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A New Refinement of Young's Type Inequalities and Applications

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Abstract

In this paper, we present multiple-term refinements of Young's type inequalities which extends and unifies two recent and important results due to L. Nasiri et al. (Int. J. Nonlinear Anal. Appl. 8, 261–267, 2017) and C. Yang et al. (Journal. Math. Inequalities 14, 401–419, 2020). As applications, we give some related inequalities for operators and matrices.

Keywords: Young's type inequality, Operator inequality, Traces, Unitarily invariant norms

2010 MSC: 47A30, 15A60

1. Introduction

We start by reviewing some important facts concerning the classical Young's inequality and its known refinements.

The well-known Young's inequality, for scalars asserts that for all positive real numbers a, b and $0 \leq \alpha \leq 1$,

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b. \tag{1.1}$$

This inequality, though very simple, has attracted researchers working in operator theory due to its applications in this field.

Refining this inequality has taken the attention of many researchers in the field, where adding a positive term to the left side is possible.

The first refinement of Young's inequality is the second version proved by Kittaneh and Manasrah [11], as follows

$$(a^{\alpha}b^{1-\alpha})^2 + r_0^2(a-b)^2 \leq (\alpha a + (1-\alpha)b)^2, \tag{1.2}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

Later, Kittaneh and Manasrah [11], obtained the other interesting refinement of Young's inequality

$$a^{\alpha}b^{1-\alpha} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \alpha a + (1-\alpha)b, \tag{1.3}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.
See ([1], [2], [4], [6], [7], [8], [9], and [15]) for improvements of Young's inequality and their recent advances.

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After a short time, H. Kai [10] gave the following Young type inequality,

$$\left[(\alpha a)^{2\alpha} b^{2-2\alpha} \chi_{(0,\frac{1}{2}]}(\alpha) + a^{2\alpha}[(1-\alpha)b]^{2-2\alpha} \chi_{(\frac{1}{2},1)}(\alpha)\right] + r_0^2(a-b)^2 \leq \alpha^2 a^2 + (1-\alpha)^2 b^2, \tag{1.4}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$ and $\chi_I(\alpha)$ the characteristic function.
Recently, Nasiri et al. [13], obtained the following refinement of inequality (1.4) as follows

Theorem 1.1. *Let $a, b > 0$ and $0 < \alpha < 1$, we have*

(1) If $0 < \alpha < \frac{1}{2}$, then

$$(\alpha a)^{2\alpha} b^{2-2\alpha} + \alpha^2(a-b)^2 + rb(\sqrt{\alpha a} - \sqrt{b})^2 \leq \alpha^2 a^2 + (1-\alpha)^2 b^2, \tag{1.5}$$

where $r = \min\{2\alpha, 1-2\alpha\}$.

(2) If $\frac{1}{2} \leq \alpha < 1$, then

$$a^{2\alpha}[(1-\alpha)b]^{2-2\alpha} + (1-\alpha)^2(a-b)^2 + ra(\sqrt{a} - \sqrt{(1-\alpha)b})^2 \leq \alpha^2 a^2 + (1-\alpha)^2 b^2, \tag{1.6}$$

where $r = \min\{2\alpha - 1, 2 - 2\alpha\}$.

In 2020, C. Yang et al. [16] obtained the following refinement of inequalities (1.5) and (1.6) as follows:

Theorem 1.2. *Let $a, b > 0$ and $0 < \alpha < 1$, we have*

(1) If $0 < \alpha < \frac{1}{4}$, then

$$(\alpha a)^{2\alpha} b^{2-2\alpha} + \alpha^2(a-b)^2 + 2\alpha b(\sqrt{\alpha a} - \sqrt{b})^2 + rb(\sqrt[4]{\alpha ab} - \sqrt{b})^2 \leq \alpha^2 a^2 + (1-\alpha)^2 b^2, \tag{1.7}$$

where $r = \min\{4\alpha, 1-4\alpha\}$.

