Montes Taurus J. Pure Appl. Math. 3 (2), 70–79, 2021

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Power Exponential Mean Labeling of Graphs

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Abstract

A (p, q) graph G is said to be a power exponential mean graph if there exist a one to one correspondence $f : V \to \{1, 2, 3, \ldots, p\}$ such that induced function $f^* : E(G) \to N$ given by

$$f^*(uv) = \left\lceil \left(f(u)^{f(u)} f(v)^{f(v)}\right)^{\frac{1}{f(u)+f(v)}} \right\rceil \qquad \text{or} \qquad f^*(uv) = \left\lfloor \left(f(u)^{f(u)} f(v)^{f(v)}\right)^{\frac{1}{f(u)+f(v)}} \right\rfloor$$

for every $uv \in E(G)$ are all distinct. In this paper the power exponential mean labeling of graphs such as path, cycle, $K_1 + C_n$ for n is odd and even, square graph, umbrella $U(m, n)$, duplicating each vertex by an edge in path P_n, comb, $C_m \bigodot \overline{K_1}$ and $C_n \bigodot \overline{K_2}$ are discussed.

Keywords: Graph, power exponential mean, path, cycle, square graph, umbrella, comb

2010 MSC: 26D10, 26D15

1. Introduction

The mean values are closely related to the mean value theorems, which are the bridge between local and global properties of functions. Inequality involving arithmetic mean and geometric mean is probably the most important and certainly a keystone to the theory of inequalities. Georghe Toader and Silvia Toader gave a brief collection of ten Greek means and their comparison, the partial derivatives of means and related results [2].

A continuous function $m : x \times x \to x$ belongs to the class m of means if it has the following properties: $x \wedge y \leq m(x, y) \leq x \vee y$ and $m(x, y) = x$, if and only if $x = y$.

If the vertices are assigned values subject to certain conditions then it is known as a graph labelling. The mean labeling was introduced by Somasundaram and Ponraj. A detailed survey on graph labeling is carried out by Gallian and its various applications by Bloom and Golomb.

In [15], Somasundaram et al. defined the geometric mean labeling of a graph as $f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil$ or $f^*(uv) = \left\lfloor \sqrt{f(u)f(v)} \right\rfloor$ for all $u, v \in E(G)$. In a graph G with p vertices and q edges the results on paths, cycles, combs, ladders

†*Article ID:* MTJPAM-D-21-00024

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Received:18 March 2021, Accepted:14 May 2021, Published:12 June 2021

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of geometric mean labeling graphs are developed. Further, disproved that $K_n(n > 4)$ and $K_{1,n}(n > 5)$ are not geometric mean graphs and proved $C_m \cup P_n$; $C_m \cup C_n$; nK_3; $nK_3 \cup P_n$; $nK_3 \cup C_m$; P_n^2 and crowns are geometric mean graphs. Also, investigated geometric mean labelings in the context of duplication of graph elements in cycle C_n and path P_n. In [14], the notion of mean labeling was introduced by Somasundaram and Ponraj. In a graph G with p vertices and q edges the results on paths, cycles, comb, splitting graph, shadow graph, bistarare discussed.

According to Beineke and Hegde, graph labeling serves as a frontier between number theory and structure of graphs. In [7], harmonic mean labeling of a graph is introduced is defined as $f^*(uv) = \left\lceil \frac{\sqrt{2f(u)f(v)}}{f(u)+f(v)} \right\rceil$ or $f^*(uv) = \left\lfloor \frac{\sqrt{2f(u)f(v)}}{f(u)+f(v)} \right\rfloor$ for all $u, v \in E(G)$ and investigated that for a polygonal chain, square of the path and dragon are harmonic mean graph of order atmost 5. In [7], authors proved that some disconnected graphs are harmonic mean graphs.

Abundant literature exists as of today concerning the structure of graphs admitting a variety of function assigning real numbers to their elements so that given conditions are satisfied. This work motivate to develop this paper. Now, recall some of the definitions which are essential.

