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Nonconvex Bifunction General Variational Inequalities

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Abstract

In this paper, we introduce and consider a new class of variational inequalities, which is called the nonconvex bifunction general variational inequality. Using the auxiliary principle technique, we suggest and analyze some iterative methods for solving the nonconvex bifunction general variational inequalities. We prove that the convergence of these methods either requires only pseudomonotonicity or partially relaxed strongly monotonicity. Our proofs of convergence are very simple. The ideas and techniques of this paper may stimulate further research in this field.

Keywords: Convex programming, Nonlinear programming, Bifunction variational inequalities, Nonconvex functions, Auxiliary principle technique, Convergence

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1. Introduction

Variational inequalities theory, which was introduced by Stampacchia [27], can be viewed as an important and significant extension of the variational principles. It is well known that the variational inequalities represent the optimality condition of the convex function. For the directional differentiable convex functions, we have another class of variational inequalities, which is known as the bifunction (directional) variational inequalities. Crespi et al.[4, 5, 6, 7], Fang et al. [8], Lalitha et al. [10] and Noor et al. [23] have studied various aspects of the bifunction variational inequalities. Noor [20] has shown that the optimality condition for a class of directional differentiable nonconvex functions on a nonconvex set can be characterized by a class of bifunction variational inequalities. This fact motivated us to introduce and consider the general nonconvex bifunction variational inequalities on uniformly prox-regular sets. It is known [3, 27] that the prox-regular sets are nonconvex and include the convex sets as a special cases. Noor [15, 16, 17, 18, 19, 20] and Bounkhel et al. [2] have considered variational inequality on the uniformly prox-regular sets. There are a substantial number of numerical methods including projection technique and its variant forms, Wiener-Hop equations, auxiliary principle and resolvent equations methods for solving variational inequalities. However, it is known that projection, Wiener-Hop equations, proximal and resolvent equations techniques cannot be extended and generalized to suggest and analyze similar iterative methods for solving the general nonconvex bifunction variational inequalities. This fact has motivated to use the auxiliary principle technique, which is mainly due to mainly due to Glowinski el al. [9]. Noor [14] and Noor et al [23, 24] have used this technique to develop

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some iterative schemes for solving various classes of variational inequalities. We point out that this technique does not involve the projection of the operator and is flexible.This technique deals with finding the auxiliary variational inequality and proving that the solution of the auxiliary problem is the solution of the original problem by using the fixed-point approach. It turned out that this technique can be used to find the equivalent differentiable optimization problems, which enables us to construct gap (merit) functions. It is well known that a substantial number of numerical methods can be obtained as special cases from this technique. We use this technique to suggest and analyze some explicit predictor-corrector methods for general variational inequalities.

Motivated and inspired by ongoing research in this field, we introduce and consider a new class of bifunction variational inequalities involving two operators, which is called nonconvex bifunction general variational inequality. We again use the auxiliary principle technique to suggest and analyze a class of iterative methods for solving the general nonconvex bifunction variational inequalities. We also prove that the convergence of these new methods either require pseudomonotonicity or partially relaxed strongly monotonicity, which are weaker conditions. In this respect, our results represent an improvement and refinement of the known results for nonconvex bifunction general variational inequalities. For suitable choice of the operator and proximate convex set, one can obtain several new and known results for variational inequalities and optimization programming. It is an interesting problem to illustrate the efficiency of the proposed methods and compare with other methods, Ww would like to emphasize that theory of bifunction general variational inequalities is quite broad. We have given the flavour of the ideas and techniques involved. The techniques used to analysis the iterative methods and other results for bifunction general variational inequalities are a beautiful blend of ideas of pure and applied mathematical sciences.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $||.||$ respectively. Let K be a nonempty and convex set in H. We, first of all, recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis, which are due to Clarke et al. [3] and Poliquin et al. [27]. Poliquin et al. [27] and Clarke et al.[3] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets.