(2) If $\frac{1}{4} \leq \alpha < \frac{1}{2}$, then

$$(\alpha a)^{2\alpha} b^{2-2\alpha} + \alpha^2(a-b)^2 + (1-2\alpha)b(\sqrt{\alpha a} - \sqrt{b})^2 + rb(\sqrt[4]{\alpha ab} - \sqrt{\alpha a})^2 \leq \alpha^2 a^2 + (1-\alpha)^2 b^2, \tag{1.8}$$

where $r = \min\{2 - 4\alpha, 4\alpha - 1\}$.

(3) If $\frac{1}{2} \leq \alpha < \frac{3}{4}$, then

$$\begin{aligned} a^{2\alpha}[(1-\alpha)b]^{2-2\alpha} + (1-\alpha)^2(a-b)^2 + (2\alpha-1)a(\sqrt{a} - \sqrt{(1-\alpha)b})^2 \\ + ra(\sqrt[4]{(1-\alpha)ab} - \sqrt{(1-\alpha)b})^2 \leq \alpha^2 a^2 + (1-\alpha)^2 b^2, \end{aligned} \tag{1.9}$$

where $r = \min\{3 - 4\alpha, 4\alpha - 2\}$.

(4) If $\frac{3}{4} \leq \alpha < 1$, then

$$\begin{aligned} a^{2\alpha}[(1-\alpha)b]^{2-2\alpha} + (1-\alpha)^2(a-b)^2 + (2-2\alpha)a(\sqrt{a} - \sqrt{(1-\alpha)b})^2 \\ + ra(\sqrt[4]{(1-\alpha)ab} - \sqrt{a})^2 \leq \alpha^2 a^2 + (1-\alpha)^2 b^2, \end{aligned} \tag{1.10}$$

where $r = \min\{4 - 4\alpha, 4\alpha - 3\}$.

Let $B(\mathcal{H})$ denote the $\mathbb{C}^*$-Algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. A operator $A \in B(\mathcal{H})$ is called positive, denoted as $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. The set of all positive operators is denoted by $B(\mathcal{H})^+$. The set of all invertible operators in $B(\mathcal{H})^+$, is denoted by $B(\mathcal{H})^{++}$.
Let $A, B \in B(\mathcal{H})^{++}$ and $\alpha \in [0, 1]$. The α-weighted operators geometric mean of A and B, denoted by $A\sharp_\alpha B$, is defined as

$$A\sharp_\alpha B = A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^\alpha A^{1/2},$$

and the α-weighted operators arithmetic mean of A and B is defined as

$$A\nabla_\alpha B = \alpha B + (1-\alpha)A.$$

If $\alpha = \frac{1}{2}$, these operators can be writing by simplification as $A\nabla B$, $A\sharp B$.
The operators version of Young's inequality states as follows:

$$A\sharp_\alpha B \leq A\nabla_\alpha B.$$

Let $\mathbf{M}_n(\mathbb{C})$ be the space of $n\times n$ complex matrices. The singular values of a matrix $A \in \mathbf{M}_n(\mathbb{C})$ are the eigenvalues of the positive semi-definite matrix $|A| = (A^*A)^{1/2}$, denoted by $s_i(A)$ for $i = 1, 2, 3, \ldots, n$. A norm $|||.|||$ on $\mathbf{M}_n(\mathbb{C})$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbf{M}_n(\mathbb{C})$, and all unitary matrices $U, V \in \mathbf{M}_n(\mathbb{C})$.
The trace norme $||.||$ is given by $||A||_1 = tr|A| = \sum_{k=1}^n s_k(A)$, where tr is the usual trace. This norms is unitarily invariant.

For further reading related to operator and matrices inequalities, the reader is referred to recent papers [16] and [13].

This paper is organised as follows:

This introduction contains some recalls and preliminaries needed for this work. The recalls concern (scalar) Young's type inequality and its new refinements recently obtained by several authors.

The purpose of this work is devoted to generalize and unify some new important results concerning both scalar and operator versions of Young's inequality.

In section 2, we present in Theorem 2.6 multiple-term refinements of Young's type inequality. This theorem will generalize and unify the result (see Theorem 1.1) obtained by L. Nasiri et al. [13] and the result (see Theorem 1.2) obtained by C. Yang et al. [16].

Section 3 is devoted to certain applications of the main results of the second section to obtain some new operator inequalities.

