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On an abstract Segal algebra under fractional convolution

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Abstract

In this work, we find approximate identities for the spaces $L^1\left(\mathbb{R}^d\right)$, $L^1_w\left(\mathbb{R}^d\right)$ and $S^\alpha_w(\mathbb{R}^d)$ under Θ convolution. Furthermore, we determine approximate identities with compactly supported fractional Fourier transforms in the spaces $L^1_w\left(\mathbb{R}^d\right)$ and $S^\alpha_w(\mathbb{R}^d)$, where w is weight function of regular growth. We give definitions of multipliers of these Banach algebras under Θ convolution. Also, we show that the space $S^\alpha_w(\mathbb{R}^d)$ under some conditions is an abstract Segal algebra with recpect to $L^1_w\left(\mathbb{R}^d\right)$.

Keywords: Fractional Fourier transform, approximate identity, Segal algebras

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1. Introduction

Throughout this article, we study on $\mathbb{R}^d$. For any function $f : \mathbb{R}^d \to \mathbb{C}$, the translation and modulation operators are defined as $T_y f(t) = f(t-y)$ and $M_\omega f(t) = e^{i\omega t} f(t)$ for all $y, \omega \in \mathbb{R}^d$, respectively, [18]. $C_c\left(\mathbb{R}^d\right)$ denotes the space of continuous complex function on $\mathbb{R}^d$ whose support is compact, [17]. Besides we write the Lebesgue space $\left(L^p\left(\mathbb{R}^d\right), \|.\|_p\right)$, for $1 \leq p < \infty$. A weight (Beurling weight) function w on $\mathbb{R}^d$ is a measurable and locally bounded function that, satisfying $w(x) \geq 1$ and $w(x+y) \leq w(x)w(y)$ (submultiplicative, [12]) for all $x, y \in \mathbb{R}^d$. All weights we give throughout the article are Beurling weights that, satisfying all the conditions are given above. A weight function w is weight function of regular growth if $w\left(\frac{x}{\rho}\right) \leq w(x)$ $(\rho \geq 1)$ and there are constants $C \geq 1$ and $\lambda > 0$ such that $w(\rho x) \leq C\rho^\lambda w(x)$ $(\rho \geq 1)$ for all $x \in \mathbb{R}^d$. If w is weight function of regular growth, then there exist constants $C \geq 1$ and $\lambda > 0$ such that

$$w(x) \leq C_1 \|x\|^\lambda \tag{1.1}$$

for $\|x\| \geq 1$, where $C_1 = C \sup\{w(x) \mid \|x\| = 1\}$. We define, for $1 \leq p < \infty$,

$$L^p_w\left(\mathbb{R}^d\right) = \left\{f \mid fw \in L^p\left(\mathbb{R}^d\right)\right\}.$$

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It is well known that $L_w^p\left(\mathbb{R}^d\right)$ is a Banach space under the norm $\|f\|_{p,w} = \|fw\|_p$, [16]. We define the Fourier transform $\widehat{f}$ (or $\mathcal{F}f$) of a function $f \in L^1(\mathbb{R})$ as

$$\widehat{f}(\omega) = \mathcal{F}f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt.$$

The fractional Fourier transform is a generalization of the Fourier transform through an angle parameter α and can be considered as a rotation by an angle α in the time-frequency plane. The fractional Fourier transform with angle α of a function $f \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}_\alpha f(u) = \int_{-\infty}^{+\infty} K_\alpha(u,t)f(t)dt,$$

where,

$$K_\alpha(u,t) = \begin{cases} \sqrt{\frac{1-i\cot\alpha}{2\pi}}e^{i\left(\frac{u^2+t^2}{2}\right)\cot\alpha - iut\csc\alpha}, & \text{if } \alpha \neq k\pi,\ k \in \mathbb{Z} \\ \delta(t-u), & \text{if } \alpha = 2k\pi,\ k \in \mathbb{Z} \\ \delta(t+u), & \text{if } \alpha = (2k+1)\pi,\ k \in \mathbb{Z} \end{cases}$$

and δ, Dirac delta function. The fractional Fourier transform with $\alpha = \frac{\pi}{2}$ corresponds to the Fourier transform, [1, 2, 4, 14, 15, 23]. The fractional Fourier transform can be extended for higher dimensions as [4]:

$$\left(\mathcal{F}_{\alpha_1,...,\alpha_d} f\right)(u_1,...,u_d) = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} K_{\alpha_1,...,\alpha_d}(u_1,...,u_d;t_1,...,t_d) f(t_1,...,t_d)\, dt_1 \cdots dt_d,$$

or shortly

$$\mathcal{F}_\alpha f(u) = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} K_\alpha(u,t) f(t)\, dt,$$

where

$$K_\alpha(u,t) = K_{\alpha_1,...,\alpha_d}(u_1,...,u_d;t_1,...,t_d) = K_{\alpha_1}(u_1,t_1) K_{\alpha_2}(u_2,t_2) ... K_{\alpha_d}(u_d,t_d).$$

Throughout this study, unless otherwise indicated, we get $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$, where $\alpha_i \neq k\pi$ for $i = 1, 2, ..., d$ and $k \in \mathbb{Z}$. Let $\gamma = (-y_1\cot\alpha_1, ..., -y_d\cot\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. The Θ convolution operation is

$$\begin{aligned} (f\Theta g)(x) &= \int_{\mathbb{R}^d} f(y)\, g(x-y)\, e^{\sum_{j=1}^{d} iy_j\left(y_j - x_j\right)\cot\alpha_j} dy \\ &= \int_{\mathbb{R}^d} f(y)\, T_y M_\gamma g(x)\, dy \end{aligned} \tag{1.2}$$

for all $f, g \in L^1\left(\mathbb{R}^d\right)$, [19, 20].

Let A and B be commutative Banach algebras and $B \subseteq A$, the space B is said to be Banach ideal of A if $\|g\|_A \leq \|g\|_B$ and $gh \in B$, the inequality $\|gh\|_B \leq C_2 \|g\|_B \|h\|_A$ holds for all $g \in B$, $h \in A$, [10]. A Banach function space $(B, \|.\|_B)$ of measurable function is said to be solid, if for every $g \in B$ and any measurable function h satisfying $|h(x)| \leq |g(x)|$ almost everywhere, $h \in B$ and $\|h\|_B \leq \|g\|_B$, [9].

$(B, \|.\|_B)$ is an abstract Segal algebra with respect to a Banach algebra $(A, \|.\|_A)$ if it satisfies the following conditions [5]:

1. B is a dense ideal in A and B is a Banach algebra under the norm $\|.\|_B$.
2. There exists $C_1 > 0$ such that $\|g\|_A \leq C_1 \|g\|_B$ for all $g \in B$.

3. There exists $C_2 > 0$ such that $\|gh\|_B \leq C_2 \|g\|_A \|h\|_B$ for all $g, h \in B$.

Let w be a weight function on $\mathbb{R}^d$. The space $S_w(\mathbb{R}^d)$ is subalgebra of $L^1_w\left(\mathbb{R}^d\right)$ satisfying the following properties [6]:

1. The space $S_w(\mathbb{R}^d)$ is dense in $L^1_w\left(\mathbb{R}^d\right)$.
2. The subalgebra $S_w(\mathbb{R}^d)$ is a Banach algebra under some norm $\|.\|_{S_w}$ and the inequality $\|g\|_{1,w} \leq \|g\|_{S_w}$ holds for all $g \in S_w(\mathbb{R}^d)$.
3. $S_w(\mathbb{R}^d)$ is translation invariant and for each $g \in S_w(\mathbb{R}^d)$ and all $y \in \mathbb{R}^d$, the inequality $\left\|T_y g\right\|_{S_w} \leq w(y) \|g\|_{S_w}$ holds.
4. The mapping $y \to T_y g$ from $\mathbb{R}^d$ into $S_w(\mathbb{R}^d)$ is continuous.

Let us take Θ convolution that is given in (1.2) instead of ordinary convolution. The space $S^\Theta_w(\mathbb{R}^d)$ under Θ convolution satisfies conditions of the space $S_w(\mathbb{R}^d)$ and the norm of this space is denoted by $\|.\|_{S^\Theta_w}$, [21].

Let w be a weight function on $\mathbb{R}^d$. The space $S^\alpha_w(\mathbb{R}^d)$ consist of all $f \in L^1_w\left(\mathbb{R}^d\right)$ such that $\mathcal{F}_\alpha f \in S^\Theta_w(\mathbb{R}^d)$. As the linear space $S^\alpha_w(\mathbb{R}^d)$ is normed by

$$\|f\|_{S^\alpha_w} = \|f\|_{1,w} + \|\mathcal{F}_\alpha f\|_{S^\Theta_w},$$

then $S^\alpha_w(\mathbb{R}^d)$ is a Banach algebra under this norm. The space $S^\alpha_w(\mathbb{R}^d)$ is a Banach ideal on $L^1_w\left(\mathbb{R}^d\right)$ if the space $S^\Theta_w(\mathbb{R}^d)$ is solid. This space is also translation and modulation invariant, [21].

Let A be a Banach algebra and T is an operator from A into A. T is a multiplier of A if the equality $x(Ty) = (Tx)y$ holds for all $x, y \in A$. The set of multipliers on A is denoted by $M(A)$. Let A be a commutative Banach algebra without order (i.e $xA = \{0\}$ implies $x = 0$). For any $T \in M(A)$, the equality $T(xy) = x(Ty) = (Tx)y$ holds for all $x, y \in A$ and T be a bounded linear operator, [13]. Investigating the multipliers of a Banach algebra it is important that this algebra be without order. The Banach algebra $L^1_w\left(\mathbb{R}^d\right)$ under the ordinary convolution is without order. The multiplier T on $L^1_w\left(\mathbb{R}^d\right)$ is a bounded linear operator that commutes with translation operator, [3].

In this study we find approximate identities for the spaces $L^1\left(\mathbb{R}^d\right)$, $L^1_w\left(\mathbb{R}^d\right)$ and $S^\alpha_w(\mathbb{R}^d)$ under Θ convolution. Furthermore, we determine approximate identities with compactly supported fractional Fourier transforms in the spaces $L^1_w\left(\mathbb{R}^d\right)$ and $S^\alpha_w(\mathbb{R}^d)$, where w is a weight function of regular growth. Also, we give definitions of multipliers of these Banach algebras under Θ convolution. In these definitions we take a new operator corresponding to the translation operator for $\alpha_i = \frac{\pi}{2}$ with $i = 1, 2, ..., d$ which commutes with multiplier. If we take $\alpha_i = \frac{\pi}{2}$ with $i = 1, 2, ..., d$, the Θ convolution and fractional Fourier transform correspond ordinary convolution and Fourier transform, respectively. Also, we show that the space $S^\alpha_w(\mathbb{R}^d)$ under some conditions is an abstract Segal algebra with recpect to $L^1_w\left(\mathbb{R}^d\right)$.

