1\| + \frac{1}{2\rho\Lambda}\left| \sum_{k=1}^{m} \alpha_k \int_0^{\tau_k} (\|u(s) - v(s)\| - \frac{1}{\Lambda}\|v_1 - u_1\|)ds \right| \\ & + \frac{1}{2\rho}\left| \int_0^t (\|u(s) - v(s)\| - \frac{1}{\Lambda}\|v_1 - u_1\|)ds \right| \\ & \leq \frac{1}{\Lambda}\|v_1 - u_1\| + \frac{1}{2}(\|u(s) - v(s)\| - \frac{1}{\Lambda}\|v_1 - u_1\|) + \frac{1}{2}(\|u(s) - v(s)\| - \frac{1}{\Lambda}\|v_1 - u_1\|) \\ & = \|u - v\|.\end{aligned}$$

Thereby,

$$\|Tu - Tv\| = \sup_{t \in J} \|Tu(t) - Tv(t)\| \leq \|u - v\|.$$

Therefore, T is relatively nonexpansive.

Step 4: Now, let $(N_1, N_2) \subseteq (\mathcal{K}_1, \mathcal{K}_2)$ be a *NBCC* and proximinal pair which is T-invariant and $\operatorname{dist}(N_1, N_2) = \operatorname{dist}(\mathcal{K}_1, \mathcal{K}_2)(= \|v_1 - u_1\|)$. It follows from Theorem 5.2 and assumption (A_2) that

$$
\begin{aligned}
\mu(T(N_1)\cup T(N_2)) &= \max\{\mu(T(N_1)),\mu(T(N_2))\}\\
&= \max\left\{\sup_{t\in J}\{\mu(\{Tu(t): u\in N_1\})\}, \sup_{t\in J}\{\mu(\{Tv(t): v\in N_2\})\}\right\}\\
&= \max\left\{\sup_{t\in J}\{\mu(\{\frac{1}{\Lambda}[v_1-\sum_{k=1}^{m}\alpha_k\int_0^{\tau_k} g_1(s,u(s),\int_0^s f_1(\eta,u(\eta))d\eta)ds]\right.\\
&+\int_0^t g_1(s,u(s),\int_0^s f_1(\eta,u(\eta))d\eta)ds : u\in N_1\})\},\\
&\sup_{t\in J}\{\mu(\{\frac{1}{\Lambda}[u_1-\sum_{k=1}^{m}\alpha_k\int_0^{\tau_k} g_2(s,v(s),\int_0^s f_2(\eta,v(\eta))d\eta)ds]\\
&\left.+\int_0^t g_2(s,v(s),\int_0^s f_2(\eta,v(\eta))d\eta)ds : v\in N_2\})\}\right\}.
\end{aligned}
$$

That is

$$
\begin{aligned}
\mu(T(N_1)\cup T(N_2)) &\le \max\left\{\sup_{t\in J}\{\mu(\{\frac{1}{\Lambda}[v_1-\sum_{k=1}^{m}\alpha_k\tau_k\,\overline{\text{con}}(g_1(s,u(s),\int_0^s f_1(\eta,u(\eta))d\eta))]\right.\\
&+t\,\overline{\text{con}}(g_1(s,u(s),\int_0^s f_1(\eta,u(\eta))d\eta)) : s\in J\})\},\\
&\sup_{t\in J}\{\mu(\{\frac{1}{\Lambda}[u_1-\sum_{k=1}^{m}\alpha_k\tau_k\,\overline{\text{con}}(g_2(s,v(s),\int_0^s f_2(\eta,v(\eta))d\eta))]\\
&\left.+t\,\overline{\text{con}}(g_2(s,v(s),\int_0^s f_2(\eta,v(\eta))d\eta)) : s\in J\})\}\right\}\\
&\le \max\left\{\rho[\mu(g_1(J\times N_1\times N_1))+\mu(g_1(J\times N_1\times N_1)],\rho[\mu(g_2(J\times N_2\times N_2))+\mu(g_2(J\times N_2\times N_2)]\right\}\\
&= 2\rho\max\{\mu(g_1(J\times N_1\times N_1),\mu(g_2(J\times N_2\times N_2)\}\\
&= 2\rho\mu(g_1(J\times N_1\times N_1)\cup g_2(J\times N_2\times N_2))\\
&< \beta(\mu(N_1\cup N_2))\mu(N_1\cup N_2).
\end{aligned}
$$

Further we define $\xi : 2^{S}\to[0,\infty)$ as $\xi(N_1\cup N_2)=1$ which implies that

$$\xi(N_1\cup N_2)\mu(T(N_1)\cup T(N_2))\le\beta(\mu(N_1\cup N_2))\mu(N_1\cup N_2).$$

Therefore, all the necessary requirements of Theorem 3.3 are satisfied. So the function T has a point of best proximity which is an optimal solution of the system of (5.1)-(5.2).