In the end and last section, we present multiple-term refinement of Young's inequalities for the norms and traces of positive definite matrices (see Theorems 4.2 and 4.3).

2. Multiple-term refinements of Young's type inequality

In this section, we are concerned by the investigation of multiple-term refinements of Young's type inequalities for scalars.
To state and prove our results, we need to recall the following definitions and lemmas which will be used in all the rest of this paper.

2.1. Definitions, notations and lemmas

In this subsection, we place all the technical details needed to prove our main result.

Definition 2.1. Let n be a positive integer. We consider the sequence $(r_n(\alpha))$ of functions on $[0, 1]$ defined by:

$$r_0(\alpha) = \min\{\alpha, 1-\alpha\},$$

$$r_n(\alpha) = \min\{2r_{n-1}(\alpha), 1 - 2r_{n-1}(\alpha)\}.$$

For all integer l, we have the following explicit formula of the function $r_l(\alpha)$, proved by D. Choi in [4].

Lemma 2.2. *Let $l \geq 0$, and $1 \leq k \leq 2^l$ be integers. If $\frac{k-1}{2^l} \leq \alpha \leq \frac{k}{2^l}$, then*

$$r_l(\alpha) = \begin{cases} 2^l\alpha - k + 1, & if,\ \frac{k-1}{2^l} \leq \alpha \leq \frac{2k-1}{2^{l+1}}, \\ k - 2^l\alpha, & if,\ \frac{2k-1}{2^{l+1}} \leq \alpha \leq \frac{k}{2^l}. \end{cases}$$

Definition 2.3. Let x and y be two positive numbers, for $l, k \in \mathbb{N}$, we define the functions $f_{l,k}(x, y)$ by

$$f_{l,k}(x, y) = \Big(\sqrt{x^{\frac{k-1}{2^l}} y^{1-\frac{k-1}{2^l}}} - \sqrt{x^{\frac{k}{2^l}} y^{1-\frac{k}{2^l}}} \Big)^2.$$

The following lemma has been proven by D. Choi in [4].

Lemma 2.4 ([4])**.** *For a positive integer N, and $0 \leq \alpha \leq \frac{1}{2}$, define $S_N(t)$ by*

$$S_N(t) = 1 - \alpha + \alpha t - \alpha(\sqrt{t} - 1)^2 - \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\sqrt{t^{\frac{(k-1)}{2^l}}} - \sqrt{t^{\frac{k}{2^l}}} \Big)^2 \chi_{(\frac{k-1}{2^l}, \frac{k}{2^l})}(\alpha),$$

for $t > 0$. Then we have

$$S_N(t) = \sum_{k=1}^{2^{N-1}} \Big((k - 2^N\alpha) t^{\frac{(k-1)}{2^N}} + (2^N\alpha - k + 1) t^{\frac{k}{2^N}} \Big) \chi_{(\frac{k-1}{2^N}, \frac{k}{2^N})}(\alpha).$$

Lemma 2.5. *For a positive integer N, and $0 < \alpha \leq \frac{1}{2}$, define $R_N(t)$ by*

$$R_N(t) = (1 - \alpha)^2 + \alpha^2 t^2 - \alpha^2(t - 1)^2 - \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\sqrt{(\alpha t)^{\frac{k-1}{2^{l-1}}}} - \sqrt{(\alpha t)^{\frac{k}{2^{l-1}}}} \Big)^2 \chi_{(\frac{k-1}{2^l}, \frac{k}{2^l})}(\alpha),$$

for $t > 0$. Then we have

$$R_N(t) = \sum_{k=1}^{2^{N-1}} \Big((k - 2^N\alpha)(\alpha t)^{\frac{(k-1)}{2^{N-1}}} + (2^N\alpha - k + 1)(\alpha t)^{\frac{k}{2^{N-1}}} \Big) \chi_{(\frac{k-1}{2^N}, \frac{k}{2^N})}(\alpha).$$