2. On an Abstract Segal Algebra under Fractional Convolution

First of all, we will show that the Banach algebra $L^1_w(\mathbb{R}^d)$ under Θ convolution is without order.

Theorem 2.1. *Let $\gamma = (-y_1 \cot \alpha_1, ..., -y_d \cot \alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$.*

1. $T_y M_\gamma g \in L^1_w(\mathbb{R}^d)$ for all $g \in L^1_w(\mathbb{R}^d)$.

2. The mapping $y \to T_y M_\gamma g$ from $\mathbb{R}^d$ into $L^1_w(\mathbb{R}^d)$ is continuous.

Proof. 1. It is obvious from the invariance of translate and modulation operators of the space $L^1_w(\mathbb{R}^d)$.

2. It is well known that the mappings $y \to T_y g$ and $y \to M_y g$ from $\mathbb{R}^d$ into $L^1_w(\mathbb{R}^d)$ are continuous by [11]. Let $\gamma = (-y_1 \cot\alpha_1, ..., -y_d \cot\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. Then, it is easy to see that the mapping $y \to M_\gamma T_y g$ from $\mathbb{R}^d$ into $L^1_w(\mathbb{R}^d)$ is continuous by using the same method of the proof of Theorem 2.9 in [21]. Also it is well known that
$$T_y M_\gamma g\,(t) = e^{-iy\gamma} M_\gamma T_y g\,(t)$$
for all $t \in \mathbb{R}^d$. Thus, we have
$$T_y M_\gamma g = e^{-iy\gamma} M_\gamma T_y g. \tag{2.1}$$
Since the mapping $y \to M_\gamma T_y g$ from $\mathbb{R}^d$ into $L^1_w(\mathbb{R}^d)$ is continuous, then by using (2.1), the mapping $y \to T_y M_\gamma g$ from $\mathbb{R}^d$ into $L^1_w(\mathbb{R}^d)$ is also continuous.

□

Proposition 2.2. *The algebra $L^1_w(\mathbb{R}^d)$ under Θ convolution has an approximate identity.*

Proof. Let F be a finite subset of $L^1_w\left(\mathbb{R}^d\right)$ such that $F = \{g_1, ..., g_n\}$. Let $\varepsilon > 0$ be given. Then by Theorem 2.1 there exist $\delta_i > 0$ such that
$$\left\|T_y M_\gamma g_i - g_i\right\|_{1,w} < \varepsilon$$
when $\|y\| < \delta_i$ for all $i = 1, ..., n$. Let $\delta = \min\{\delta_1, ..., \delta_n\}$. Thus we have
$$\left\|T_y M_\gamma g_i - g_i\right\|_{1,w} < \varepsilon \tag{2.2}$$
when $\|y\| < \delta$ for all $i = 1, 2, ..., n$. Let us take a positive function $h \in C_c\left(\mathbb{R}^d\right)$ such that $\operatorname{supp} h \subset B(0, \delta)$ and $\int_{\mathbb{R}^d} h(x)dx = 1$. Therefore we write
$$(h\Theta g_i)(x) - g_i(x) = \int_{\mathbb{R}^d} h(y)\left(T_y M_\gamma g_i(x) - g_i(x)\right) dy$$
for all $x \in \mathbb{R}^d$ and $i = 1, ..., n$. Then by using (2.2), we have
$$\begin{aligned}\|(h\Theta g_i) - g_i\|_{1,w} &= \left\|\int_{\mathbb{R}^d} h(y)\left(T_y M_\gamma g_i - g_i\right) dy\right\|_{1,w} \\ &\le \int_{\operatorname{supp} h} |h(y)| \left\|T_y M_\gamma g_i - g_i\right\|_{1,w} dy \\ &< \varepsilon \int_{\operatorname{supp} h} h(y)dy = \varepsilon\end{aligned}$$
for all $i = 1, ..., n$. Hence $L^1_w(\mathbb{R}^d)$ under Θ convolution has an approximate identity by 1.3. Proposition in [8]. □

Since the algebra $L^1_w(\mathbb{R}^d)$ under Θ convolution has an approximate identity, then it is an algebra without order. Now, we will give a definition of multipliers for the Banach algebra $L^1_w(\mathbb{R}^d)$ under Θ convolution.

Definition 2.3. A multiplier T is a continuous linear operator from $L^1_w(\mathbb{R}^d)$ into $L^1_w(\mathbb{R}^d)$ which commutes with operator $T_y M_\gamma$, where $\gamma = (-y_1 \cot\alpha_1, ..., -y_d \cot\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. The set of multipliers on $L^1_w(\mathbb{R}^d)$ is denoted by $M\left(L^1_w\left(\mathbb{R}^d\right)\right)$.

Proposition 2.4. *Let $\gamma = (-y_1 \cot\alpha_1, ..., -y_d \cot\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. Then we have*
$$T_y M_\gamma f \Theta g = f \Theta T_y M_\gamma g \tag{2.3}$$
for all $f, g \in L^1_w\left(\mathbb{R}^d\right)$.

Proof. Let $f, g \in L^1_w\left(\mathbb{R}^d\right)$. We may write

$$\begin{aligned}\left(T_y M_\gamma f \Theta g\right)(x) &= \int_{\mathbb{R}^d} T_y M_\gamma f(z)\, g(x-z) e^{\sum_{j=1}^{d} i z_j (z_j - x_j)\cot\alpha_j} dz \\ &= \int_{\mathbb{R}^d} f(z-y)\, g(x-z) e^{\sum_{j=1}^{d} i y_j (y_j - z_j)\cot\alpha_j} e^{\sum_{j=1}^{d} i z_j (z_j - x_j)\cot\alpha_j} dz\end{aligned}$$

for all $x \in \mathbb{R}^d$. By substitution $u = z - y$, we obtain

$$\begin{aligned}\left(T_y M_\gamma f \Theta g\right)(x) &= \int_{\mathbb{R}^d} f(u)\, g(x-u-y) e^{\sum_{j=1}^{d} -i u_j y_j \cot\alpha_j} e^{\sum_{j=1}^{d} i (u_j + y_j)(u_j + y_j - x_j)\cot\alpha_j} du \\ &= \int_{\mathbb{R}^d} f(u)\, g(x-u-y) e^{\sum_{j=1}^{d} i y_j (y_j + u_j - x_j)\cot\alpha_j} e^{\sum_{j=1}^{d} i u_j (u_j - x_j)\cot\alpha_j} du \\ &= \int_{\mathbb{R}^d} f(u)\, T_y M_\gamma g(x-u) e^{\sum_{j=1}^{d} i u_j (u_j - x_j)\cot\alpha_j} du \\ &= f \Theta T_y M_\gamma g(x)\end{aligned}$$

for all $x \in \mathbb{R}^d$. □

Theorem 2.5. *Let T be an operator from $L^1_w\left(\mathbb{R}^d\right)$ into $L^1_w\left(\mathbb{R}^d\right)$. Then $T \in M\left(L^1_w\left(\mathbb{R}^d\right)\right)$ if and only if*

$$T(f\Theta g) = Tf\Theta g = f\Theta Tg \tag{2.4}$$

for all $f, g \in L^1_w\left(\mathbb{R}^d\right)$.

Proof. Let T be an operator from $L^1_w\left(\mathbb{R}^d\right)$ into $L^1_w\left(\mathbb{R}^d\right)$. Firstly, we assume that equality (2.4) holds for all $f, g \in L^1_w\left(\mathbb{R}^d\right)$. Let $h \in L^1_w\left(\mathbb{R}^d\right)$ and $\lambda, \beta \in \mathbb{C}$. Thus by using (2.4), we have

$$\begin{aligned}f\Theta T(\lambda g + \beta h) &= Tf\Theta(\lambda g + \beta h) = \lambda(Tf\Theta g) + \beta(Tf\Theta h) = (f\Theta\lambda Tg) + (f\Theta\beta Th) \\ &= f\Theta(\lambda Tg + \beta Th).\end{aligned}$$

Since $L^1_w\left(\mathbb{R}^d\right)$ is an algebra without order, then we may write

$$T(\lambda g + \beta h) = \lambda Tg + \beta Th.$$

Hence T is linear.

Let $h \in L^1_w\left(\mathbb{R}^d\right)$ and $(f_n)_{n\in\mathbb{N}}$ is a convergent sequence such that

$$\|f_n - f\|_{1,w} \to 0 \text{ and } \|Tf_n - g\|_{1,w} \to 0, \tag{2.5}$$

where $f, g \in L^1_w\left(\mathbb{R}^d\right)$. Then we have

$$\begin{aligned}\|h\Theta g - h\Theta Tf\|_{1,w} &\leq \|h\Theta g - h\Theta Tf_n\|_{1,w} + \|h\Theta Tf_n - h\Theta Tf\|_{1,w} \\ &= \|h\Theta(g - Tf_n)\|_{1,w} + \|Th\Theta f_n - Th\Theta f\|_{1,w} \\ &\leq \|h\|_{1,w}\|Tf_n - g\|_{1,w} + \|Th\|_{1,w}\|f_n - f\|_{1,w}.\end{aligned} \tag{2.6}$$

By combining (2.5) and (2.6) we obtain

$$h\Theta g - h\Theta Tf = h\Theta(g - Tf) = 0.$$

Since $L^1_w\left(\mathbb{R}^d\right)$ is an algebra without order, then we may write $Tf = g$. Then by closed graph theorem T is continuous.

Finally, we will show that T commutes with operator T_yM_γ, where $\gamma = (-y_1\cot\alpha_1, ..., -y_d\cot\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. Let $f, g \in L^1_w\left(\mathbb{R}^d\right)$. Then by using (2.3) and (2.4) we have

$$TT_yM_\gamma f\Theta g = T\left(T_yM_\gamma f\Theta g\right) = Tf\Theta T_yM_\gamma g = T_yM_\gamma Tf\Theta g,$$

where $\gamma = (-y_1\cot\alpha_1, ..., -y_d\cot\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. Since $L^1_w\left(\mathbb{R}^d\right)$ is an algebra without order, then we obtain

$$TT_yM_\gamma f = T_yM_\gamma Tf.$$

As a result, $T \in M\left(L^1_w\left(\mathbb{R}^d\right)\right)$.