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = ||u - u^*||\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v \in K} ||v - u||.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.2. *Let K be a nonempty, closed and convex subset in H. Then $\zeta \in N_K^P(u)$, if and only if, there exists a constant $\alpha > 0$ such that*

$$\langle \zeta, v - u \rangle \leq \alpha ||v - u||^2, \quad \forall v \in K.$$

Definition 2.3. For a given $r \in (0, \infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal cone to K_r can be realized by an r-ball, that is,$\forall u \in K_r$ and $0 \neq \xi \in N_{K_r}^P(u)$, one has

$$\langle (\xi)/||\xi||, v - u \rangle \leq (1/2r)||v - u||^2, \quad \forall v \in K_r.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see Clarke et al. [3] and Poliquin et al. [27]. It is well-known that the union of two disjoint intervals [a,b] and [c,d] is a prox-regular set with $r = \frac{c-b}{2}$. For other examples of prox-regular sets, see Noor [20] . Obviously, for $r = \infty$, the uniformly prox-regularity of K_r is equivalent to the convexity of K. This class of uniformly prox-regular sets have played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. It is known that if K_r is a uniformly prox-regular set, then the proximal normal cone $N^P_{K_r}(u)$ is closed as a set-valued mapping.

For given bifunction $B(.,.) : H \times H \longrightarrow H$ and nonlinear operator $g : H \longrightarrow H$, we consider the problem of finding $u \in H : g(u) \in K_r$ such that

$$B(g(u), g(v) - g(u)) \geq 0, \quad \forall v \in H : g(v) \in K_r, \tag{2.1}$$

which is called the *nonconvex bifunction general variational inequality.*

We now discuss some important special cases nonconvex bifunction general variational inequality.

Special Cases

(I). We note that, if $K_r \equiv K$, the convex set in H, then problem (2.1) is equivalent to finding $u \in H : g(u) \in K$ such that

$$B(g(u), g(v) - g(u)) \geq 0, \qquad \forall v \in H : g(v) \in K. \tag{2.2}$$

Inequality of type (2.2) is called the *bifunction general variational inequality,* which appears to be a new one.

(II). If $B(g(u), g(v) - g(u)) = \langle Tu, g(v) - g(u)\rangle$, where T is a nonlinear operator, then problem (2.1) is equivalent to finding $u \in H : g(u) \in K_r$ such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K_r, \tag{2.3}$$

which is called the general nonconvex variational inequality, see Noor [15, 16, 17, 18, 19, 20].

(III). If $B(g(u), g(v) - g(u)) = \langle Tu, g(v) - g(u)\rangle$, where T is a nonlinear operator and $K_r = K$, the convex set, then problem (2.3) is equivalent to finding $u \in H : g(u) \in K_r$ such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.4}$$

which is called the general variational inequality, introduced and studied by Noor [11, 12, 13, 14]. It has been shown a wide class of nonsymmetric and odd-order obstacle boundary value and initial value problems can be studied in the general framework of general variational inequalities (2.4). For the applications, numerical methods, sensitivity analysis, dynamical system, merit functions and other aspects of general variational inequalities, see Al-Said et al. [1], Noor at al. [21, 22, 23, 24, 25, 26] and the references therein.

(IV). If $g \equiv I$, the identity operator, then problem (2.4) reduces to finding $u \in K_r$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K_r, \tag{2.5}$$

which is called the nonconvex variational inequality, see Noor [16, 20].

(V). If $K_r \equiv K$, the convex set, then problem (2.5) reduces to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K, \tag{2.6}$$

which is called the classical variational inequality, introduced and studied by Stampacchia [28]. It turned out that a number of unrelated obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via variational inequalities. It is worth mentioning that for suitable and appropriate choice of the operators, nonconvex sets and spaces, one can obtain a wide class of variational inequalities and optimization programming. This shows that the nonconvex bifunction general variational inequalities are quite flexible and unified ones.

3. Main Results

In this section, we use the auxiliary principle technique of Glowinski et al.[9] as developed by Noor [14] and Noor et al [23, 24] suggest and analyze a some iterative methods for solving the nonconvex bifunction general nonconvex bifunction variational inequality (2.1). This technique does not involve the concept of the projection, which is the main advantage of this technique.