Proof. Replacing t, by $(\alpha t)^2$ in Lemma 2.4, we have

$$\begin{aligned} (1 - \alpha) + \alpha^3 t^2 - \alpha(\alpha t - 1)^2 - \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^{l-1}} \Big(\sqrt{(\alpha t)^{\frac{(k-1)}{2^{l-1}}}} - \sqrt{(\alpha t)^{\frac{k}{2^{l-1}}}} \Big)^2 \chi_{(\frac{k-1}{2^l}, \frac{k}{2^l})}(\alpha) \\ = \sum_{k=1}^{2^{N-1}} \Big((k - 2^N\alpha)(\alpha t)^{\frac{(k-1)}{2^{N-1}}} + (2^N\alpha - k + 1)(\alpha t)^{\frac{k}{2^{N-1}}} \Big) \chi_{(\frac{k-1}{2^N}, \frac{k}{2^N})}(\alpha). \end{aligned}$$

Therefore,

$$\begin{aligned} R_N(t) &= (1 - \alpha)^2 + \alpha^2 t^2 - \alpha^2(t - 1)^2 - \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^{l-1}} \Big(\sqrt{(\alpha t)^{\frac{(k-1)}{2^{l-1}}}} - \sqrt{(\alpha t)^{\frac{k}{2^{l-1}}}} \Big)^2 \chi_{(\frac{k-1}{2^l}, \frac{k}{2^l})}(\alpha), \\ &= (1 - \alpha)^2 + \alpha^2 t^2 - \alpha^2(t - 1)^2 - ((1 - \alpha) + \alpha^3 t^2 - \alpha(\alpha t - 1)^2) \\ &+ \sum_{k=1}^{2^{N-1}} \Big((k - 2^N\alpha)(\alpha t)^{\frac{(k-1)}{2^{N-1}}} + (2^N\alpha - k + 1)(\alpha t)^{\frac{k}{2^{N-1}}} \Big) \chi_{(\frac{k-1}{2^N}, \frac{k}{2^N})}(\alpha) \\ &= \sum_{k=1}^{2^{N-1}} \Big((k - 2^N\alpha)(\alpha t)^{\frac{(k-1)}{2^{N-1}}} + (2^N\alpha - k + 1)(\alpha t)^{\frac{k}{2^{N-1}}} \Big) \chi_{(\frac{k-1}{2^N}, \frac{k}{2^N})}(\alpha). \end{aligned}$$

□

2.2. Multiple-term refinements of Young's type inequality

The following theorem shows a multiple-term refinements of Young's type inequalities due to Kai, Nasiri and Yang ([10], [13] and [16]).

Theorem 2.6. *Let $a, b > 0$ and $0 \leq \alpha \leq 1$. Then for all a positive integer N, we have*

$$\begin{aligned}&\Big[(\alpha a)^{2\alpha} b^{2-2\alpha} \chi_{(0,\frac{1}{2}]} \quad + \quad a^{2\alpha}[(1-\alpha)b]^{2-2\alpha}\chi_{(\frac{1}{2},1)}\Big] + r_0^2(a-b)^2 \\ &+ \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(b f_{l-1,k}(\alpha a, b)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) + a f_{l-1,k}((1-\alpha)b, a)\chi_{(\frac{2^l-k}{2^l},\frac{2^l-k+1}{2^l})}(\alpha)\Big) \\ &\leq \alpha^2 a^2 + (1-\alpha)^2 b^2. \end{aligned} \tag{2.1}$$

Proof. Suppose that $0 < \alpha \leq \frac{1}{2}$. We claim that

$$(\alpha a)^{2\alpha} b^{2-2\alpha} + \alpha^2(a-b)^2 + \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} b f_{l-1,k}(\alpha a, b)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) \leq \alpha^2 a^2 + (1-\alpha)^2 b^2. \tag{2.2}$$

Putting $t = \frac{a}{b}$, in inequality (2.2), then (2.2) can be rewritten as

$$(1-\alpha)^2 + \alpha^2 t^2 \geq (\alpha t)^{2\alpha} + \alpha^2(t-1)^2 + \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\sqrt{(\alpha t)^{\frac{k-1}{2^{l-1}}}} - \sqrt{(\alpha t)^{\frac{k}{2^{l-1}}}}\Big)^2 \chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha),$$

