Montes Taurus J. Pure Appl. Math. 3 (2), 47–50, 2021

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

# Lucas Cube vs Zeckendorf's Lucas Code

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**Abstract**

The theorem of Zeckendorf states that every positive integer $n$ can be uniquely decomposed as a sum of non consecutive Lucas numbers in the form $n = \sum_i b_i L_i$, where $b_i \in \{0, 1\}$ and satisfy $b_0 b_2 = 0$. The Lucas string is a binary string that do not contain two consecutive 1's in a circular way. In this note, we derive a bijection between the set of Zeckendorf's Lucas codes and the set of vertices of the Lucas' strings.

*Keywords:* Fibonacci numbers, Lucas numbers, Fibonacci cube, Lucas cube

*2010 MSC:* 11B39, 05C99

## 1. Introduction

The Fibonacci numbers $(F_n)_{n\geq 0}$ given by $F_0 = 0$, $F_1 = 1$ and

$$F_{n+1} = F_n + F_{n-1}, n \geq 1,$$

constitute a basis element of the binary Fibonacci numeration system, see for instance [11]. Every integer $0 \leq N < F_{n+1}$ has a unique binary representation $a_2 a_3 \cdots a_n$, such that

$$N = \sum_{i=2}^{n} a_i F_i,$$

with $a_i \in \{0, 1\}$ and $a_i a_{i+1} = 0$, for $2 \leq i \leq n - 1$. The binary string $a_2 \cdots a_n$ is called a Zeckendorf's Fibonacci code. According to this representation, a Zeckendorf's Fibonacci code does not contain two consecutive 1's. It follows that, $F_{n+2}$ can be regarded as the cardinality of the set of binary strings of length $n$ without two consecutive ones.
The Lucas numbers, $(L_n)_{n\geq 0}$ are defined by the recursion $L_{n+2} = L_{n+1} + L_n$ with initial conditions $L_0 = 2$ and $L_1 = 1$. For some properties on the Fibonacci and Lucas numbers see Kochy [7]. In [2, 11], authors showed that the Lucas

†*Article ID:* MTJPAM-D-21-00025

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Received:29 March 2021, Accepted:12 April 2021, Published:15 May 2021
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numbers are complete in the sense that every positive integer can be expressed as sum of distinct Lucas numbers. In fact, every integer has a unique representation

$$N = \sum_{i=0}^{n} b_i L_i,$$

where each $b_0 \cdots b_n$ is a binary string satisfying $b_i b_{i+1} = 0$ for $0 \leq i \leq n-1$ and $b_0 b_2 = 0$. Let us mention that the condition $b_i b_{i+1} = 0$ prevents the use of two successive Lucas numbers, the additional condition reflects the particularity of the Lucas sequence.

The Fibonacci cube of dimension $n$, denoted $\Gamma_n$, was introduced by Hsu [3] and defined as a graph whose vertex set is the set of Zeckendorf's Fibonacci codes of length $n$ also called Fibonacci strings where two vertices are adjacent if their Hamming distance is equal to 1, see [5] for a survey.
The Lucas string is a binary string that do not contain two consecutive 1's in a circular way, i.e. it is a Fibonacci string with the additional condition that the first and the last bit are not equal to 1 simultaneously. The Lucas cube, introduced by Munarini et *al.* [9] and denoted by $\Lambda_n$, is a subgraph of the $n$-cube induced by the set of all Lucas strings of length $n$. $\Lambda_n$ can also be regarded as a subgraph of $\Gamma_n$, obtained by removing all the vertices that start and end with 1. The number of vertices of $\Lambda_n$ is $L_n$ and the number of edges is expressed in terms of Fibonacci and Lucas numbers, $\#E(\Lambda_n) = \sum_{i=1}^{n-1} F_i L_{n-1-i}$. Since their introduction, Lucas cubes have been the subject of numerous works in where structural and enumerative properties have been obtained, see for instance [6, 8, 1, 4, 10].

The main question we answer in this note is: "Given the Zeckendorf's Lucas code of any integer how can we find the corresponding Lucas' string and conversely?"