For a given $u \in H : g(u) \in K_r$ satisfying (2.1), consider the problem of finding $w \in H : g(w) \in K_r$ such that

$$\rho B(g(w), g(v) - g(w)) + \langle w - u, v - w\rangle \geq 0, \forall v \in H : g(v) \in K_r, \tag{3.1}$$

where $\rho > 0$ is a constant. Inequality of type (3.1) is called the auxiliary nonconvex bifunction general variational inequality. Note that if $w = u$, then w is a solution of (2.1). This simple observation enables us to suggest the following iterative method for solving the general nonconvex bifunction variational inequalities (2.1).

Algorithm 3.1. *For a given $u_0 \in K_r$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(g(u_{n+1}), g(v) - g(u_{n+1}) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in H : g(v) \in K_r. \tag{3.2}$$

Algorithm 3.1 is called the proximal point algorithm for solving general nonconvex bifunction variational inequality (2.1).
In particular, if $r = \infty$, then the uniformly prox-regular set K_r becomes the standard convex set K, and consequently Algorithm 3.1 reduces to:

Algorithm 3.2. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(g(u_{n+1}), g(v) - g(u_{n+1})) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in H : g(v) \in K,$$

which is known as the proximal point algorithm for solving bifunction variational inequalities (2.2) and has been studied extensively, see Noor [11, 12, 13, 14, 15, 16, 17, 17, 18, 19, 20].

For the convergence analysis of Algorithm 3.1, we recall the following concepts and results.

Definition 3.3. A bifunction $B(.,.) : H \times H \to H$ with respect to the operator g is said to be:
(i) *monotone*, if and only if,

$$B(g(u), g(v) - g(u)) + B(g(v), g(u) - g(v)) \leq 0, \quad \forall u, v \in H.$$

(ii) *pseudomonotone*, if and only if,

$$B(g(u), g(v) - g(u)) \geq 0 \quad \text{implies that} \quad -B(g(v), g(u) - g(v)) \geq 0, \quad \forall u, v \in H.$$

(iii) *partially relaxed strongly monotone,* if and only if, there exists a constant $\alpha > 0$ such that

$$B(g(u), g(v) - g(u)) + B(g(v), g(z) - g(v)) \leq \alpha\|z - u\|^2, \quad \forall u, v, z \in H.$$

Note that for $z = u$, partially relaxed strongly monotonicity reduces to monotonicity. It is known that cocoercivity implies partially relaxed strongly monotonicity, but the converse is not true. It is known that monotonicity implies pseudomonotonicity; but the converse is not true.

We also recall the well-known result.

$$2\langle u, v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2. \tag{3.3}$$

We now consider the convergence criteria of Algorithm 3.1 and this is the main motivation of our next result.

Theorem 3.4. *Let the operator $B(.,.) : K_r \times K_r \longrightarrow H$ be pseudomonotone. If u_{n+1} is the approximate solution obtained from Algorithm 3.1 and $u \in K_r$ is a solution of (2.1), then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2. \tag{3.4}$$

Proof. Let $u \in H : g(u) \in K_r$ be a solution of (2.1). Then

$$-B(g(v), g(u) - g(v)) \geq 0, \quad \forall v \in H : g(v) \in K_r, \tag{3.5}$$

since $B(.,.)$ is pseudomonotone.

Taking $v = u_{n+1}$ in (3.5), we have

$$-B(g(u_{n+1}), g(u) - g(u_{n+1})) \geq 0. \tag{3.6}$$

Setting $v = u$ in (2.2), and using (3.2), we have

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq -\rho B(g(u_{n+1}), g(u_{n+1}) - g(u)) \geq 0. \tag{3.7}$$

Setting $v = u - u_{n+1}$ and $u = u_{n+1} - u_n$ in (2.3), we obtain

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2 - \|u - u_{n+1}\|^2. \tag{3.8}$$

From (3.3) and (3.8), we obtain (3.4), which is the required result. □

Theorem 3.5. *Let H be a finite dimension subspace and let u_{n+1} be the approximate solution obtained from Algorithm 3.1. If $u \in K_r$ is a solution of (2.1), then* $\lim_{n\longrightarrow\infty} u_n = u$.