It follows that by Lemma 2.5, the above inequality is equivalent to

$$R_N(t) \geq (\alpha t)^{2\alpha}.$$

Hence, it suffices to prove that, if $\alpha \in (\frac{k-1}{2^N}, \frac{k}{2^N})$, then

$$(k - 2^N\alpha)(\alpha t)^{\frac{(k-1)}{2^{N-1}}} + (2^N\alpha - k + 1)(\alpha t)^{\frac{k}{2^{N-1}}} \geq (\alpha t)^{2\alpha}.$$

Replacing t by $\frac{t^{2^{N-1}}}{\alpha}$ and letting $\alpha = 2^N\alpha$, the above inequality is equivalent to

$$(k-\alpha)t^{k-1} + (\alpha - k + 1)t^k \geq t^\alpha.$$

By Young inequality, we have

$$(k-\alpha)t^{k-1} + (\alpha - k + 1)t^k \geq t^{(k-1)(k-\alpha)+k((\alpha-k+1))} = t^\alpha.$$

Therefore,

$$(k - 2^N\alpha)(\alpha t)^{\frac{(k-1)}{2^{N-1}}} + (2^N\alpha - k + 1)(\alpha t)^{\frac{k}{2^{N-1}}} \geq (\alpha t)^{2\alpha}.$$

If $\alpha \in [\frac{1}{2}, 1]$, then $1 - \alpha \in [0, \frac{1}{2}]$. So by changing two elements a, b and two weights $\alpha, 1-\alpha$ in inequality (2.2), and note that $r_l(1-\alpha) = r_l(\alpha)$ the desired inequality is obtained. □

Remark 2.7. If we set $N = 2$ in Theorem 2.6, then we recapture Theorem 1.1.

If we set $N = 3$ in Theorem 2.6, then we recapture Theorem 1.2.

3. Inequalities for operators

In this section, we are concerned by the investigation of multiple-term refinements of operator version of Young's type inequalities for operators.

To prove the main result of this section, we need the following lemma

Lemma 3.1 ([14, p. 3]). *Let $T \in B(\mathcal{H})$ be self-adjoint. If f and g are both continuous functions with $f(t) \geq g(t)$ for $t \in Sp(T)$ (where the sign $Sp(T)$ denotes the spectrum of operator T), then $f(T) \geq g(T)$.*

The main result of this section is the following theorem

Theorem 3.2. *Let $A, B \in B(\mathcal{H})^{++}$, and $0 \leq \alpha \leq 1$, then for all a positive integer N, we have*

1. *If $0 < \alpha \leq \frac{1}{2}$, then*

$$\begin{aligned} \alpha^{2\alpha} A\sharp_{1-\alpha} B \quad &+ \quad \alpha^2 (A + B - 2A\sharp B) \\ &+ \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\alpha^{\frac{k-1}{2^{l-1}}} A\sharp_{\frac{k-1}{2^l}} B + \alpha^{\frac{k}{2^{l-1}}} A\sharp_{\frac{k}{2^l}} B - 2\alpha^{\frac{2k-1}{2^l}} A\sharp_{\frac{2k-1}{2^{l+1}}} B \Big) \chi_{(\frac{k-1}{2^l}, \frac{k}{2^l})}(\alpha) \\ &\leq (1-\alpha)^2 A + \alpha^2 B. \end{aligned} \tag{3.1}$$

2. *If $\frac{1}{2} < \alpha \leq 1$, then*

$$\begin{aligned} (1-\alpha)^{2-2\alpha} A\sharp_{1-\alpha} B \quad &+ \quad (1-\alpha)^2 (A + B - 2A\sharp B) \\ &+ \sum_{l=1}^{N-1} r_l(\alpha) \Big[\sum_{k=1}^{2^{l-1}} \Big((1-\alpha)^{\frac{k-1}{2^{l-1}}} A\sharp_{\frac{2^l-k+1}{2^l}} B + (1-\alpha)^{\frac{k}{2^{l-1}}} A\sharp_{\frac{2^l-k}{2^l}} B \\ &- 2(1-\alpha)^{\frac{2k-1}{2^l}} A\sharp_{\frac{2^{l+1}-2k+1}{2^{l+1}}} B \Big) \Big] \chi_{(\frac{2^l-k}{2^l}, \frac{2^l-k+1}{2^l})}(\alpha) \\ &\leq (1-\alpha)^2 A + \alpha^2 B. \end{aligned} \tag{3.2}$$