Conversely, we suppose that $T \in M\left(L^1_w\left(\mathbb{R}^d\right)\right)$. Let $\phi \in L^\infty_w\left(\mathbb{R}^d\right)$. It is easy to see that the mapping $g \to \int_{\mathbb{R}^d} Tg(x)\overline{\phi(x)}dx$ is a linear functional on $L^1_w\left(\mathbb{R}^d\right)$. Also, we may write

$$\begin{aligned}\left|\int_{\mathbb{R}^d} Tg(x)\overline{\phi(x)}dx\right| &\leq \int_{\mathbb{R}^d} |Tg(x)|\, w(x)\,\frac{|\phi(x)|}{w(x)}dx \\ &= \|\phi\|_{\infty,w}\|Tg\|_{1,w} \\ &\leq \|\phi\|_{\infty,w}\,\|T\|\,\|g\|_{1,w}\end{aligned}$$

such that $\|T\|$ is an operator norm. Hence, this functional is bounded. Since $L^\infty_w\left(\mathbb{R}^d\right)$ is dual space of $L^1_w\left(\mathbb{R}^d\right)$ (see [16], p. 121), then there exists a function $\kappa \in L^\infty_w\left(\mathbb{R}^d\right)$ (i.e $\frac{\kappa}{w} \in L^\infty\left(\mathbb{R}^d\right)$) such that

$$\int_{\mathbb{R}^d} Tg(x)\overline{\phi(x)}dx = \int_{\mathbb{R}^d} g(x)\overline{\kappa(x)}dx \tag{2.7}$$

for all $g \in L^1_w\left(\mathbb{R}^d\right)$. Let $g, h \in L^1_w\left(\mathbb{R}^d\right)$. Thus by using (2.7), we get

$$\begin{aligned}\int_{\mathbb{R}^d} (g\Theta Th)(x)\overline{\phi(x)}dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} TT_yM_\gamma h(x)\overline{\phi(x)}dx g(y)\,dy \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} T_yM_\gamma h(x)\overline{\kappa(x)}dx g(y)\,dy \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} T_yM_\gamma h(x)g(y)\,dy\overline{\kappa(x)}dx \\ &= \int_{\mathbb{R}^d} (g\Theta h)(x)\overline{\kappa(x)}dx \\ &= \int_{\mathbb{R}^d} T(g\Theta h)(x)\overline{\phi(x)}dx.\end{aligned}$$

Hence, by the Hahn-Banach theorem, we obtain

$$T(g\Theta h) = g\Theta Th.$$

Consequently, by commutativity of Θ convolution, we have equality (2.4). □

Now, we will investigate approximate identities with compactly supported fractional Fourier transforms in the spaces $L^1\left(\mathbb{R}^d\right)$ and $L^1_w\left(\mathbb{R}^d\right)$, respectively. Before that, we will show that there exists a function g with compactly supported fractional Fourier transform in the space $L^1(\mathbb{R})$. Throughout these examples we will set $\alpha \neq k\pi$, $k \in \mathbb{Z}$.

Example 2.6. Let g be a function on $\mathbb{R}$ such that $g(t) = \frac{1}{\sqrt{2\pi}}\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2 e^{-\frac{i}{2}t^2\cot\alpha}$ for $t \neq 0$ and $g(0) = \frac{1}{\sqrt{2\pi}}$, where $\alpha \neq k\pi$, $k \in \mathbb{Z}$. It is easy to see that this function is continuous. Also, since we have

$$\int_{-\infty}^{+\infty} |g(t)| dt = \int_{-\infty}^{+\infty} \left| \frac{1}{\sqrt{2\pi}} \left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2 \right| dt < \infty,$$

then $g \in L^1(\mathbb{R})$. Let us take a function on $\mathbb{R}$ as $f(t) = \frac{1}{\sqrt{2\pi}}\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$ for $t \neq 0$ and $f(0) = \frac{1}{\sqrt{2\pi}}$. Then we write $g(t) = e^{-\frac{i}{2}t^2\cot\alpha} f(t)$. By using Example 1.1.8 in [16] and inverse Fourier transform, we obtain

$$\hat{f}(\omega) = \begin{cases} 1 - |\omega|, & |\omega| \leq 1 \\ 0, & |\omega| > 1 \end{cases}. \tag{2.8}$$

Hence, we get

$$\mathcal{F}_\alpha g(u) = Be^{\frac{i}{2}u^2\cot\alpha} \int_{-\infty}^{+\infty} e^{-iutu\csc\alpha} f(t) dt = Be^{\frac{i}{2}u^2\cot\alpha} \hat{f}(u\csc\alpha),$$

where $B = \sqrt{\frac{1-i\cot\alpha}{2\pi}}$. Consequently, by using (2.8), we obtain

$$\mathcal{F}_\alpha g(u) = \begin{cases} Be^{\frac{i}{2}u^2\cot\alpha}(1 - |u\csc\alpha|), & |u\csc\alpha| \leq 1 \\ 0, & |u\csc\alpha| > 1 \end{cases} \tag{2.9}$$

and then g has compactly supported fractional Fourier transform.

Now, by using the function f which is denoted in Example 2.6, we will show that there exists a function l with compactly supported fractional Fourier transform in the space $L^1\left(\mathbb{R}^d\right)$.

Example 2.7. Let l be a function on $\mathbb{R}^d$ as

$$l(t_1, ..., t_d) = g(t_1) g(t_2) \cdots g(t_d) \tag{2.10}$$

such that

$$g\left(t_j\right) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sin\frac{t_j}{2}}{\frac{t_j}{2}}\right)^2 e^{-\frac{i}{2}t_j^2\cot\alpha_j}$$

and $g(0) = \frac{1}{\sqrt{2\pi}}$, where $\alpha_j \neq k\pi$, $k \in \mathbb{Z}$ with $j = 1, 2, ..., d$ for all $t = (t_1, ..., t_d) \in \mathbb{R}^d$. Let us take the function f which is given in Example 2.6. Then we have

$$g\left(t_j\right) = e^{-\frac{i}{2}t_j^2\cot\alpha_j} f\left(t_j\right),$$

where $\alpha_j \neq k\pi$, $k \in \mathbb{Z}$ with $j = 1, 2, ..., d$. Therefore by Example 2.6, $g\left(._j\right) \in L^1(\mathbb{R})$ for all $j = 1, 2, ..., d$. Hence by using (2.10) and Example 1.1.12 in [16], we write $l \in L^1\left(\mathbb{R}^d\right)$. Also we may write

$$\begin{aligned} \mathcal{F}_\alpha l(u_1, ..., u_d) &= \int_{-\infty}^{+\infty} K_{\alpha_1}(u_1, t_1) g(t_1) dt_1 \cdots \int_{-\infty}^{+\infty} K_{\alpha_d}(u_d, t_d) g(t_d) dt_d \\ &= \mathcal{F}_{\alpha_1} g(u_1) \cdots \mathcal{F}_{\alpha_d} g(u_d) \end{aligned} \tag{2.11}$$

for all $u = (u_1, ..., u_d) \in \mathbb{R}^d$. Thus by using (2.9), we obtain $\mathcal{F}_\alpha l \in C_c\left(\mathbb{R}^d\right)$. Consequently, the function l has compactly supported fractional Fourier transform.

Now, we will define a sequence $(k_n)_{n\in\mathbb{N}}$ that has integral equal to 1 and compactly supported fractional Fourier transforms in the space $L^1\left(\mathbb{R}^d\right)$.

Example 2.8. Firstly, let us take the function f which is given in Example 2.6. Let us set a sequence $(k_n)_{n\in\mathbb{N}}$ by

$$k_n(t_1,...,t_d) = h_n(t_1)\,h_n(t_2)\cdots h_n(t_d) \tag{2.12}$$

such that

$$h_n\left(t_j\right) = \frac{n}{A_n^j} e^{-\frac{i}{2}t_j^2 \cot\alpha_j} f\left(nt_j\right),$$

where

$$A_n^j = \int_{-\infty}^{+\infty} n e^{-\frac{i}{2}t_j^2 \cot\alpha_j} f\left(nt_j\right) dt_j \neq 0$$

for all $n \in \mathbb{N}$, $t = (t_1,...,t_d) \in \mathbb{R}^d$ and $\alpha_j \neq k\pi$, $k \in \mathbb{Z}$ with $j = 1,2,...,d$. Let us take functions

$$g_n\left(t_j\right) = n e^{-\frac{i}{2}t_j^2 \cot\alpha_j} f\left(nt_j\right),$$

where $\alpha_j \neq k\pi$, $k \in \mathbb{Z}$ with $j = 1,2,...,d$ for all $n \in \mathbb{N}$. Then we write $h_n\left(t_j\right) = \frac{g_n\left(t_j\right)}{A_n^j}$ for all $n \in \mathbb{N}$ and $j = 1,2,...,d$. Hence by using definition of the function f, it is easy to see that functions $g_n\left(._j\right)$ are continuous and

$$\int_{-\infty}^{+\infty} \left|g_n\left(t_j\right)\right| dt_j = n \int_{-\infty}^{+\infty} \left|f\left(nt_j\right)\right| dt_j = \|f\|_1 < \infty$$

for all $n \in \mathbb{N}$ and $j = 1,2,...,d$. Thus $g_n\left(._j\right) \in L^1(\mathbb{R})$ and then $h_n\left(._j\right) \in L^1(\mathbb{R})$ for all $n \in \mathbb{N}$ and $j = 1,2,...,d$. Therefore by using (2.12) and Example 1.1.12 in [16], we have $(k_n)_{n\in\mathbb{N}} \subset L^1\left(\mathbb{R}^d\right)$, $\int_{\mathbb{R}^d} k_n(t)\,dt = 1$ and

$$\mathcal{F}_\alpha k_n(u_1,...,u_d) = \mathcal{F}_{\alpha_1} h_n(u_1)\,\mathcal{F}_{\alpha_2} h_n(u_2)\cdots \mathcal{F}_{\alpha_d} h_n(u_d) \tag{2.13}$$

for all $u = (u_1,...,u_d) \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Also we may write