Graph: A graph G is a pair (V, E), where V is a nonempty set and E is a set of unordered pairs of elements taken from the set V. A graph which does not contain loops and multiple edges is a simple graph, a finite number of vertices and edges in a graph is a finite graph and undirected with p vertices and q edges. The cardinality of vertex set V of a graph is the order and the cardinality of edge set E is called the size of the graph G. The graph $G - e$ is obtained from G by deleting an edge e.

Union: The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G = G_1 \cup G_2$ with vertex set $V = V_1 \cup V_2$ and the edge set $E = E_1 \cup E_2$.

Corona: The corona of two graphs G_1 and G_2 is the graph $G = G_1 \bigodot G_2$ formed by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex is the i^{th} copy of G_2.

Umbrella: For any integers $m > 2$ and $n > 1$, the umbrella graph $U(m, n)$ whose vertex set is defined as $V(U(m, n)) = \{x_1, x_2, \ldots x_m, y_1, y_2, \ldots, y_n\}$

$$V(U(m,n)) = \begin{cases} (x_i, x_{i+1}), & if\ i = 1,\ 2,\ 3, \ldots, (m-1) \\ (y_i, y_{i+1}), & if\ i = 1,\ 2,\ 3, \ldots, (n-1) \\ (x_i, y_1), & if\ i = 1,\ 2,\ 3, \ldots, m \end{cases} \tag{1.1}$$

For other terminology and notations refer [6].

2. Power exponential mean labeling of a graphs

Here, the power exponential mean and its labeling of graphs are discussed.

Definition 2.1 ([8, 9]). For $a, b > 0$, the power exponential mean is denoted by $Z(a, b)$ and is defined as $Z = Z(a, b) = (a^a b^b)^{\frac{1}{a+b}}$.

Definition 2.2 ([12]). A graph G with n vertices and m edges is called a power exponential mean graph if the function $f : V(G) \to A \subseteq N$ to label the vertices $u \in V(G)$ with distinct labels $f(u)$ and each edge $e = uv$ is labeled with $f^*(uv) = \left\lceil \left(f(u)^{f(u)} f(v)^{f(v)}\right)^{\frac{1}{f(u)+f(v)}} \right\rceil$ or $f^*(uv) = \left\lfloor \left(f(u)^{f(u)} f(v)^{f(v)}\right)^{\frac{1}{f(u)+f(v)}} \right\rfloor$ for every $u_i, u_j \in V(G)$ and $u_i \neq u_j$ are all distinct.

3. Main results

In this section, the results on power exponential mean labeling of graphs are discussed.

Theorem 3.1. *Any path is a power exponential mean graph.*

Proof. A path P_n with q edges to be a power exponential mean graph if there is an injective function f from the vertices of G to N such that when each edge are labeled with the induced function $f^* : E(G) \to N$ given by

$$f^*(uv) = \left\lceil \left(f(u)^{f(u)} f(v)^{f(v)}\right)^{\frac{1}{f(u)+f(v)}} \right\rceil = \left\lfloor \left(f(u)^{f(u)} f(v)^{f(v)}\right)^{\frac{1}{f(u)+f(v)}} \right\rfloor$$

for every $u, v \in V(G)$ and $u_i \neq u_j$, then the resulting edges are distinct. Therefore the path is a power exponential mean labeling. Note that if the vertices are labelled with odd numbers the edges admits even labelling with power exponential mean labeling, If the vertices are labelled with even numbers the edges admits odd labelling with power exponential mean labeling. □

Example 3.2. Consider the path P_7 with 7 vertices labeled by even and odd numbers. The edges shows power exponential mean labeling is as shown in the Figure 1.