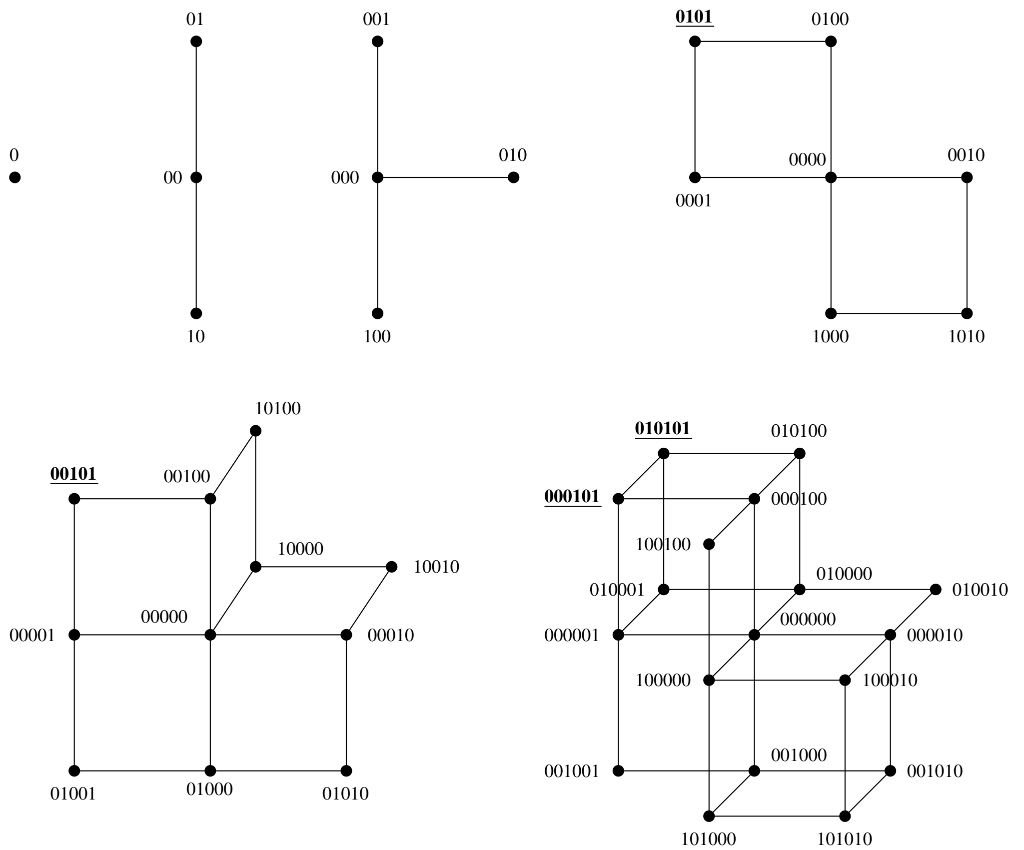


Figure 1. Representation of $\Lambda_n$ for $n \leq 6$, the vertices in bold underlined do not correspond to Zeckendorf's Lucas codes.

## 2. The main result

Comparing the set of Zeckendorf's Lucas code and the set of Lucas string of length $n$, as illustrated in Table 1 and Figure 1, we can see that not all of the Zeckendorf's Lucas codes correspond to the Lucas strings, and conversely. That motivates us to construct a bijective mapping between the set of the Lucas strings and the set of the Zeckendorf's Lucas codes. Notice that to get Zeckendorf's Lucas code of length $n$, we add zeros at the left of Zeckendorf's Lucas code.

| Integer | Zeckendorf's Lucas code | Integer | Zeckendorf's Lucas code |
|---|---|---|---|
| 1 | 10 | 11 | 100000 |
| 2 | 01 | 12 | 100010 |
| 3 | 100 | 13 | 100001 |
| 4 | 1000 | 14 | 100100 |
| 5 | 1010 | 15 | 101000 |
| 6 | 1001 | 16 | 101010 |
| 7 | 10000 | 17 | 101001 |
| 8 | 10010 | 18 | 1000000 |
| 9 | 10001 | 19 | 1000010 |
| 10 | 10100 | 20 | 1000001 |

Table 1. Zeckendorf's Lucas code of 20 first integers.

Before stating the main theorem, we set for $n \geq 0$ :
$\mathcal{F}_n$ : the set of all the Zeckendorf's Fibonacci codes of length $n$;
$\mathcal{L}_n$ : the set of all the Zeckendorf's Lucas codes of length $n$;
$C_n$ : the set of all vertices of Lucas's cube of length $n$;
$\mathcal{A}_n = \{\alpha_{n-1}0; \alpha_n \in \mathcal{F}_n\}$: the set of all Fibonacci codes ending by 0.

**Theorem 2.1.** *For any integer $n \geq 0$, the function $f : C_n \to \mathcal{L}_n$ defined for $\beta_n \in \mathcal{A}_n$, by*

$$f(x_n) = \begin{cases} 10\beta_{n-4}01 & \text{for } x_n \text{ of the form } 0\beta_{n-4}101, \\ x_n & \text{otherwise.} \end{cases}$$

*is bijective.*

*Proof.* We show that $f$ is well defined and bijective function. To prove that it is well defined, it suffices to observe that $f(x_n)$ is in $\mathcal{L}_n$ for any $x_n \in C_n$.
Now, we show that $f$ is injective. Given $x_n$ and $y_n$ two vertices of Lucas cube of length $n$, such that $f(x_n) = f(y_n)$.
If $x_n \neq 0\beta_{n-4}101$ and $y_n \neq 0\alpha_{n-4}101$ then $f(x_n) = f(y_n)$ implies $x_n = y_n$.
If $x_n = 0\beta_{n-4}101$ and $y_n = 0\alpha_{n-4}101$ then $f(x_n) = f(y_n)$ implies $10\beta_{n-4}01 = 10\alpha_{n-4}01$, which gives $\beta_{n-4} = \alpha_{n-4}$ thus $x_n = y_n$.
If $x_n = 0\beta_{n-4}101$ and $y_n \neq 0\alpha_{n-4}101$, then $f(x_n) = f(y_n)$ gives $10\beta_{n-4}01 = y_n$ which contradicts $y_n \in C_n$, so this case is rejected. Therefore, $f$ is injective.
Finally, we prove that $f$ is surjective. Let $y_n \in \mathcal{L}_n$ be a Zeckendorf's Lucas code of length $n$ then, $y_n$ is a Zeckendorf's Fibonacci code of length $n$ and $y_n \neq \alpha_{n-3}101$, $\alpha_{n-3} \in \mathcal{A}_{n-3}$. We attest that $\mathcal{L}_n = \mathcal{F}_n \setminus \{\alpha_{n-3}101, \alpha_{n-3} \in \mathcal{A}_{n-3}\}$, with $\#\mathcal{F}_n = F_{n+2}$ and $\#\mathcal{A}_n = F_{n+1}$. It follows that

$$\#\mathcal{L}_n = \#\mathcal{F}_n - \#\{\alpha_{n-3}101, \alpha_{n-3} \in \mathcal{A}_{n-3}\} = F_{n+2} - F_{n-2} = L_n.$$

We have shown that $f$ is injective and $\#\mathcal{L}_n = \#C_n = L_n$. Therefore, $f$ is bijective mapping as desired.

## Acknowledgements

The paper was partially supported by the DGRSDT grant number C0656701.

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