$$\mathcal{F}_{\alpha_j} h_n(u_j) = \frac{n}{A_n^j} B_j e^{\frac{i}{2}u_j^2 \cot\alpha_j} \int_{-\infty}^{+\infty} e^{-iu_j t_j \csc\alpha_j} f\left(nt_j\right) dt_j,$$

where $B_j = \sqrt{\frac{1-i\cot\alpha_j}{2\pi}}$ for all $n \in \mathbb{N}$ and $j = 1,2,...,d$. By substitution $nt_j = x_j$, we obtain

$$\begin{aligned}
\mathcal{F}_{\alpha_j} h_n(u_j) &= \frac{B_j}{A_n^j} e^{\frac{i}{2}u_j^2 \cot\alpha_j} \int_{-\infty}^{+\infty} e^{-iu_j \frac{x_j}{n} \csc\alpha_j} f\left(x_j\right) dx_j \\
&= \frac{B_j}{A_n^j} e^{\frac{i}{2}u_j^2 \cot\alpha_j} \hat{f}\left(\frac{u_j}{n} \csc\alpha_j\right),
\end{aligned}$$

where $B_j = \sqrt{\frac{1-i\cot\alpha_j}{2\pi}}$ for all $n \in \mathbb{N}$ and $j = 1,2,...,d$. Hence by using (2.8), we get

$$\mathcal{F}_{\alpha_j} h_n(u_j) = \begin{cases} \frac{B_j}{A_n^j} e^{\frac{i}{2}u_j^2 \cot\alpha_j} \left(1 - \left|\frac{u_j}{n} \csc\alpha_j\right|\right), & \left|\frac{u_j}{n} \csc\alpha_j\right| \le 1 \\ 0, & \left|\frac{u_j}{n} \csc\alpha_j\right| > 1, \end{cases}$$

where $B_j = \sqrt{\frac{1-i\cot\alpha_j}{2\pi}}$ for all $n \in \mathbb{N}$ and $j = 1,2,...,d$. Thus, $\mathcal{F}_{\alpha_j} h_n \in C_c(\mathbb{R})$ for all $n \in \mathbb{N}$ and $j = 1,2,...,d$. Then by using (2.13), we obtain $\mathcal{F}_\alpha k_n \in C_c\left(\mathbb{R}^d\right)$ for all $n \in \mathbb{N}$.

Proposition 2.9. *The sequence $(k_n)_{n\in\mathbb{N}}$ which is given in Example 2.8 is an approximate identity with compactly supported fractional Fourier transforms in the algebra $L^1(\mathbb{R}^d)$ under Θ convolution.*

Proof. Let us take the sequence $(k_n)_{n\in\mathbb{N}}$ that is given in Example 2.8. Also, let $g \in L^1\left(\mathbb{R}^d\right)$ and $\gamma = (-y_1 \cot\alpha_1, ..., -y_d \cot\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. Then we may write

$$\begin{aligned}(k_n \Theta g)(x) - g(x) &= \int\limits_{\mathbb{R}^d} k_n(y)\, T_y M_\gamma g(x)\, dy - g(x) \\ &= \int\limits_{\mathbb{R}^d} k_n(y) \left(T_y M_\gamma g(x) - g(x)\right) dy\end{aligned}$$

for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Thus we get

$$\begin{aligned}\|k_n \Theta g - g\|_1 &\le \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} |k_n(y)| \left|T_y M_\gamma g(x) - g(x)\right| dy dx \\ &= \int\limits_{\mathbb{R}^d} |k_n(y)| \left\|T_y M_\gamma g - g\right\|_1 dy \\ &= \int\limits_{\mathbb{R}^d} \left|\frac{n^d}{A_n^1 \cdots A_n^d}\right| |f(ny_1) \cdots f(ny_d)| \left\|T_y M_\gamma g - g\right\|_1 dy\end{aligned}$$

for all $n \in \mathbb{N}$. By substitution $ny_j = z_j$ for all $j = 1, 2, ..., d$, we obtain

$$\|k_n \Theta g - g\|_1 \le \frac{1}{\left|A_n^1\right| \cdots \left|A_n^d\right|} \int\limits_{d} |f(z_1) \cdots f(z_d)| \left\|T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g\right\|_1 dz, \tag{2.14}$$

where $z = (z_1, ..., z_d) \in \mathbb{R}^d$ and $\tau = (-z_1 \cot\alpha_1, ..., -z_d \cot\alpha_d)$. It is known by definition of the sequence $(k_n)_{n\in\mathbb{N}}$ that

$$A_n^j = \int\limits_{-\infty}^{+\infty} n e^{-\frac{i}{2} y_j^2 \cot\alpha_j} f\left(ny_j\right) dy_j$$

for all $n \in \mathbb{N}$ and $j = 1, 2, ..., d$. By substitution $ny_j = z_j$, we obtain

$$A_n^j = \int\limits_{-\infty}^{+\infty} e^{-\frac{i}{2} \frac{z_j^2}{n^2} \cot\alpha_j} f\left(z_j\right) dz_j$$

for all $n \in \mathbb{N}$ and $j = 1, 2, ..., d$. Also we have

$$\lim_{n\to\infty} \left(e^{-\frac{i}{2} \frac{z_j^2}{n^2} \cot\alpha_j} f\left(z_j\right)\right) = f\left(z_j\right)$$

and

$$\left|e^{-\frac{i}{2} \frac{z_j^2}{n^2} \cot\alpha_j} f\left(z_j\right)\right| = \left|f\left(z_j\right)\right|.$$

Thus, Dominated Convergence Theorem implies that the sequence $\left(\left|\frac{1}{A_n^1 \cdots A_n^d}\right|\right)_{n\in\mathbb{N}}$ is convergent. Hence, this sequence is bounded, that is to say, there exists $M > 0$ such that

$$\left|\frac{1}{A_n^1 \cdots A_n^d}\right| \le M \tag{2.15}$$

for all $n \in \mathbb{N}$. Moreover, let $g \in L^1\left(\mathbb{R}^d\right)$ and $\gamma = (-y_1 \cot \alpha_1, ..., -y_d \cot \alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. Then the mapping $y \to T_y M_\gamma g$ from $\mathbb{R}^d$ into $L^1\left(\mathbb{R}^d\right)$ is continuous by Theorem 2.1 (in the case of $w = 1$). Therefore, we get

$$\left\|T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g\right\|_1 \to 0.$$

Let $s(z) = f(z_1) \cdots f(z_d)$ for all $z = (z_1, ..., z_d) \in \mathbb{R}^d$. Then $s \in L^1\left(\mathbb{R}^d\right)$ by Example 1.1.12 in [16]. Thus, we have

$$|s(z)| \left\|T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g\right\|_1 \to 0$$

also,

$$\left\|T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g\right\|_1 \le \left\|T_{\frac{z}{n}} M_{\frac{\tau}{n}} g\right\|_1 + \|g\|_1 = 2\|g\|_1 = C_1$$

and then

$$|s(z)| \left\|T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g\right\|_1 \le C_1 |s(z)|.$$

Hence, Dominated Convergence Theorem implies that

$$\int_{\mathbb{R}^d} |s(z)| \left\|T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g\right\|_1 dz \to 0. \tag{2.16}$$

Finally, combining (2.14), (2.15) and (2.16), we obtain

$$\|k_n \Theta g - g\|_1 \le \frac{1}{\left|A_n^1\right| \cdots \left|A_n^d\right|} \int_{\mathbb{R}^d} |s(z)| \left\|T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g\right\|_1 dz \to 0.$$

This is the desired result. □

Now, we will show that there exists a function l with compactly supported fractional Fourier transform in the space $L_w^1(\mathbb{R})$, where w is a weight function of regular growth on $\mathbb{R}$.

Example 2.10. Let $\alpha \neq m\pi$, $m \in \mathbb{Z}$. Let l be a continuous function as $l = \mathcal{F}_{-\alpha} h$ such that $h(t) = e^{\frac{i}{2} t^2 \cot \alpha} g(t)$ for any nonzero function g on $\mathbb{R}$ with compact support and possessing a continuous kth derivative ($k \geq 2$). Let $M = \sqrt{\frac{1 + i \cot \alpha}{2\pi}}$. Let us take a function f on $\mathbb{R}$ such that

$$f(x) = M \int_{-\infty}^{+\infty} g(t) e^{ixt \csc \alpha} dt \tag{2.17}$$

for all $x \in \mathbb{R}$. Then by definition of the function l we write

$$l(x) = e^{-\frac{i}{2} x^2 \cot \alpha} f(x) \tag{2.18}$$

for all $x \in \mathbb{R}$. Also, since g has a continuous kth derivative, then integration by parts k times yields. Assume that supp $g = [a, b]$. Therefore, we may write

$$f(x) = M(-ix \csc \alpha)^{-1} \int_a^b g'(t) e^{ixt \csc \alpha} dt$$

for $x \neq 0$. Continuing in this way, we obtain

$$f(x) = M(-ix \csc \alpha)^{-k} \int_{-\infty}^{+\infty} g^{(k)}(t) e^{ixt \csc \alpha} dt \tag{2.19}$$

for $x \neq 0$. Let us take a constant B is given by

$$B = \max\left\{2^k |M| \int_{-\infty}^{+\infty} |g(t)|\, dt, 2^k |M| |\csc\alpha|^{-k} \int_{-\infty}^{+\infty} \left|g^{(k)}(t)\right| dt\right\}. \tag{2.20}$$

Also we have

$$|x| \leq 1 \Rightarrow 1 + |x| \leq 2 \tag{2.21}$$

and

$$|x| \geq 1 \Rightarrow 1 + |x| \leq |x| + |x| = 2|x|. \tag{2.22}$$

Hence, combining (2.17), (2.20) and (2.21) we obtain

$$|f(x)| \leq \frac{B}{(1+|x|)^k} \tag{2.23}$$

for all $|x| \leq 1$ and also combining (2.19), (2.20) and (2.22) obtain the inequality (2.23) for all $|x| \geq 1$. Since the improper integral $\int_{-\infty}^{+\infty} \frac{B}{(1+|x|)^k} dx$ is convergent for $k \geq 2$, then we may write

$$\int_{-\infty}^{+\infty} |l(x)|\, dx = \int_{-\infty}^{+\infty} \left|e^{-\frac{i}{2}x^2 \cot\alpha}\right| |f(x)|\, dx = \int_{-\infty}^{+\infty} |f(x)|\, dx < \infty$$

and so $l \in L^1(\mathbb{R})$. Since $l = \mathcal{F}_{-\alpha} h$ and $l \in L^1(\mathbb{R})$, then we obtain $\mathcal{F}_\alpha l = h$. This means that l has compactly supported fractional Fourier transform.