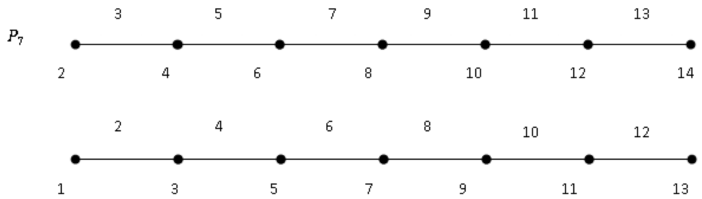

Figure 1. Path P_7

Theorem 3.3. *Any cycle $C_n, n \geq 3$ is a power exponential mean graph.*

Proof. A cycle $C_n, n \geq 3$ are power exponential mean graph if there is an injective function f from the vertices of G to N such that when each edges are labeled with the induced function $f^* : E(G) \to N$ given by

$$f^*(uv) = \left\lceil \left(f(u)^{f(u)} f(v)^{f(v)}\right)^{\frac{1}{f(u)+f(v)}} \right\rceil$$

for every $u, v \in V(G)$ and $u_i \neq u_j$, then the resulting edges are distinct. Therefore the cycles are power exponential mean labeling. □

Example 3.4. Consider the cycle C_n with $n \geq 3$ vertices labeled by odd numbers. The edges shows power exponential mean labeling is as shown in the Figure 2.

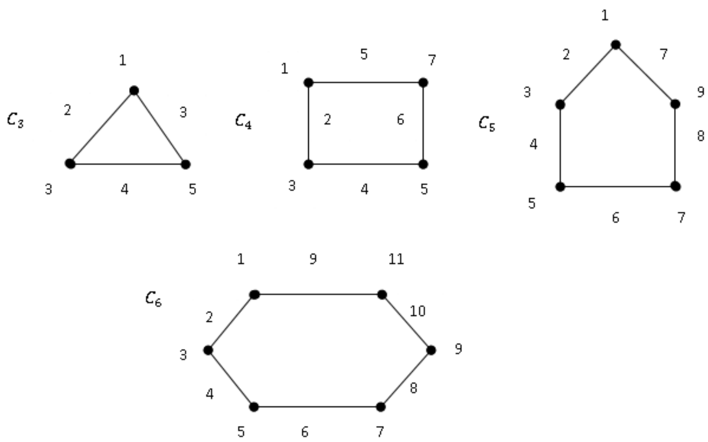

Figure 2. Cyclie C_n

Theorem 3.5. *The square graph P_n^2 is a power exponential mean graph.*

Proof. If a path P_n of n vertices $x_1, x_2, x_3, \ldots x_n$ then P_n^2 has n vertices and $(2n-3)$ edges is a graph obtained by joining the vertices whenever $d(u, v) \leq 2$.
Define $f : V(P_n^2) \rightarrow N$ by $f(x_i) = 2i - 1$, $1 \leq i \leq n$. The edges are labeled by

$$f^*(uv) = \left\lfloor \left(f(u)^{f(u)} f(v)^{f(v)} \right)^{\frac{1}{f(u)+f(v)}} \right\rfloor$$

for every $u, v \in V(G)$ and $x_i \neq x_j$, for all $\{uv\} = e \in E(P_n^2)$ such that $f^*(u) \neq f^*(v)$. Hence P_n^2 is a power exponential mean graph. □

Example 3.6. Consider a graph P_8^2 with 8 vertices obtained by joining the vertices when ever $d(u, v) \leq 2$. The labeling is as shown in the Figure 3.

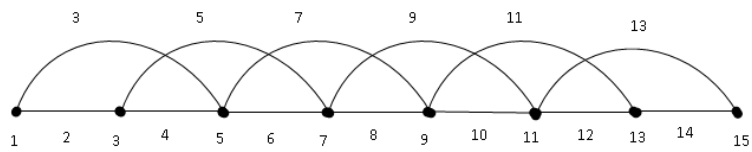

Figure 3. square graph P_8^2

Theorem 3.7. *Comb is power exponential mean graph.*

Proof. Let G be a comb. Let P_n be a path $v_1, v_2, \ldots .v_n$. Define a function $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ by $f(u_i) = 2i$ and $f(v_i) = 2i - 1$ such that $1 \leq i \leq n$. The label of the edge $\{u_i, v_i\}$ is