□

The next corollary follows from Theorem 5.3 in which we obtain a result for actuality of solution for a system of non-local integro-differential equations.

Corollary 5.4. *Under the above notations and the assumptions of Theorem 5.3, if $u_1 = u_2$, then the system*

$$
\begin{cases}
u'(s)=g_1(s,u(s),\int_0^s f_1(r,u(r))dr), & \sum_{k=1}^{m}\alpha_k u(\tau_k)=u_1,\\
v'(s)=g_2(s,v(s),\int_0^s f_2(r,v(r))dr), & \sum_{k=1}^{m}\alpha_k v(\tau_k)=u_1,
\end{cases}
\tag{5.4}
$$

has a solution.

6. Application

In cell life (plants and micro-cell), it stores food by cyclic and by methods depend on different factors of the environments (light, Oxygen etc.). This system can be formulated by a coupled dynamic system as follows. The importance of the system is to investigate the cyclic and non-cyclic behavior. Entropy actions the rate of growth in dynamical complexity as the system progresses with time. This is not to be disordered with other concepts of entropy linked with another complexity.

Example 6.1. In this example, we present the optimum solution of a system using entropy integral formula via Tsallis concept. For continuous probability distributions, we define the entropy as follows:

$$T_\lambda[f](t) = \frac{1}{\lambda - 1}\left(1 - \int_0^t (f(\varsigma))^\lambda \, d\varsigma\right), \quad \lambda \neq 1,$$

where f indicates a probability density function [29]. The maximum value [25] of Tsallis entropy is

$$T_\lambda^{max}[f] = \frac{1-(1/f)^{1-\lambda}}{\lambda - 1},$$

where $\lambda > 1$ implies that $T_\lambda^{max}[f] = \frac{1}{\lambda-1}$.

The integro-differential entripic system is given by

$$\begin{cases} u'(s) = g_1(s, u(s), T_\lambda[u](s)), & \sum_{k=1}^m \alpha_k u(\tau_k) = u_1 < 1, \ s \in (0,1) \\ v'(s) = g_2(s, v(s), T_\lambda[v](s)), & \sum_{k=1}^m \alpha_k v(\tau_k) = u_1 < 1, \ s \in (0,1), \end{cases} \tag{6.1}$$

where for $u, v \in C((0,1),[0,1])$

$$g_1(s, u(s), T_\lambda[u](s)) = u(s)T_\lambda[u](s), \quad g_2(s, v(s), T_\lambda[v](s)) = v(s)T_\lambda[v](s).$$

It is clear that, by using the maximum value of the entropy, we have

(A1) For $u, v \in C((0,1),[0,1])$, we obtain

$$\|g_1(s, u(s), T_\lambda[u](s)) - g_2(s, v(s), T_\lambda[v](s))\| \leq \frac{1}{\lambda - 1}\|u - v\|, \quad \lambda > 1$$

provided $\frac{1}{2\rho} = \frac{1}{\lambda-1}$ that is $\rho = \frac{\lambda-1}{2}$.

(A2) By letting $\beta(\chi) = 2\chi$ then

$$\mu(g_1 \cup g_2) = \max(\mu(g_1), \mu(g_2)) = \max(\|g_1\|, \|g_2\|) = \frac{1}{(\lambda-1)} < \frac{2}{2\rho} = \frac{1}{\rho} = \frac{2}{\lambda - 1}.$$

(A3)

$$\rho = \frac{2M_1\rho}{2max(g_i)} = \frac{2\rho(1/(\lambda-1))}{2(1/(\lambda-1))} = \frac{\lambda-1}{2}.$$

Hence, all the hypothesis of Corollary 5.4 hold then System (6.1) has an optimal solution given by Tsallis entropy maxima.

As a numerical evaluation, when $\lambda = 9$ and $(u_0, v_0) = (0.1, 0.1)$, we have a stable solution (see Fig.1). Fig.2 describes the behavior of solution when $\lambda = 4(r = 0.2)$ and initially stands with the point $(u_0, v_0) = (0.1, 0.1)$. Obviously, the system is still stable. The stable system in Fig.3 is presented when $\lambda = 4(r = 0.2)$ and $(u_0, v_0) = (0, 0)$.