Now, let w be a weight function of regular growth on $\mathbb{R}$. Firstly, let $K = \{x \in \mathbb{R} |\ |x| \leq 1\}$. Since the function w is locally bounded, then there exists $T > 0$ such that $|w(x)| \leq T$ for all $x \in K$. Therefore we may write

$$\int_K |l(x)|\, w(x)\, dx \leq T \int_K |l(x)|\, dx < \infty. \tag{2.24}$$

Also, let $L = \{x \in \mathbb{R} |\ |x| \geq 1\}$. Thus by using (1.1) and (2.23), there exist some constants B, C_1 and $\lambda > 0$ such that

$$\int_L |l(x)|\, w(x)\, dx \leq BC_1 \int_L \frac{|x|^\lambda}{(1+|x|)^k} dx \leq BC_1 \int_L \frac{(1+|x|)^\lambda}{(1+|x|)^k} dx.$$

If the number k is taken as $k \geq [\lambda] + 3$, we get

$$\int_L |l(x)|\, w(x)\, dx \leq BC_1 \int_L \frac{1}{(1+|x|)^{k-\lambda}} dx < \infty. \tag{2.25}$$

By (2.24) and (2.25) we obtain

$$\int_{-\infty}^{+\infty} |l(x)|\, w(x)\, dx = \int_K |l(x)|\, w(x)\, dx + \int_L |l(x)|\, w(x)\, dx < \infty. \tag{2.26}$$

Finally, since $l \in L^1(\mathbb{R})$ and (2.26) holds, then $l \in L^1_w(\mathbb{R})$.

Now, by using the function l which is given in Example 2.10, we will show that there exists a function r with compactly supported fractional Fourier transform in the space $L^1_w\left(\mathbb{R}^d\right)$, where w is a weight function of regular growth on $\mathbb{R}^d$.

Example 2.11. Firstly, let us take the funcion g that is denoted in Example 2.10. Let r be a function on $\mathbb{R}^d$ as

$$r(x_1,...,x_d) = l(x_1)l(x_2)\cdots l(x_d) \tag{2.27}$$

such that

$$l\left(x_j\right) = \sqrt{\frac{1+i\cot\alpha_j}{2\pi}} e^{-\frac{i}{2}x_j^2\cot\alpha_j} \int_{-\infty}^{+\infty} g\left(t_j\right) e^{ix_jt_j\csc\alpha_j} dt_j, \tag{2.28}$$

where $\alpha_j \neq k\pi$, $k \in \mathbb{Z}$ with $j = 1,2,...,d$ for all $x = (x_1,...,x_d) \in \mathbb{R}^d$. By using the same method in Example 2.10, it is easy to see that $l\left(._j\right) \in L^1(\mathbb{R})$ for all $j = 1,2,...,d$. Then by using (2.27) and Example 1.1.12 in [16], we have $r \in L^1\left(\mathbb{R}^d\right)$. Also, let w be a weight function of regular growth on $\mathbb{R}^d$. If we take the maximum norm that is given by

$$\|x\|_\infty = \max\{|x_1|,|x_2|,...,|x_d|\}$$

for all $x \in \mathbb{R}^d$ instead of Euclid norm on $\mathbb{R}^d$, then the inequality (1.1) holds for this maximum norm. Let $K = \{x \in \mathbb{R} \mid \|x\|_\infty \leq 1\}$. Since the function w is locally bounded, then there exists $T > 0$ such that $|w(x)| \leq T$ for all $x \in K$. Therefore by using $l\left(._j\right) \in L^1(\mathbb{R})$, we may write

$$\begin{aligned} \int_K |r(x)|\, w(x)\, dx &\leq T\int_K |r(x)|\, dx \\ &\leq T\int_{-\infty}^{+\infty} |l(x_1)|\, dx_1 \int_{-\infty}^{+\infty} |l(x_2)|\, dx_2 \cdots \int_{-\infty}^{+\infty} |l(x_d)|\, dx_d < \infty. \end{aligned} \tag{2.29}$$

Let $\|x\|_\infty = \left|x_{j_0}\right|$ for all $x \in \mathbb{R}^d$. Also, let us take $L = \{x \in \mathbb{R} \mid \|x\|_\infty \geq 1\}$. Then by using (1.1), there exist some constants C_1 and $\lambda > 0$ such that

$$\begin{aligned} \int_L |r(x)|\, w(x)\, dx &\leq C_1\int_L |r(x)|\, \|x\|_\infty^\lambda\, dx \\ &\leq C_1\int_{-\infty}^{+\infty} |l(x_1)|\, dx_1 \cdots \int_{-\infty}^{+\infty} \left|l(x_{j_0})\right| \left|x_{j_0}\right|^\lambda dx_{j_0} \cdots \int_{-\infty}^{+\infty} |l(x_d)|\, dx_d. \end{aligned} \tag{2.30}$$

Also similar to Example 2.10, let us take a function f such that

$$f\left(x_j\right) = \sqrt{\frac{1+i\cot\alpha_j}{2\pi}} \int_{-\infty}^{+\infty} g\left(t_j\right) e^{ix_jt_j\csc\alpha_j} dt_j,$$

where $\alpha_j \neq k\pi$, $k \in \mathbb{Z}$ with $j = 1,2,...,d$ for all $x_j \in \mathbb{R}$. Thus by using (2.28) we write

$$l\left(x_j\right) = e^{-\frac{i}{2}x_j^2\cot\alpha_j} f\left(x_j\right) \tag{2.31}$$

for all $x_j \in \mathbb{R}$. Let $M_j = \sqrt{\frac{1+i\cot\alpha_j}{2\pi}}$. By using the same method in Example 2.10, we get

$$\left|f\left(x_j\right)\right| \leq \frac{B_j}{\left(1+\left|x_j\right|\right)^k}, \tag{2.32}$$

where

$$B_j = \max\left\{2^k\left|M_j\right| \int_{-\infty}^{+\infty} \left|g\left(t_j\right)\right| dt_j, 2^k\left|M_j\right| \left|\csc\alpha_j\right|^{-k} \int_{-\infty}^{+\infty} \left|g^{(k)}\left(t_j\right)\right| dt_j\right\}$$

for all $j = 1, 2, ..., d$. Let $\prod_{j=1}^{d}\left(\int_{-\infty}^{+\infty}\left|l\left(x_j\right)\right|dx_j\right) = C_2$ for $j \neq j_0$. Hence by using (2.30) and (2.32), we get

$$\begin{aligned}\int_L |r(x)|\, w(x)\, dx &\leq C_1C_2\int_{-\infty}^{+\infty}\left|l(x_{j_0})\right|\left|x_{j_0}\right|^{\lambda}dx_{j_0}\\ &\leq B_{j_0}C_1C_2\int_{-\infty}^{+\infty}\frac{\left(1+\left|x_{j_0}\right|\right)^{\lambda}}{\left(1+\left|x_{j_0}\right|\right)^{k}}dx_{j_0}.\end{aligned}$$

If the number k is taken as $k \geq [\lambda] + 3$, then we have

$$\int_L |r(x)|\, w(x)\, dx \leq B_{j_0}C_1C_2\int_{-\infty}^{+\infty}\frac{1}{\left(1+\left|x_{j_0}\right|\right)^{k-\lambda}}dx_{j_0} < \infty. \tag{2.33}$$

Therefore combining (2.29) and (2.33), we obtain

$$\int_{\mathbb{R}^d} |r(x)|\, w(x)\, dx = \int_K |r(x)|\, w(x)\, dx + \int_L |r(x)|\, w(x)\, dx < \infty. \tag{2.34}$$

Since $r \in L^1\left(\mathbb{R}^d\right)$ and (2.34) holds, then $r \in L^1_w\left(\mathbb{R}^d\right)$. Furthermore, by using (2.27) we may write

$$\mathcal{F}_{\alpha}r(u_1, ..., u_d) = \mathcal{F}_{\alpha_1}l(u_1)\mathcal{F}_{\alpha_2}l(u_2)\cdots\mathcal{F}_{\alpha_d}l(u_d). \tag{2.35}$$

for all $u = (u_1, ..., u_d) \in \mathbb{R}^d$. Let us take a function h such that

$$h\left(t_j\right) = e^{-\frac{i}{2}x_j^2\cot\alpha_j}g\left(t_j\right),$$

where $\alpha_j \neq k\pi$, $k \in \mathbb{Z}$ with $j = 1, 2, ..., d$ for all $x_j \in \mathbb{R}$. Then by using (2.28), we have $\mathcal{F}_{-\alpha_j}h = l$ for all $j = 1, 2, ..., d$. Since $l\left(\cdot_j\right) \in L^1\left(\mathbb{R}^d\right)$ for all $j = 1, 2, ..., d$, then $\mathcal{F}_{\alpha_j}l = h \in C_c(\mathbb{R})$. Hence by using (2.35) we obtain $\mathcal{F}_{\alpha}r \in C_c\left(\mathbb{R}^d\right)$.

Now, we will set a sequence $(p_n)_{n\in\mathbb{N}}$ that has integral equal to 1 and compactly supported fractional Fourier transforms in the space $L^1_w\left(\mathbb{R}^d\right)$, where w be a weight function of regular growth on $\mathbb{R}^d$.