Proof. Let $u \in H : g(u) \in K_r$ be a solution of(2.1). Then it follows from (3.4) that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \|u_n - u_{n+1}\|^2 \leq \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_n - u_{n+1}\| = 0. \tag{3.9}$$

Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$ and let the subsequence $\{u_j\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in K_r$. replacing u_n by u_{n_j} in (3.9) and taking the limit $n_j \longrightarrow \infty$ and using (3.2), we have

$$B(g(\hat{u}), g(v) - g(\hat{u})) \geq 0, \quad \forall \quad v \in H : g(v) \in K_r,$$

which implies that $\hat{u}$ solves the general nonconvex bifunction variational inequality (2.1) and

$$\|u_n - u_{n+1}\|^2 \leq \|\hat{u} - u_n\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\longrightarrow\infty} u_n = \hat{u}$. the required result. □

We note that for $r = \infty$, the r-prox-regular set K becomes a convex set and the nonconvex bifunction variational inequality (2.1) collapses to the bifunction variational inequality (2.2). Thus our results include the previous known results as special cases.

It is well-known that to implement the proximal point methods, one has to calculate the approximate solution implicitly, which is in itself a difficult problem. To overcome this drawback, we suggest another iterative method, the convergence of which requires only partially relaxed strongly monotonicity, which is a weaker condition that cocoercivity.

For a given $u \in H : g(u) \in K_r$ satisfying (2.1), consider the problem of finding $w \in H : g(w) \in K_r$ such that

$$\rho B(g(u), g(v) - g(w)) + \langle w - u, v - w\rangle \geq 0, \quad \forall \quad v \in H : g(v) \in K_r, \tag{3.10}$$

which is also called the auxiliary nonconvex bifunction general variational inequality. Note that problems (2.3) and (3.10) are quite different. If $w = u$, then clearly w is a solution of the nonconvex bifunction general variational inequality (2.1). This fact enables us to suggest and analyze the following iterative method for solving the nonconvex bifunction general variational inequality (2.1).

Algorithm 3.6. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(g(u_n), g(v) - g(u_{n+1})) + \langle u_{n+1} - u_n, v - u_{n+1} \geq 0, \forall v \in H : g(v) \in K_r. \tag{3.11}$$

Note that, for $r = \infty$, the uniformly prox-regular set K_r becomes a convex set K and Algorithm 3.6 reduces to:

Algorithm 3.7. *For a given $u_0 \in K$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(g(u_n), g(v) - g(u_{n+1})) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in H : g(v) \in K,$$

which is known as the iterative method for solving bifunction variational inequalities (2.2).

We now study the convergence of Algorithm 3.6 and this is the main motivation of our next result.

Theorem 3.8. *Let the operator $B(.,.)$ be partially relaxed strongly monotone with constant $\alpha > 0$. If u_{n+1} is the approximate solution obtained from Algorithm 3.6 and $u \in H : g(u) \in K_r$ is a solution of (2.1), then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \{1 - 2\rho\alpha\}\|u_n - u_{n+1}\|^2. \tag{3.12}$$

Proof. Let $u \in H : g(v) \in K_r$ be a solution of (2.1). Then

$$B(g(u), g(v) - g(u)) \geq 0, \quad \forall v \in H : g(v) \in K_r. \tag{3.13}$$

Taking $v = u_{n+1}$ in (3.13), we have

$$B(g(u), g(u_{n+1}) - g(u)) \geq 0. \tag{3.14}$$

Letting $v = u$ in (3.11), we obtain

$$\rho B(g(u_n), g(u) - g(u_{n+1})) + \langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq 0,$$

which implies that

$$\begin{aligned} \langle u_{n+1} - u_n, u - u_{n+1}\rangle &\geq -\rho B(g(u_n), g(u) - g(u_{n+1})) \\ &\geq -\rho\{B(g(u_n), g(u) - g(u_{n+1})) + B(g(u), g(u_{n+1}) - g(u))\} \\ &\geq -\alpha\rho\|u_n - u_{n+1}\|^2. \end{aligned} \tag{3.15}$$

since $B(.,.)$ is partially relaxed strongly monotone with constant $\alpha > 0$.