Proof. Suppose that $0 < \alpha \leq \frac{1}{2}$, we claim that

$$\begin{aligned} \alpha^{2\alpha} A\sharp_{1-\alpha} B + \alpha^2 (A + B - 2A\sharp B) + \sum_{l=1}^{N-1} r_l(v) \Big[\sum_{k=1}^{2^{l-1}} \Big(\alpha^{\frac{k-1}{2^{l-1}}} A\sharp_{\frac{2^l-k+1}{2^l}} B + \alpha^{\frac{k}{2^{l-1}}} A\sharp_{\frac{2^l-k}{2^l}} B - 2\alpha^{\frac{2k-1}{2^l}} A\sharp_{\frac{2^{l+1}-2k+1}{2^{l+1}}} B \Big) \chi_{(\frac{k-1}{2^l}, \frac{k}{2^l})}(\alpha) \\ \leq \alpha^2 A + (1-\alpha)^2 B. \end{aligned} \tag{3.3}$$

Taking $a = 1$, $b^2 = t > 0$ in Theorem 2.6, then we have

$$\begin{aligned} \alpha^2 + (1-\alpha)^2 t \quad &\geq \quad \alpha^{2\alpha} t^{1-\alpha} + \alpha^2 (\sqrt{t} - 1)^2 + \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^{l-1}} \sqrt{t} \Big(\sqrt{\alpha^{\frac{k-1}{2^{l-1}}} \sqrt{t}^{1-\frac{k-1}{2^{l-1}}}} - \sqrt{\alpha^{\frac{k}{2^{l-1}}} \sqrt{t}^{1-\frac{k}{2^{l-1}}}} \Big)^2 \chi_{(\frac{k-1}{2^l}, \frac{k}{2^l})}(\alpha) \\ &= \quad \alpha^{2\alpha} t^{1-\alpha} + \alpha^2 (t - 2\sqrt{t} + 1) + \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^{l-1}} \Big(\alpha^{\frac{k-1}{2^{l-1}}} t^{1-\frac{k-1}{2^l}} + \alpha^{\frac{k}{2^{l-1}}} t^{1-\frac{k}{2^l}} - 2\alpha^{\frac{2k-1}{2^l}} t^{1-\frac{2k-1}{2^{l+1}}} \Big) \chi_{(\frac{k-1}{2^l}, \frac{k}{2^l})}(\alpha) \end{aligned}$$

For $X := A^{-1/2} B A^{-1/2}$, by Lemma 3.1, we have

$$\begin{aligned} \alpha^2 I + (1-\alpha)^2 A^{-1/2} B A^{-1/2} \quad &\geq \quad \alpha^{2\alpha} (A^{-1/2} B A^{-1/2})^{1-\alpha} + \alpha^2 (A^{-1/2} B A^{-1/2} - 2\sqrt{A^{-1/2} B A^{-1/2}} + I) \\ &+ \quad \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^{l-1}} \Big(\alpha^{\frac{k-1}{2^{l-1}}} (A^{-1/2} B A^{-1/2})^{1-\frac{k-1}{2^{l+1}}} + \alpha^{\frac{k}{2^{l-1}}} (A^{-1/2} B A^{-1/2})^{1-\frac{k}{2^{l+1}}} \\ &- \quad 2\alpha^{\frac{2k-1}{2^l}} (A^{-1/2} B A^{-1/2})^{1-\frac{2k-1}{2^{l+2}}} \Big) \chi_{(\frac{k-1}{2^l}, \frac{k}{2^l})}(\alpha). \end{aligned}$$

So, multiplying the above inequality by $A^{\frac{1}{2}}$ on the left-hand side and on the right hand side, we can deduce the result. If $\alpha \in [\frac{1}{2}, 1]$, then $1 - \alpha \in [0, \frac{1}{2}]$. So by changing two operators A, B and two weights $\alpha, 1-\alpha$ in inequality (3.1) and note that $A\sharp_\alpha B = B\sharp_{1-\alpha} A$, $r_l(1-\alpha) = r_l(\alpha)$ the desired inequality is obtained. □