Example 2.12. Firstly, let us take the functions l and f are denoted in Example 2.11. Let us define a sequence $(p_n)_{n\in\mathbb{N}}$ by

$$p_n(t_1, ..., t_d) = k_n(t_1)\,k_n(t_2)\cdots k_n(t_d) \tag{2.36}$$

such that

$$k_n\left(t_j\right) = \frac{n}{A_n^j}e^{-\frac{i}{2}t_j^2\cot\alpha_j}f\left(nt_j\right),$$

where

$$A_n^j = \int_{-\infty}^{+\infty} ne^{-\frac{i}{2}t_j^2\cot\alpha_j}f\left(nt_j\right)dt_j \neq 0$$

for all $n \in \mathbb{N}$, $t = (t_1, ..., t_d) \in \mathbb{R}^d$ and $\alpha_j \neq k\pi$, $k \in \mathbb{Z}$ with $j = 1, 2, ..., d$. Let us take functions

$$g_n\left(t_j\right) = ne^{-\frac{i}{2}t_j^2\cot\alpha_j}f\left(nt_j\right),$$

where $\alpha_j \neq k\pi$, $k \in \mathbb{Z}$ with $j = 1, 2, ..., d$ for all $n \in \mathbb{N}$. Then we write $k_n\left(t_j\right) = \frac{g_n(t_j)}{A_n^j}$ for all $n \in \mathbb{N}$ and $j = 1, 2, ..., d$. Also by using continuity of the function l and (2.31), we say that the function f is continuous. Then functions $g_n\left(\cdot_j\right)$ are continuous and

$$\int_{-\infty}^{+\infty}\left|g_n\left(t_j\right)\right|dt_j = n\int_{-\infty}^{+\infty}\left|f\left(nt_j\right)\right|dt_j = \|f\|_1 < \infty$$

for all $n \in \mathbb{N}$ and $j = 1, 2, ..., d$. Thus $g_n(\cdot_j) \in L^1(\mathbb{R})$ and then $k_n(\cdot_j) \in L^1(\mathbb{R})$ for all $n \in \mathbb{N}$ and $j = 1, 2, ..., d$. Hence by using (2.36) and Example 1.1.12 in [16], we have $(p_n)_{n\in\mathbb{N}} \subset L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} p_n(t)\,dt = 1$ for all $n \in \mathbb{N}$. Also, using the same method in the proof of Proposition 2.9, there exists $C > 0$ such that

$$\left|\frac{1}{A_n^1 \cdots A_n^d}\right| \leq C \tag{2.37}$$

for all $n \in \mathbb{N}$. Let w be a weight function of regular growth on $\mathbb{R}^d$. First of all, let $K = \left\{x \in \mathbb{R}^d \,\middle|\, \|x\|_\infty \leq 1\right\}$. Since the function w is locally bounded, then there exists $T > 0$ such that $|w(x)| \leq T$ for all $x \in K$. Then, by using (2.37) and substitution $z_j = nt_j$ for all $j = 1, 2, ..., d$, we may write

$$\begin{aligned}
\int_K |p_n(t)|\,w(t)\,dt &\leq T\left|\frac{n^d}{A_n^1 \cdots A_n^d}\right| \int_{\mathbb{R}^d} |k_n(t_1)|\,|k_n(t_2)| \cdots |k_n(t_d)|\,dt \\
&\leq TCn^d \int_{-\infty}^{+\infty} |f(nt_1)|\,dt_1 \cdots \int_{-\infty}^{+\infty} |f(nt_d)|\,dt_d \\
&= TC \int_{-\infty}^{+\infty} |f(z_1)|\,dz_1 \cdots \int_{-\infty}^{+\infty} |f(z_d)|\,dz_d \\
&= TC\,\|f\|_1^d < \infty.
\end{aligned} \tag{2.38}$$

Let $\|x\|_\infty = \left|x_{j_0}\right|$ for all $x \in \mathbb{R}^d$. Also, let us take $L = \{x \in \mathbb{R} \,|\, \|x\|_\infty \geq 1\}$. Then by using (1.1) and (2.37), there exist some constants C_1 and $\lambda > 0$ such that

$$\begin{aligned}
\int_L |p_n(t)|\,w(t)\,dt &\leq C_1 \int_L |p_n(t)|\,\|t\|_\infty^\lambda\,dt \\
&\leq C_1 \int_{\mathbb{R}^d} |p_n(t)|\,\left|t_{j_0}\right|^\lambda dt \\
&\leq C_1 C n^d \int_{-\infty}^{+\infty} |f(nt_1)|\,dt_1 \cdots \int_{-\infty}^{+\infty} \left|f\left(nt_{j_0}\right)\right| \left|t_{j_0}\right|^\lambda dt_{j_0} \cdots \int_{-\infty}^{+\infty} |f(nt_d)|\,dt_d
\end{aligned} \tag{2.39}$$

for all $n \in \mathbb{N}$. By substitution $z_j = nt_j$ for all $j = 1, 2, ..., d$, we get

$$\begin{aligned}
\int_L |p_n(t)|\,w(t)\,dt &\leq CC_1 \int_{-\infty}^{+\infty} |f(z_1)|\,dz_1 \cdots \int_{-\infty}^{+\infty} \left|f\left(z_{j_0}\right)\right| \left|\frac{z_{j_0}}{n}\right|^\lambda dz_{j_0} \cdots \int_{-\infty}^{+\infty} |f(z_d)|\,dz_d \\
&\leq CC_1 \int_{-\infty}^{+\infty} |f(z_1)|\,dz_1 \cdots \int_{-\infty}^{+\infty} \left|f\left(z_{j_0}\right)\right| \left|z_{j_0}\right|^\lambda dz_{j_0} \cdots \int_{-\infty}^{+\infty} |f(z_d)|\,dz_d
\end{aligned}$$

for all $n \in \mathbb{N}$. Let $\prod_{j=1}^{d} \left(\int_{-\infty}^{+\infty} \left|f\left(z_j\right)\right| dz_j \right) = C_2$ for $j \neq j_0$. Thus, combining (2.39) and (2.32), we obtain

$$\begin{aligned}
\int_L |p_n(t)|\,w(t)\,dt &\leq CC_1C_2 \int_{-\infty}^{+\infty} \left|f\left(z_{j_0}\right)\right| \left|z_{j_0}\right|^\lambda dz_{j_0} \\
&\leq CC_1C_2B_{j_0} \int_{-\infty}^{+\infty} \frac{\left(1 + \left|z_{j_0}\right|\right)^\lambda}{\left(1 + \left|z_{j_0}\right|\right)^k} dz_{j_0}
\end{aligned} \tag{2.40}$$

for all $n \in \mathbb{N}$. If the number k is taken as $k \geq [\lambda] + 3$ and

$$C_3 = CC_1C_2B_{j_0} \int_{-\infty}^{+\infty} \frac{1}{\left(1 + \left|z_{j_0}\right|\right)^{k-\lambda}} dz_{j_0},$$

then we have

$$\int_L |p_n(t)| \, w(t) \, dt \leq C_3 < \infty \tag{2.41}$$

for all $n \in \mathbb{N}$. By combining (2.38) and (2.41) we obtain

$$\int_{\mathbb{R}^d} |p_n(x)| \, w(x) \, dx = \int_K |p_n(x)| \, w(x) \, dx + \int_L |p_n(x)| \, w(x) \, dx < \infty. \tag{2.42}$$

for all $n \in \mathbb{N}$. Since $(p_n)_{n\in\mathbb{N}} \subset L^1\left(\mathbb{R}^d\right)$ and (2.42) holds, then $(p_n)_{n\in\mathbb{N}} \subset L^1_w\left(\mathbb{R}^d\right)$. Let $C_4 = TC \|f\|_1^d + C_3$. Then, combining (2.38) and (2.41), we have

$$\|p_n\|_{1,w} = \int_{\mathbb{R}^d} |p_n(x)| \, w(x) \, dx \leq C_4$$

for all $n \in \mathbb{N}$. This means that the sequence $(p_n)_{n\in\mathbb{N}}$ is bounded in the space $L^1_w\left(\mathbb{R}^d\right)$. Besides by using (2.36), we have

$$\mathcal{F}_\alpha p_n(u_1, ..., u_d) = \mathcal{F}_{\alpha_1} k_n(u_1) \mathcal{F}_{\alpha_2} k_n(u_2) \cdots \mathcal{F}_{\alpha_d} k_n(u_d) \tag{2.43}$$

for all $u = (u_1, ..., u_d) \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Also we write

$$\mathcal{F}_{\alpha_j} k_n(u_j) = \frac{n}{A_n^j} N_j e^{\frac{i}{2} u_j^2 \cot \alpha_j} \int_{-\infty}^{+\infty} e^{-iu_j t_j \csc \alpha_j} f\left(nt_j\right) dt_j,$$

where $N_j = \sqrt{\frac{1 - i \cot \alpha_j}{2\pi}}$ for all $n \in \mathbb{N}$ and $j = 1, 2, ..., d$. By substitution $nt_j = y_j$, we have

$$\mathcal{F}_{\alpha_j} k_n(u_j) = \frac{N_j}{A_n^j} e^{\frac{i}{2} u_j^2 \cot \alpha_j} \int_{-\infty}^{+\infty} e^{-i \frac{u_j}{n} y_j \csc \alpha_j} f\left(y_j\right) dy_j, \tag{2.44}$$

where $N_j = \sqrt{\frac{1 - i \cot \alpha_j}{2\pi}}$ for all $n \in \mathbb{N}$ and $j = 1, 2, ..., d$. Let us take the functions h and g are denoted in Example 2.11. Then $\mathcal{F}_{\alpha_j} l = h$ for all $j = 1, 2, ..., d$. By using the definition of fractional Fourier transform, we get

$$g\left(u_j\right) = N_j \int_{-\infty}^{+\infty} f\left(t_j\right) e^{-iu_j t_j \csc \alpha_j} dt_j,$$

where $N_j = \sqrt{\frac{1 - i \cot \alpha_j}{2\pi}}$ for all $j = 1, 2, ..., d$. Therefore by using (2.44), we write

$$\mathcal{F}_{\alpha_j} k_n(u_j) = \frac{1}{A_n^j} e^{\frac{i}{2} u_j^2 \cot \alpha_j} g\left(\frac{u_j}{n}\right)$$

for all $n \in \mathbb{N}$ and $j = 1, 2, ..., d$. Since the function $g \in C_c(\mathbb{R})$, then $\mathcal{F}_{\alpha_j} k_n \in C_c(\mathbb{R})$ for all $n \in \mathbb{N}$ and $j = 1, 2, ..., d$. Thus, by using (2.43), we obtain $\mathcal{F}_\alpha p_n \in C_c\left(\mathbb{R}^d\right)$ for all $n \in \mathbb{N}$.