From above, we conclude that the optimal solution of integro-differential systems can be optimized by using fractal entropy. The fractal power is playing an important role to state the stability and maximization of solutions.

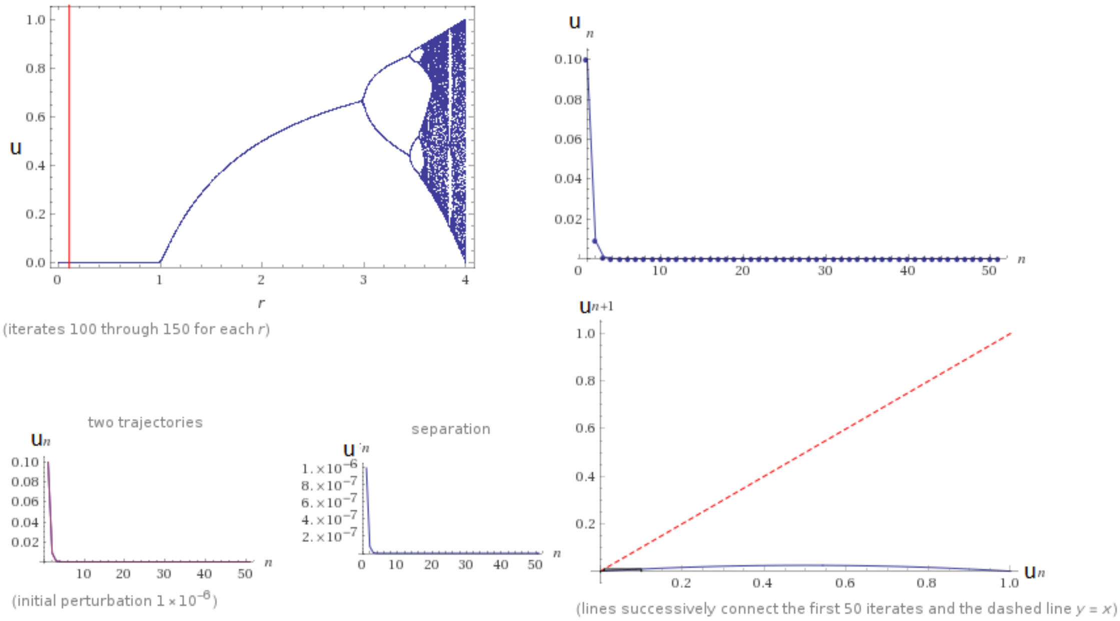

Figure 1. The behavior of solution of System(6.1) for the variable u (similarly for v) in the upper left graph, where $r = 1/(\lambda - 1) = 0.1$; the lower left indicates two trajectories and separate trajectory; the right graphs represent the iteration solutions and fixed points of the system with the initial condition $(u_0, v_0) = (0.1, 0.1)$. The system has a limit cycle from the first iteration with one period.