$$f^*(u_i v_i) = \left\lfloor \left(f(u_i)^{f(u_i)} f(v_i)^{f(v_i)} \right)^{\frac{1}{f(u_i)+f(v_i)}} \right\rfloor = 2i - 1; \quad 1 \leq i \leq n$$

for every $u_i, v_i \in V(G)$ and $u_i \neq v_i$. The label of the edge $\{v_i, v_{i+1}\}$ is

$$f^*(v_i v_{i+1}) = \left\lfloor \left(f(v_i)^{f(v_i)} f(v_{i+1})^{f(v_{i+1})} \right)^{\frac{1}{f(v_i)+f(v_{i+1})}} \right\rfloor = 2i; \quad 1 \leq i \leq n - 1$$

for every $v_i \in V(G)$ and $v_i \neq v_{i+1}$ are all distinct.
Hence combs are power exponential mean graph. □

Example 3.8. Consider a graph comb with 4 vertices. The labeling is as shown in the Figure 4.

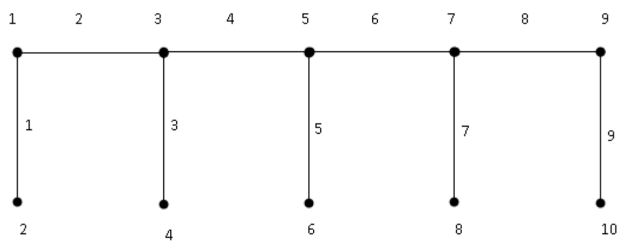

Figure 4. Comb graph

Theorem 3.9. *The graph $K_1 + C_n$ is power exponential mean graph.*

Proof. The graph $K_1 + C_n$ has $n + 1$ vertices and $2n$ edges. Let v be vertex of K_1 and $v_1, v_2, \ldots, v_n$ be the vertices of the cycle.

The ordinary labeling of $K_1 + C_7$ and $K_1 + C_6$ are given in the Figure 5 and Figure 6.

Define a vertex labeling $f : V(K_1 + C_n) \to \{1, 3, 5, \ldots, (2q - 1)\}$ by $f(u) = 1$.

Case-1: If n is odd, the vertices are labeled by

$$f(v_i) = 4i - 1; \quad 1 \leq i \leq n. \tag{3.1}$$

Case-2: If n is even, the vertices are labeled by

$$f(v_i) = \begin{cases} 3, & \text{if } i = 1 \\ 2i + 3, & \text{if } i \text{ is even} \\ q + 2i + 1, & \text{if } i \text{ is odd} \end{cases} \tag{3.2}$$

where q is no of edges in the graph.
Clearly, for n is odd, the edges are labeled by the power exponential mean

$$f^*(uv) = \left\lceil \left(f(u)^{f(u)} f(v)^{f(v)} \right)^{\frac{1}{f(u)+f(v)}} \right\rceil$$

for every $u, v \in V(G)$ and $u_i \neq v_j$.
The edge

$$f^*(x_1 x_n) = \left\lfloor \left(f(u)^{f(u)} f(v)^{f(v)} \right)^{\frac{1}{f(u)+f(v)}} \right\rfloor$$

and the edges in K_1 by the power exponential mean of the labels on end vertices

$$f^*(uv) = \left\lfloor \left(f(u)^{f(u)} f(v)^{f(v)} \right)^{\frac{1}{f(u)+f(v)}} \right\rfloor$$

for every $u, v \in V(G)$ and $u_i \neq v_j$. For n is even, labels of the edges received by the power exponential mean of the labels on end vertices

$$f^*(uv) = \left\lfloor \left(f(u)^{f(u)} f(v)^{f(v)} \right)^{\frac{1}{f(u)+f(v)}} \right\rfloor$$

for every $u, v \in V(G)$ and $u_i \neq v_j$, the edges in K_1 by the power exponential mean of the labels on end vertices

$$f^*(uv) = \left\lceil \left(f(u)^{f(u)} f(v)^{f(v)} \right)^{\frac{1}{f(u)+f(v)}} \right\rceil$$

for every $u, v \in V(G)$ and $u_i \neq v_j$ are all distinct. □

Example 3.10. a. Consider the graph $C_7 + K_1$. The following Figure 5 shows the power exponential mean labeling of a graph.