Proposition 2.13. *Let w be a weight function of regular growth on $\mathbb{R}^d$. Then the sequence $(p_n)_{n\in\mathbb{N}}$ which is given in Example 2.12 is a bounded approximate identity with compactly supported fractional Fourier transforms in the algebra $L^1_w(\mathbb{R}^d)$ under Θ convolution.*

Proof. Let us take the sequence $(p_n)_{n\in\mathbb{N}}$ that is given in Example 2.12. Also, let $g \in L^1_w\left(\mathbb{R}^d\right)$ and $\gamma = (-y_1 \cot\alpha_1, ..., -y_d \cot\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. Then we may write

$$\begin{aligned}(p_n\Theta g)(x) - g(x) &= \int_{\mathbb{R}^d} p_n(y)\, T_y M_\gamma g(x)\, dy - g(x) \\ &= \int_{\mathbb{R}^d} p_n(y) \left(T_y M_\gamma g(x) - g(x)\right) dy\end{aligned}$$

for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Thus we get

$$\begin{aligned}\|p_n\Theta g - g\|_{1,w} &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |p_n(y)| \left|T_y M_\gamma g(x) - g(x)\right| dy w(x)\, dx \\ &= \int_{\mathbb{R}^d} |p_n(y)| \left\|T_y M_\gamma g - g\right\|_{1,w} dy \\ &= \int_{\mathbb{R}^d} \left|\frac{n^d}{A_n^1 \cdots A_n^d}\right| |f(ny_1) \cdots f(ny_d)| \left\|T_y M_\gamma g - g\right\|_{1,w} dy\end{aligned}$$

for all $n \in \mathbb{N}$. By substitution $ny_j = z_j$ for $j = 1, 2, ..., d$, we obtain

$$\|p_n\Theta g - g\|_{1,w} \leq \frac{1}{\left|A_n^1\right| \cdots \left|A_n^d\right|} \int_{\mathbb{R}^d} |f(z_1) \cdots f(z_d)| \left\|T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g\right\|_{1,w} dz, \tag{2.45}$$

where $z = (z_1, ..., z_d) \in \mathbb{R}^d$ and $\tau = (-z_1 \cot\alpha_1, ..., -z_d \cot\alpha_d)$. Also, using the same method in the proof of the Proposition 2.9, there exists $C > 0$ such that

$$\left|\frac{1}{A_n^1 \cdots A_n^d}\right| \leq C \tag{2.46}$$

for all $n \in \mathbb{N}$. Moreover, let $g \in L^1_w\left(\mathbb{R}^d\right)$ and $\gamma = (-y_1 \cot\alpha_1, ..., -y_d \cot\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. Then the mapping $y \to T_y M_\gamma g$ from $\mathbb{R}^d$ into $L^1_w\left(\mathbb{R}^d\right)$ is continuous by Theorem 2.1. Therefore, we get

$$\left\|T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g\right\|_{1,w} \to 0.$$

Let $s(z) = f(z_1) \cdots f(z_d)$ for all $z = (z_1, ..., z_d) \in \mathbb{R}^d$. Hence, it is known that $s \in L^1_w\left(\mathbb{R}^d\right)$ by Example 2.11. Thus, we have

$$|s(z)| \left\|T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g\right\|_{1,w} \to 0.$$

Since w is a weight function of regular growth on $\mathbb{R}^d$, then we write

$$\begin{aligned}\left\|T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g\right\|_{1,w} &\leq w\left(\frac{z}{n}\right)\|g\|_{1,w} + \|g\|_{1,w} \\ &\leq w\left(\frac{z}{n}\right)\|g\|_{1,w} + w\left(\frac{z}{n}\right)\|g\|_{1,w} \\ &\leq 2\|g\|_{1,w} w(z).\end{aligned}$$

Let $C_1 = 2\|g\|_{1,w}$. Thus we get

$$|s(z)| \left\|T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g\right\|_{1,w} \leq C_1 |s(z)|\, w(z).$$

Since $sw \in L^1\left(\mathbb{R}^d\right)$, then Dominated Convergence Theorem implies that

$$\int_{\mathbb{R}^d} |s(z)| \left\| T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g \right\|_{1,w} dz \to 0. \tag{2.47}$$

Consequently, combining (2.45), (2.46) and (2.47), we obtain

$$\|p_n \Theta g - g\|_{1,w} \leq \frac{1}{\left|A_n^1\right| \cdots \left|A_n^d\right|} \int_{\mathbb{R}^d} |s(z)| \left\| T_{\frac{z}{n}} M_{\frac{\tau}{n}} g - g \right\|_{1,w} dz \to 0.$$

This is the desired result. □

Corollary 2.14. *Let w be a weight function of regular growth on $\mathbb{R}^d$. Then the set $F_{0,w}^{\alpha}\left(\mathbb{R}^d\right) = \left\{ f \in L_w^1\left(\mathbb{R}^d\right) \middle| \mathcal{F}_{\alpha} f \in C_c\left(\mathbb{R}^d\right) \right\}$ is dense in $L_w^1\left(\mathbb{R}^d\right)$.*

Proof. Let w be a weight function of regular growth on $\mathbb{R}^d$. Also, let us take the sequence $(p_n)_{n\in\mathbb{N}}$ is given in Example 2.12 and $g \in L_w^1\left(\mathbb{R}^d\right)$. It is known that the sequence $(p_n)_{n\in\mathbb{N}}$ is an approximate identity with compactly supported fractional Fourier transforms by Proposition 2.13. Let $\varepsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that

$$\left\| (p_{n_0} \Theta g) - g \right\|_{1,w} < \varepsilon \tag{2.48}$$

for all $n \geq n_0$. Therefore we may write

$$\mathcal{F}_{\alpha}(p_{n_0} \Theta g)(u) = M e^{\sum_{j=1}^{d} -\frac{i}{2} u_j^2 \cot \alpha_j} F_{\alpha} p_{n_0}(u) F_{\alpha} g(u), \tag{2.49}$$

where $M = \left[\prod_{j=1}^{d} \sqrt{\frac{2\pi}{1 - i \cot \alpha_j}} \right]$ for all $u \in \mathbb{R}^d$ by Theorem 7 in [20]. Since $\mathcal{F}_{\alpha} p_{n_0} \in C_c\left(\mathbb{R}^d\right)$, then by using (2.49), we have $\mathcal{F}_{\alpha}(p_{n_0} \Theta g) \in C_c\left(\mathbb{R}^d\right)$. This means the function $\mathcal{F}_{\alpha}(p_{n_0} \Theta g)$ belongs to $F_{0,w}^{\alpha}\left(\mathbb{R}^d\right)$. Hence, the set

$$F_{0,w}^{\alpha}\left(\mathbb{R}^d\right) = \left\{ f \in L_w^1\left(\mathbb{R}^d\right) \middle| \mathcal{F}_{\alpha} f \in C_c\left(\mathbb{R}^d\right) \right\}$$

is dense in $L_w^1\left(\mathbb{R}^d\right)$. □

Proposition 2.15. *Let w be a weight function of regular growth on $\mathbb{R}^d$. If $C_c\left(\mathbb{R}^d\right) \subset S_w^{\Theta}(\mathbb{R}^d)$, then $S_w^{\alpha}(\mathbb{R}^d)$ is dense in $L_w^1\left(\mathbb{R}^d\right)$.*

Proof. Let w be a weight function of regular growth on $\mathbb{R}^d$ and $C_c\left(\mathbb{R}^d\right) \subset S_w^{\Theta}(\mathbb{R}^d)$. We have

$$F_{0,w}^{\alpha}\left(\mathbb{R}^d\right) \subset S_w^{\alpha}(\mathbb{R}^d) \subset L_w^1\left(\mathbb{R}^d\right) \tag{2.50}$$

by the definition of the space $S_w^{\alpha}(\mathbb{R}^d)$. Also, it is known that the set

$$F_{0,w}^{\alpha}\left(\mathbb{R}^d\right) = \left\{ f \in L_w^1\left(\mathbb{R}^d\right) \middle| \mathcal{F}_{\alpha} f \in C_c\left(\mathbb{R}^d\right) \right\}$$

is dense in $L_w^1\left(\mathbb{R}^d\right)$ by Corollary 2.14. Let $\varepsilon > 0$ be given. Then there exists a function $h \in F_{0,w}^{\alpha}\left(\mathbb{R}^d\right)$ such that

$$\|g - h\|_{1,w} < \varepsilon$$

for all $g \in L_w^1\left(\mathbb{R}^d\right)$. Therefore by using (2.50), the function h also belongs to $S_w^{\alpha}(\mathbb{R}^d)$. This is the desired result. □

Proposition 2.16. *Let w be a weight function of regular growth on $\mathbb{R}^d$. If $S_w^{\Theta}(\mathbb{R}^d)$ is a solid space and $C_c\left(\mathbb{R}^d\right) \subset S_w^{\Theta}(\mathbb{R}^d)$, then $S_w^{\alpha}(\mathbb{R}^d)$ is an abstract Segal algebra with respect to $L_w^1\left(\mathbb{R}^d\right)$.*

Proof. Let w be a weight function of regular growth on $\mathbb{R}^d$. Then, it is known that $S_w^{\alpha}(\mathbb{R}^d)$ is a Banach algebra and is a Banach ideal on $L_w^1\left(\mathbb{R}^d\right)$, also inequalities $\|g\|_{1,w} \leq \|g\|_{S_w^{\alpha}}$ and $\|g\Theta h\|_{S_w^{\alpha}} \leq \|g\|_{S_w^{\alpha}} \|h\|_{1,w}$ holds for all $g, h \in S_w^{\alpha}(\mathbb{R}^d)$ by [21]. Furthermore, $S_w^{\alpha}(\mathbb{R}^d)$ is dense in $L_w^1\left(\mathbb{R}^d\right)$ by Proposition 2.15. Hence, $S_w^{\alpha}(\mathbb{R}^d)$ is an abstract Segal algebra with respect to $L_w^1\left(\mathbb{R}^d\right)$. □

Theorem 2.17. *Let $S_w^{\Theta}(\mathbb{R}^d)$ be a solid space and $\gamma = (-y_1 \cot\alpha_1, ..., -y_d \cot\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$.*

1. *$T_y M_\gamma g \in S_w^{\alpha}(\mathbb{R}^d)$ and*
$$\left\|T_y M_\gamma g\right\|_{S_w^{\alpha}} \leq w(y) \|g\|_{S_w^{\alpha}} \tag{2.51}$$
for all $g \in S_w^{\alpha}(\mathbb{R}^d)$.
2. *Let $C_c(\mathbb{R}^d) \cap S_w^{\Theta}(\mathbb{R}^d)$ is dense in $S_w^{\Theta}(\mathbb{R}^d)$. Then the mapping $y \to T_y M_\gamma g$ from $\mathbb{R}^d$ into $S_w^{\alpha}(\mathbb{R}^d)$ is continuous.*