4. Inequalities for matrix

In this section, we are concerned by the investigation of multiple-term refinements of operator version of Young's type inequalities for matrices.
A Young inequality matrix due to Ando [3] asserts that for $\alpha \in [0,1]$

$$s_j(A^\alpha B^{1-\alpha}) \leq s_j(\alpha A + (1-\alpha)B),$$

the above singular value inequality entails the following unitarily invariant norm inequality

$$|||A^\alpha B^{1-\alpha}||| \leq |||\alpha A + (1-\alpha)B|||.$$

We recall the following lemma which will be used in our proof. From Theorems 4.2 and 4.3.

Lemma 4.1 ([12]). *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive semi-definite matrices. Then we have*

$$|||A^\alpha X B^{1-\alpha}||| \leq |||AX|||^\alpha |||XB|||^{1-\alpha}. \tag{4.1}$$

In particular

$$tr|A^\alpha B^{1-\alpha}| \leq (trA)^\alpha (trB)^{1-\alpha}. \tag{4.2}$$

is a Heinz-Kato type inequality for unitarily invariant norms.
The first result of this section concerns the traces of positive definite matrices which can be reads as follows:

Theorem 4.2. *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive definite matrices and $0 \leq \alpha \leq 1$. Then for all a positive integer N, we have*

$$\begin{aligned}
\Big[\alpha^{2\alpha} tr(|A^\alpha B^{1-\alpha}|)^2 \chi_{(0,\frac{1}{2}]}(\alpha) + (1-\alpha)^{2-2\alpha} tr(|A^\alpha B^{1-\alpha}|)^2 \chi_{(\frac{1}{2},1)}(\alpha)\Big] &+ r_0^2(tr(A)-tr(B))^2 + 2\alpha(1-\alpha)tr(A)tr(B) \\
&+ \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big[tr(A) f_{l-1,k}\Big(\alpha tr(A), tr(B)\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) \\
&+ tr(A) f_{l-1,k}\Big((1-\alpha)tr(B), tr(A)\Big)\chi_{(\frac{2^l-k}{2^l},\frac{2^l-k+1}{2^l})}(\alpha)\Big] \\
&\leq \; tr(\alpha A + (1-\alpha)B)^2.
\end{aligned}$$

Proof. By using Theorem 2.6 and Lemma 4.1, we have

$$\begin{aligned}
tr(\alpha A + (1-\alpha)B)^2 \;&=\; \alpha^2 tr(A)^2 + (1-\alpha)^2 tr(B)^2 + 2\alpha(1-\alpha)tr(A)tr(B) \\
&\geq \Big[\alpha^{2\alpha} tr(A)^{2\alpha} tr(B)^{2(1-\alpha)} \chi_{(0,\frac{1}{2}]}(\alpha) + (1-\alpha)^{2-2\alpha} tr(A)^{2\alpha} tr(B)^{2(1-\alpha)} \chi_{(\frac{1}{2},1)}(\alpha)\Big] \\
&+ r_0^2(tr(A)-tr(B))^2 + 2\alpha(1-\alpha)tr(A)tr(B) \\
&+ \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big[tr(A) f_{l-1,k}\Big(\alpha tr(A), tr(B)\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) \\
&+ tr(A) f_{l-1,k}\Big((1-\alpha)tr(B), tr(A)\Big)\chi_{(\frac{2^l-k}{2^l},\frac{2^l-k+1}{2^l})}(\alpha)\Big] \\
&\geq \Big[\alpha^{2\alpha} tr(|A^\alpha B^{1-\alpha}|)^2 \chi_{(0,\frac{1}{2}]}(\alpha) + (1-\alpha)^{2-2\alpha} tr(|A^\alpha B^{1-\alpha}|)^2 \chi_{(\frac{1}{2},1)}(\alpha)\Big] \\
&+ r_0^2(tr(A)-tr(B))^2 + 2\alpha(1-\alpha)tr(A)tr(B) \\
&+ \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big[tr(A) f_{l-1,k}\Big(\alpha tr(A), tr(B)\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) \\
&+ tr(A) f_{l-1,k}\Big((1-\alpha)tr(B), tr(A)\Big)\chi_{(\frac{2^l-k}{2^l},\frac{2^l-k+1}{2^l})}(\alpha)\Big].
\end{aligned}$$