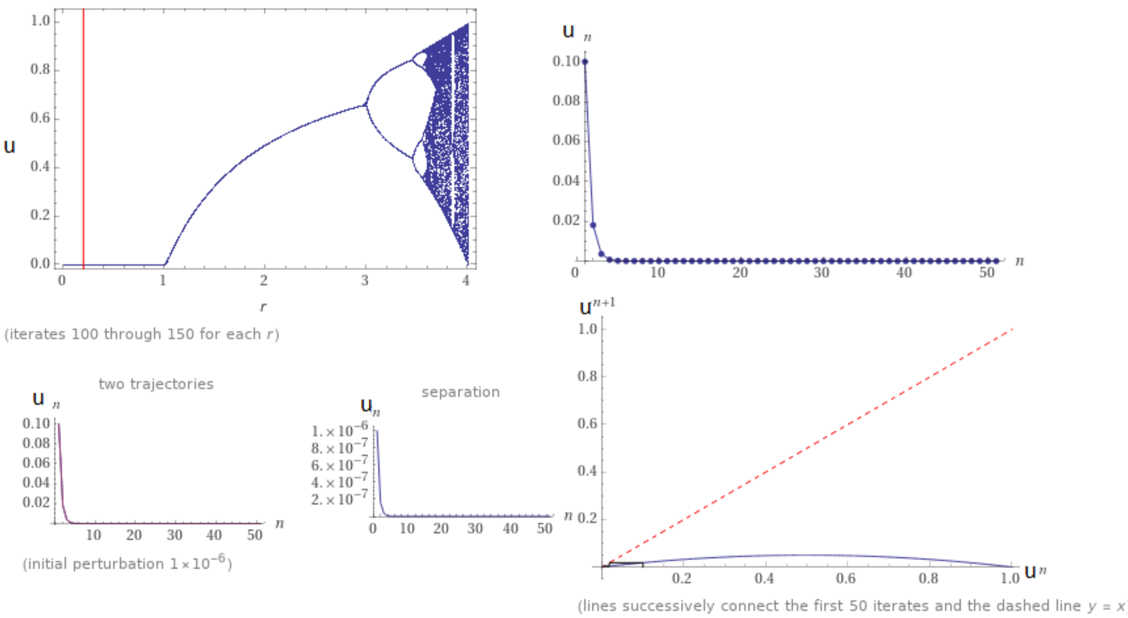

Figure 2. The behavior of solution of System(6.1) for the variable u (similarly for v) in the upper left graph, where $r = 1/(\lambda - 1) = 0.2$; the lower left indicates two trajectories and separate trajectory; the right graphs represent the iteration solutions and fixed points of the system with the initial condition $(u_0, v_0) = (0.1, 0.1)$. The system has a limit cycle from the first iteration with one period.

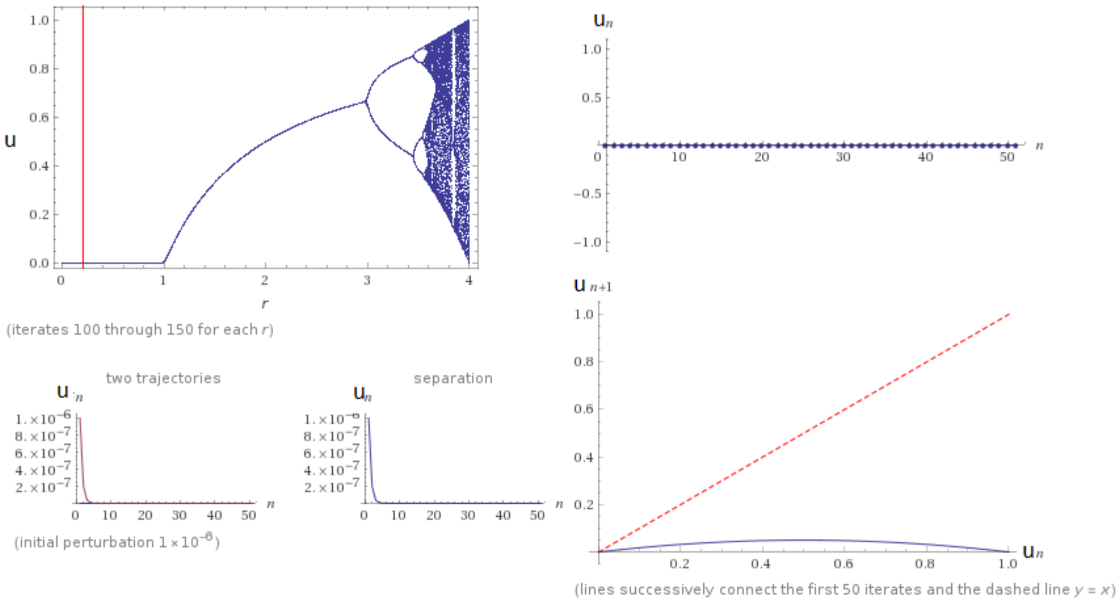

Figure 3. The behavior of solution of System(6.1) for the variable u (similarly for v) in the upper left graph, where $r = 1/(\lambda - 1) = 0.2$; the lower left indicates two trajectories and separate trajectory; the right graphs represent the iteration solutions and fixed points of the system with the initial condition $(u_0, v_0) = (0, 0)$. The system has a limit cycle from the first iteration with one period.

7. Conclusion

From what we introduced above, we generalized the theory of measure of non-compactness and associated fixed point theorems. This developments are indicated as tools in non-linear functional analysis including partial differential equations, integral and integro-differential equations and optimal control theory. We considered a class of cyclic (non-cyclic) relatively Geraghty condensing functions and study the existence of points (pairs) of best proximity as well as coupled points of best proximity in Banach spaces. As an application of the obtained results, the actuality of optimal solutions for a system of non-local integro-differential equations is demonstrated. As numerical illustrations, we present the optimal solution of integro-differential systems, which can be optimized by using fractal entropy. The fractal power is playing an important role to state the stability and maximization of solutions.

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