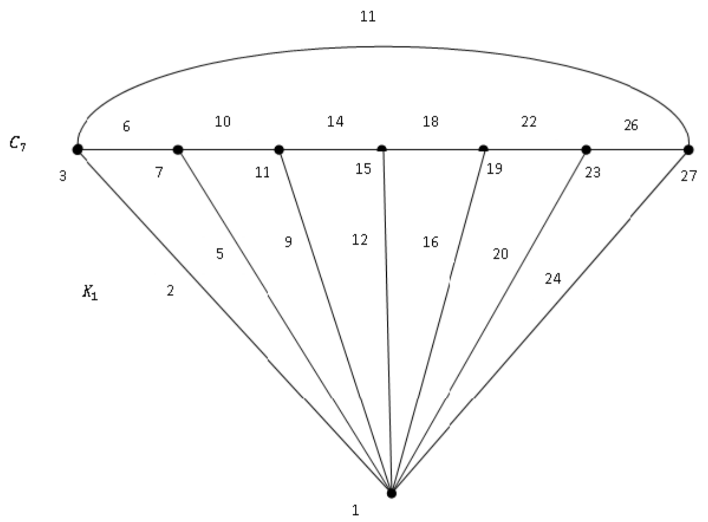

Figure 5. Graph $C_7 + K_1$

Example 3.11. b. Consider the graph $C_6 + K_1$. The following Figure 6 shows the power exponential mean labeling of a graph.

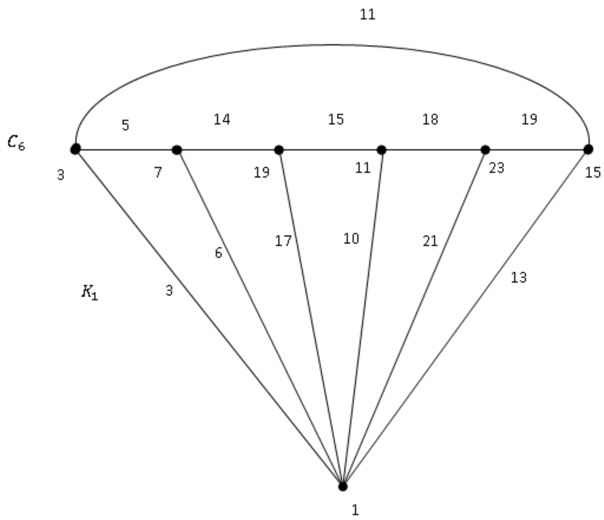

Figure 6. Graph $C_6 + K_1$

Theorem 3.12. *For any integer $m > 2$ and $n > 1$, the umbrella $U(m, n)$ is a power exponential mean graph.*

Proof. The graph $U(m, n)$ has $m + n$ vertices and $2m + n - 2$ edges. Define a vertex labeling

$$f : V(U(m, n)) \to \{1, 3, 5, \ldots, 2q - 1\}$$

by $f(x_i) = 4i - 1$, $i = 1, 2, \ldots, m$ and $f(y_i) = 4i - 3$, $i = 1, 2, \ldots, n$.
Clearly, labels of the edges received by the power exponential mean of the labels on end vertices using

$$f^*(x_i x_{i+1}) = \left\lceil \left(f(x_i)^{f(x_i)} f(x_{i+1})^{f(x_{i+1})} \right)^{\frac{1}{f(x_i)+f(x_{i+1})}} \right\rceil$$

for every $x_i, x_{i+1} \in V(G)$ and $x_i \neq x_{i+1}$.

$$f^*(x_i y_1) = \left\lfloor \left(f(x_i)^{f(x_i)} f(y_1)^{f(y_1)} \right)^{\frac{1}{f(x_i)+f(y_1)}} \right\rfloor$$

for every $x_i, y_1 \in V(G)$ and $x_i \neq y_1$.