□

The second result of this section concerns the norms of positive semi-definite matrices which can be reads as follows:

Theorem 4.3. *Let $A, X, B \in \mathbf{M}_n(\mathbb{C})$ be positive semi-definite matrices and $0 \leq \alpha \leq 1$. Then for all a positive integer N, we have*

$$\begin{aligned}
&\Big[\alpha^{2\alpha}|||A^\alpha XB^{1-\alpha}|||^2\chi_{(0,\frac{1}{2}]}(\alpha) + (1-\alpha)^{2-2\alpha}|||A^\alpha XB^{1-\alpha}|||^2\chi_{(\frac{1}{2},1)}(\alpha)\Big] + r_0^2(|||AX||| - |||XB|||)^2 + 2\alpha(1-\alpha)|||AX||||||XB||| \\
&\qquad + \sum_{l=1}^{N-1} r_l(\alpha)\sum_{k=1}^{2^{l-1}}\Big[|||XB|||f_{l-1,k}\Big(\alpha|||AX|||, |||XB|||\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) \\
&\qquad + |||AX|||f_{l-1,k}\Big((1-\alpha)|||XB|||, |||AX|||\Big)\chi_{(\frac{2^l-k}{2^l},\frac{2^l-k+1}{2^l})}(\alpha)\Big] \\
&\qquad \leq (\alpha|||AX||| + (1-\alpha)|||XB|||)^2.
\end{aligned}$$

Proof. By using Theorem 2.6 and Lemma 4.1, we have

$$\begin{aligned}
(\alpha|||AX||| + (1-\alpha)|||XB|||)^2 &= \alpha^2|||AX|||^2 + (1-\alpha)^2|||XB|||^2 + 2\alpha(1-\alpha)|||AX||||||XB||| \\
&\geq \Big[\alpha^{2\alpha}|||AX|||^{2\alpha}|||XB|||^{2(1-\alpha)}\chi_{(0,\frac{1}{2}]}(\alpha) + (1-\alpha)^{2-2\alpha}|||AX|||^{2\alpha}|||XB|||^{2(1-\alpha)}\chi_{(\frac{1}{2},1)}(\alpha)\Big] \\
&+ r_0^2(|||AX||| - |||XB|||)^2 + 2\alpha(1-\alpha)|||AX||||||XB||| \\
&+ \sum_{l=1}^{N-1} r_l(\alpha)\sum_{k=1}^{2^{l-1}}\Big[|||XB|||f_{l-1,k}\Big(\alpha|||AX|||, |||XB|||\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) \\
&+ |||AX|||f_{l-1,k}\Big((1-\alpha)|||XB|||, |||AX|||\Big)\chi_{(\frac{2^l-k}{2^l},\frac{2^l-k+1}{2^l})}(\alpha)\Big].
\end{aligned}$$

So,

$$\begin{aligned}
(\alpha|||AX||| + (1-\alpha)|||XB|||)^2 &\geq \Big[\alpha^{2\alpha}|||A^\alpha XB^{1-\alpha}|||^2\chi_{(0,\frac{1}{2}]}(\alpha) + (1-\alpha)^{2-2\alpha}|||A^\alpha XB^{1-\alpha}|||^2\chi_{(\frac{1}{2},1)}(\alpha)\Big] \\
&+ r_0^2(|||AX||| - |||XB|||)^2 + 2\alpha(1-\alpha)|||AX||||||XB||| \\
&+ \sum_{l=1}^{N-1} r_l(\alpha)\sum_{k=1}^{2^{l-1}}\Big[|||XB|||f_{l-1,k}\Big(\alpha|||AX|||, |||XB|||\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) \\
&+ |||AX|||f_{l-1,k}\Big((1-\alpha)|||XB|||, |||AX|||\Big)\chi_{(\frac{2^l-k}{2^l},\frac{2^l-k+1}{2^l})}(\alpha)\Big].
\end{aligned}$$

□

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