Combining (3.14) and (3.15), we obtain the required result (3.12). □

Using essentially the technique of Theorem 3.5, one can study the convergence analysis of Algorithm 3.6.

Using again the auxiliary principle technique, we can consider the following the following problems: For a given $u \in H : g(u) \in K_r$ satisfying (2.1), consider the problem of finding $w \in H : g(w) \in K_r$ such that

$$\rho B(g(w), g(v) - g(w)) + w - u + \alpha(u - u), v - w\rangle \geq 0, \forall v \in H : g(v) \in K_r, \tag{3.16}$$

where $\rho > 0$ and α are constants. Note that, if $w = u$, then w is a solution of (2.1). Consequently, one can suggest and analyze the following iterative method for solving the nonconvex bifunction general variational inequality (2.1).

Algorithm 3.9. *For a given $u_0 \in K_r$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(g(u_{n+1}), g(v) - g(u_{n+1}) + \langle u_{n+1} - u_n + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in H : g(v) \in K_r. \tag{3.17}$$

Algorithm 3.9 is called the inertial proximal point algorithm for solving nonconvex bifunction general variational inequality (2.1).

Using again the auxiliary principle technique, we can consider the following the following problems: For a given $u \in H : g(u) \in K_r$ satisfying (2.1), consider the problem of finding $w \in H : g(w) \in K_r$ such that

$$\rho B(g(u), g(v) - g(w)) + w - u + \alpha(u - u), v - w\rangle \geq 0, \forall v \in H : g(v) \in K_r, \tag{3.18}$$

where $\rho > 0$ and α are constants. Note that, if $w = u$, then w is a solution of (2.1). Consequently, one can suggest and analyze the following iterative method for solving the nonconvex bifunction general variational inequality (2.1).

Algorithm 3.10. *For a given $u_0 \in K_r$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(g(u_n), g(v) - g(u_{n+1}) + \langle u_{n+1} - u_n + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in H : g(v) \in K_r. \tag{3.19}$$

Algorithm 3.10 is called the inertial explicit algorithm for solving nonconvex bifunction general variational inequality (2.1).

For a given $u \in H : g(u) \in K_r$ satisfying (2.1), consider the problem of finding $w \in H : g(w) \in K_r$ such that

$$\rho B(g(w), g(v) - g(w)) + \langle g(w) - g(u), g(v) - g(w)\rangle \geq 0, \forall v \in H : g(v) \in K_r, \tag{3.20}$$

where $\rho > 0$ is a constant. Inequality of type (3.20) is called the auxiliary nonconvex bifunction general variational inequality. Note that if $w = u$, then w is a solution of (2.1). This simple observation enables us to suggest the following iterative method for solving the general nonconvex bifunction variational inequalities (2.1).

Algorithm 3.11. *For a given $u_0 \in K_r$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(g(u_{n+1}), g(v) - g(u_{n+1}) + \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1}))\rangle \geq 0, \quad \forall v \in H : g(v) \in K_r.$$

Algorithm 3.11 is called the general proximal point algorithm for solving general nonconvex bifunction variational inequality (2.1).
Convergence analysis of Algorithm 3.9,Algorithm 3.10 and Algorithm 3.11 can be studied using the above ideas and techniques of Noor [16] and Noor at al. [23, 24].

4. Conclusion

In this paper, we have introduced and considered a new class of nonconvex bifunction general variational inequalities involving two arbitrary operators. Some special cases are discussed. The auxiliary principle technique is used to suggest some iterative methods for nonconvex bifunction general variational inequalities. Convergence analysis of the proposed methods is investigated under pseudo-monotonicity and partially relaxed strongly monotonicity. It is an open problem to compare the efficiency of the inertial and proximal methods with other methods and this is another direction for future research. Comparison of these methods need further efforts.

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