$$f^*(y_i y_{i+1}) = \left\lfloor \left(f(y_i)^{f(y_i)} f(y_{i+1})^{f(y_{i+1})} \right)^{\frac{1}{f(y_i)+f(y_{i+1})}} \right\rfloor$$

for every $y_i, y_{i+1} \in V(G)$ and $y_i \neq y_{i+1}$ such that $f^*(e_i) \neq f^*(e_j)$ for $i \neq j$. Hence $U(m, n)$ is a power exponential mean graph. □

Example 3.13. Consider a graph $U(5, 3)$ with 8 vertices. The labeling is as shown in the Figure 7.

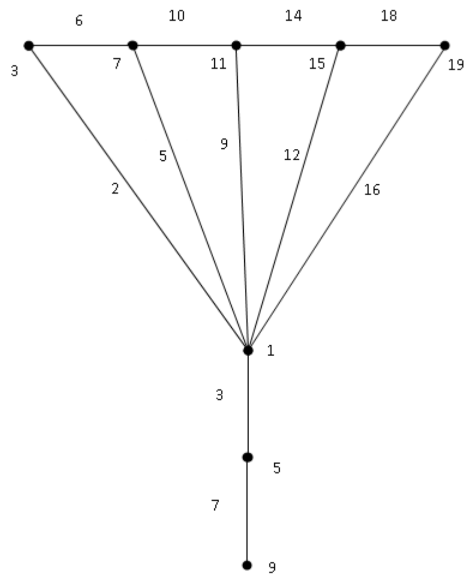

Figure 7. Umbrella graph $U(5, 3)$

Theorem 3.14. *Duplicating each vertex by an edge in path P_n is a power exponential mean graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of a path P_n. Let G be the graph obtained by duplicating each vertex v_i of P_n by an edge x_i and y_i at a time, $1 \leq i \leq n$. Here $V(G) = 3n$ and $E(G) = 4n - 1$.
Define a function $f : V(G) \to \{1, 2, 3, \ldots, q + 1\}$ by

$$f(u_i) = 4i - 2, \quad 1 \leq i \leq n$$
$$f(x_i) = 4i - 3, \quad 1 \leq i \leq n$$
$$f(y_i) = 4i - 1, \quad 1 \leq i \leq n$$

Then the edge $\{u_i u_{i+1}\}$ labeled with

$$f^*(u_i u_{i+1}) = \left\lfloor \left(f(u_i)^{f(u_i)} f(u_{i+1})^{f(u_{i+1})} \right)^{\frac{1}{f(u_i)+f(u_{i+1})}} \right\rfloor$$

for every $u_i, u_{i+1} \in V(G)$ and $u_i \neq u_{i+1}$. The edge $\{x_iu_i\}$ labeled with

$$f^*(x_iu_i) = \left\lfloor \left(f(x_i)^{f(x_i)} f(u_i)^{f(u_i)} \right)^{\frac{1}{f(x_i)+f(u_i)}} \right\rfloor$$

for every $x_i, u_i \in V(G)$ and $x_i \neq u_i$. The edge $\{x_iy_i\}$ labeled with

$$f^*(x_iy_i) = \left\lfloor \left(f(x_i)^{f(x_i)} f(y_i)^{f(y_i)} \right)^{\frac{1}{f(x_i)+f(y_i)}} \right\rfloor$$

for every $x_i, y_i \in V(G)$ and $x_i \neq y_i$. The edge $\{y_iu_i\}$ labeled with

$$f^*(y_iu_i) = \left\lfloor \left(f(y_i)^{f(y_i)} f(u_i)^{f(u_i)} \right)^{\frac{1}{f(y_i)+f(u_i)}} \right\rfloor$$

for every $y_i, u_i \in V(G)$ and $y_i \neq u_i$ are all distinct. Therefore f^* is injective, the duplicating each vertex by an edge in path P_n is power exponential mean labeling. Hence the graph obtained by duplicating each vertex by an edge in path P_n is a power exponential mean graph. □

Example 3.15. Consider a graph with duplicating each vertex by an edge in path P_4. The labeling is as shown in the Figure 8.

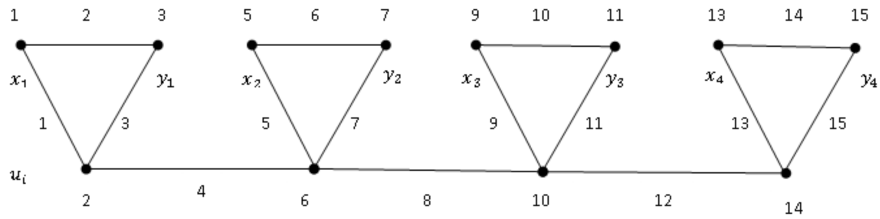

Figure 8. Duplicating each vertex by an edge in path P_4

Theorem 3.16. *The graph $C_m \odot \overline{K_1}$ is a power exponential mean graph.*

Proof. Let $u_1u_2u_3 \ldots u_mu_1$ be the cycle C_m and let v_i be the vertex which is joined to the vertex u_i of the cycle C_m for $1 \leq i \leq m$. Let $G = C_m \odot \overline{K_1}$. Define a function $f : V(G) \to \{1, 2, \ldots, 2q\}$ by $f(u_i) = 4i - 1$ and $f(v_i) = 4i$ such that $1 \leq i \leq n$. The label of the edge $\{u_i, v_i\}$ is

$$f^*(u_iv_i) = \left\lfloor \left(f(u_i)^{f(u_i)} f(v_i)^{f(v_i)} \right)^{\frac{1}{f(u_i)+f(v_i)}} \right\rfloor = 4i - 1, \quad 1 \leq i \leq n$$

for every $u_i, v_i \in V(G)$ and $u_i \neq v_i$. The label of the edge $\{u_i, u_{i+1}\}$ is

$$f^*(u_iu_{i+1}) = \left\lfloor \left(f(u_i)^{f(u_i)} f(u_{i+1})^{f(u_{i+1})} \right)^{\frac{1}{f(u_i)+f(u_{i+1})}} \right\rfloor = 4i + 1, \quad 1 \leq i \leq n - 1$$

for every $v_i \in V(G)$ and $v_i \neq v_{i+1}$ and the label of the edge $\{u_n, u_1\}$ is $f^*(u_nu_1) = 4i + 1$ which are all distinct. Therefore the graph $C_m \odot \overline{K_1}$ is a power exponential mean labeling. Hence $C_m \odot \overline{K_1}$ is a power exponential mean graph. □

Example 3.17. Consider a graph $C_8 \odot \overline{K_1}$. The labeling is as shown in the Figure 9.

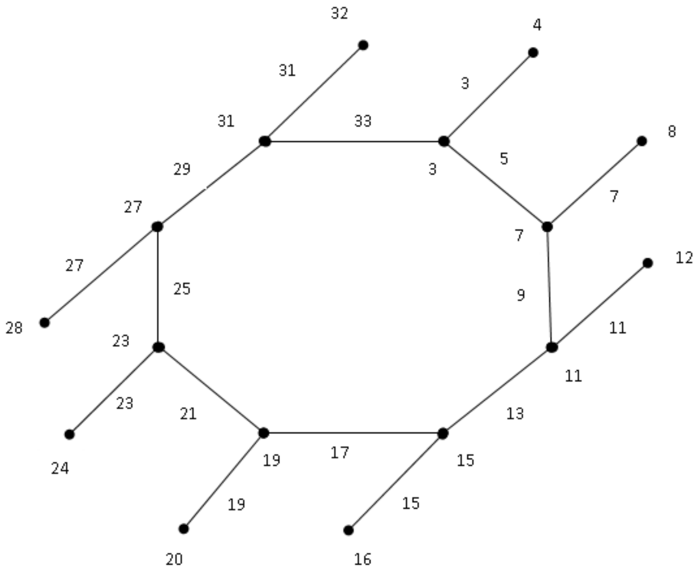

Figure 9. The graph $C_8 \odot \overline{K_1}$

Theorem 3.18. *The graph $C_n \odot \overline{K_2}$ is a power exponential mean graph.*

Proof. Let $u_1u_2u_3\ldots u_nu_1$ be the cycle C_n. For $1 \le i \le n$, let v_i, w_i be the vertices of K_2 which are attached to the vertices of C_n and $G = C_n \odot \overline{K_2}$. Define a function $f : V(G) \to \{1, 2, \ldots, 2q+1\}$ by $f(u_i) = 3i-1$, $f(v_i) = 3i-2$ and $f(w_i) = 3i$ such that $1 \le i \le n$. The edge $\{u_i, v_i\}$ is labeled by

$$f^*(u_iv_i) = \left\lfloor \left(f(u_i)^{f(u_i)} f(v_i)^{f(v_i)} \right)^{\frac{1}{f(u_i)+f(v_i)}} \right\rfloor$$

for every $u_i, v_i \in V(G)$ and $u_i \ne v_i$ such that $1 \le i \le n$. The label of the edge $\{u_i, u_{i+1}\}$ is

$$f^*(u_iu_{i+1}) = \left\lfloor \left(f(u_i)^{f(u_i)} f(u_{i+1})^{f(u_{i+1})} \right)^{\frac{1}{f(u_i)+f(u_{i+1})}} \right\rfloor$$

for every $v_i \in V(G)$ and the label of the edge $\{u_n, u_1\}$ is $f^*(u_nu_1) = 4i+1$ such that $1 \le i \le n-1$. The label of the edge $\{u_i, w_i\}$ is

$$f^*(u_iw_i) = \left\lfloor \left(f(u_i)^{f(u_i)} f(w_i)^{f(w_i)} \right)^{\frac{1}{f(u_i)+f(w_i)}} \right\rfloor$$

for every $u_i, w_i \in V(G)$ and $u_i \ne w_i$ are all distinct such that $1 \le i \le n$. Therefore the graph $C_n \odot \overline{K_2}$ is a power exponential mean labeling. Hence $C_n \odot \overline{K_2}$ is a power exponential mean graph. □

Example 3.19. Consider a graph $C_8 \odot \overline{K_2}$. The labeling is as shown in the Figure 10.

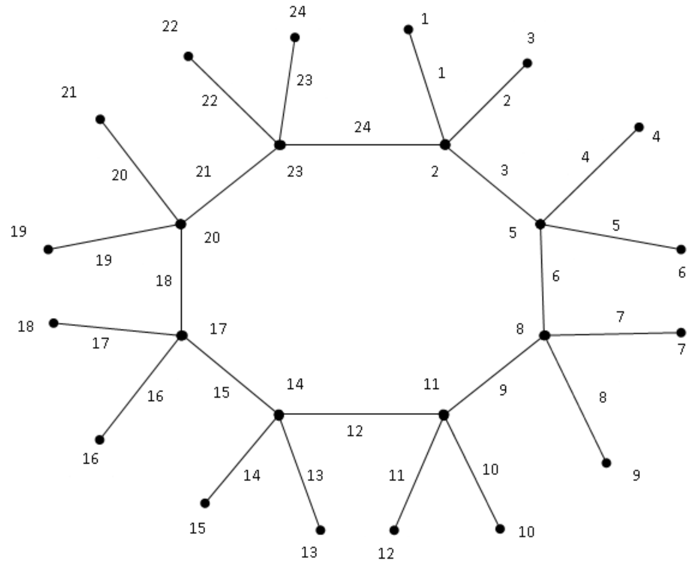

Figure 10. The graph $C_8 \bigodot \overline{K_2}$

Acknowledgments

The authors wish to thank the reviewers for their suggestion to improve the quality of this article.

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