Proof. 1. Let $g \in S_w^{\alpha}(\mathbb{R}^d)$. Then $g \in L_w^1\left(\mathbb{R}^d\right)$ and $\mathcal{F}_\alpha g \in S_w^{\Theta}(\mathbb{R}^d)$. It is easy to see that $\left\|M_\gamma g\right\|_{1,w} = \|g\|_{1,w}$ and $M_\gamma g \in L_w^1(\mathbb{R}^d)$. Also it is well known that the space $L_w^1(\mathbb{R}^d)$ is translation invariant and holds $\left\|T_y g\right\|_{1,w} \leq w(y)\|g\|_{1,w}$ for all $y \in \mathbb{R}^d$. Hence we have
$$\left\|T_y M_\gamma g\right\|_{1,w} \leq w(y) \|g\|_{1,w}. \tag{2.52}$$

By using Proposition 3 in [20], we get
$$\mathcal{F}_\alpha\left(T_y M_\gamma g\right)(u) = e^{\sum_{j=1}^{d} \frac{i}{2} y_j^2 \sin\alpha_j \cos\alpha_j} e^{\sum_{j=1}^{d} -iu_j y_j \sin\alpha_j} \mathcal{F}_\alpha\left(M_\gamma g\right)(u-b) \tag{2.53}$$

and
$$\begin{aligned}\mathcal{F}_\alpha\left(M_\gamma g\right)(u-b) &= e^{\sum_{j=1}^{d} -\frac{i}{2}\gamma_j^2 \sin\alpha_j \cos\alpha_j} e^{\sum_{j=1}^{d} i(u_j-b_j)\gamma_j \cos\alpha_j} \mathcal{F}_\alpha g(u-b-c) \\ &= e^{\sum_{j=1}^{d} -\frac{i}{2} y_j^2 \frac{\cos^3\alpha_j}{\sin\alpha_j}} e^{\sum_{j=1}^{d} -iu_j y_j \frac{\cos^2\alpha_j}{\sin\alpha_j} + iy_j^2 \frac{\cos^3\alpha_j}{\sin\alpha_j}} \mathcal{F}_\alpha g(u) \\ &= e^{\sum_{j=1}^{d} \frac{i}{2} y_j^2 \frac{\cos^3\alpha_j}{\sin\alpha_j}} e^{\sum_{j=1}^{d} -iu_j y_j \frac{\cos^2\alpha_j}{\sin\alpha_j}} \mathcal{F}_\alpha g(u)\end{aligned} \tag{2.54}$$

such that $b = (y_1 \cos\alpha_1, ..., y_d \cos\alpha_d)$ and
$$c = (\gamma_1 \sin\alpha_1, ..., \gamma_d \sin\alpha_d) = (-y_1 \cot\alpha_1 \sin\alpha_1, ..., -y_d \cot\alpha_d \sin\alpha_1) = -b.$$

Combining (2.53) and (2.54) we write
$$\begin{aligned}\mathcal{F}_\alpha\left(T_y M_\gamma g\right)(u) &= e^{\sum_{j=1}^{d} \frac{i}{2} y_j^2 \cos\alpha_j\left(\sin\alpha_j + \frac{\cos^2\alpha_j}{\sin\alpha_j}\right)} e^{\sum_{j=1}^{d} -iu_j y_j\left(\sin\alpha_j + \frac{\cos^2\alpha_j}{\sin\alpha_j}\right)} \mathcal{F}_\alpha g(u) \\ &= e^{\sum_{j=1}^{d} \frac{i}{2} y_j^2 \cot\alpha_j} e^{\sum_{j=1}^{d} -iu_j y_j \csc\alpha_j} \mathcal{F}_\alpha g(u).\end{aligned}$$

Let $\tau = (-y_1 \csc\alpha_1, ..., -y_d \csc\alpha_d)$. Therefore we have
$$\mathcal{F}_\alpha\left(T_y M_\gamma g\right)(u) = e^{\sum_{j=1}^{d} \frac{i}{2} y_j^2 \cot\alpha_j} M_\tau \mathcal{F}_\alpha g(u). \tag{2.55}$$

Since $S_w^{\Theta}(\mathbb{R}^d)$ is a solid space, then it is strongly character invariant by Lemma 2.4 in [7]. Thus we get

$$e^{\sum_{j=1}^{d} \frac{i}{2} y_j^2 \cot \alpha_j} M_\tau \mathcal{F}_\alpha g \in S_w^{\Theta}(\mathbb{R}^d).$$

By using (2.55), we obtain $M_\tau F_\alpha f \in S_w^{\Theta}(\mathbb{R}^d)$ and

$$\begin{aligned} \left\| \mathcal{F}_\alpha \left(T_y M_\gamma g \right) \right\|_{S_w^{\Theta}} &= \left\| e^{\sum_{j=1}^{d} \frac{i}{2} y_j^2 \cot \alpha_j} M_\tau \mathcal{F}_\alpha g \right\|_{S_w^{\Theta}} = \left| e^{\sum_{j=1}^{d} \frac{i}{2} y_j^2 \cot \alpha_j} \right| \| M_\tau \mathcal{F}_\alpha g \|_{S_w^{\Theta}} \\ &= \| M_\tau \mathcal{F}_\alpha g \|_{S_w^{\Theta}} = \| \mathcal{F}_\alpha g \|_{S_w^{\Theta}}. \end{aligned} \tag{2.56}$$

Finally, by (2.52) and (2.56), we have

$$\left\| T_y M_\gamma g \right\|_{1,w} \leq w(y) \|g\|_{1,w}.$$

2. First of all we will show continuity at 0. It is easy to see that mapping $y \to \gamma$ from $\mathbb{R}^d$ into $\mathbb{R}^d$ is continuous. Also mapping $y \to M_y g$ is from $\mathbb{R}^d$ into $S_w^{\alpha}(\mathbb{R}^d)$ continuous for all $g \in S_w^{\alpha}(\mathbb{R}^d)$ by Theorem 2.12 in [21]. Hence the composition mapping $y \to M_\gamma g$ from $\mathbb{R}^d$ into $S_w^{\alpha}(\mathbb{R}^d)$ is continuous. Let $\varepsilon > 0$ be given. There exists $\delta_1 > 0$ such that
$$\left\| M_\gamma g - g \right\|_{S_w^{\alpha}} < \frac{\varepsilon}{2} \tag{2.57}$$
when $\|y\| < \delta_1$ and there exist $\delta_2 > 0$ such that
$$\left\| T_y \left(M_\gamma g \right) - M_\gamma g \right\|_{S_w^{\alpha}} < \frac{\varepsilon}{2} \tag{2.58}$$
when $\|y\| < \delta_2$ by Theorem 2.10 in [21]. Let $\delta_3 = \min\{\delta_1, \delta_2\}$. Thus, combining (2.57) and (2.58) we get
$$\begin{aligned} \left\| T_y \left(M_\gamma g \right) - g \right\|_{S_w^{\alpha}} &= \left\| T_y \left(M_\gamma g \right) - M_\gamma g + M_\gamma g - g \right\|_{S_w^{\alpha}} \\ &\leq \left\| T_y \left(M_\gamma g \right) - M_\gamma g \right\|_{S_w^{\alpha}} + \left\| M_\gamma g - g \right\|_{S_w^{\alpha}} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$
when $\|y\| < \delta_3$ and this proves that the mapping $y \to T_y M_\gamma g$ is continuous at 0. Now, let us take any fixed point $y^* = \left(y_1^*, ..., y_d^* \right) \in \mathbb{R}^d$. Therefore, we have
$$\begin{aligned} T_{y-y^*} M_{\gamma-\gamma^*} \left(T_{y^*} M_{\gamma^*} g \right)(x) &= M_{\gamma-\gamma^*} \left(T_{y^*} M_{\gamma^*} g \right)(x - y + y^*) \\ &= e^{(i(\gamma-\gamma^*)(x-y+y^*))} \left(T_{y^*} M_{\gamma^*} g \right)(x - y + y^*) \\ &= e^{(i(\gamma-\gamma^*)(x-y+y^*))} e^{(i\gamma^*(x-y))} g(x-y) \\ &= e^{(iy^*\gamma - iy^*\gamma^*)} e^{(i\gamma(x-y))} g(x-y) \\ &= e^{(iy^*\gamma - iy^*\gamma^*)} T_y M_\gamma g(x) \end{aligned}$$
such that $\gamma^* = \left(-y_1^* \cot \alpha_1, ..., -y_d^* \cot \alpha_d \right)$. Then we write
$$\left\| T_y M_\gamma g - T_{y^*} M_{\gamma^*} g \right\|_{S_w^{\alpha}} = \left\| e^{(iy^*\gamma^* - iy^*\gamma)} T_{y-y^*} M_{\gamma-\gamma^*} \left(T_{y^*} M_{\gamma^*} g \right) - T_{y^*} M_{\gamma^*} g \right\|_{S_w^{\alpha}}.$$
By the first part of this theorem, let us take $T_{y^*} M_{\gamma^*} g = h \in S_w^{\alpha}(\mathbb{R}^d)$. Hence, we obtain
$$\begin{aligned} \left\| T_y M_\gamma g - T_{y^*} M_{\gamma^*} g \right\|_{S_w^{\alpha}} &= \left\| e^{(iy^*\gamma^* - iy^*\gamma)} T_{y-y^*} M_{\gamma-\gamma^*} h - h \right\|_{S_w^{\alpha}} \\ &\leq \left\| e^{(iy^*\gamma^* - iy^*\gamma)} T_{y-y^*} M_{\gamma-\gamma^*} h - e^{(iy^*\gamma^* - iy^*\gamma)} h \right\|_{S_w^{\alpha}} \\ &\quad + \left\| e^{(iy^*\gamma^* - iy^*\gamma)} h - h \right\|_{S_w^{\alpha}} \\ &= \left\| T_{y-y^*} M_{\gamma-\gamma^*} h - h \right\|_{S_w^{\alpha}} + \|h\|_{S_w^{\alpha}} \left| e^{iy^*\gamma} - e^{iy^*\gamma^*} \right|. \end{aligned} \tag{2.59}$$

Moreover, since the point y^* be an arbitrary fixed point, then mapping $y \to e^{iy^* y}$ from $\mathbb{R}^d$ into $\mathbb{C}$ is obviously continuous and also it is easy to see that mapping $y \to \gamma$ from $\mathbb{R}^d$ into $\mathbb{R}^d$ is continuous. Hence, composition mapping $y \to e^{iy^*\gamma}$ is also continuous. Let $\varepsilon > 0$ be given. There exists $\delta_4 > 0$ such that

$$\left|e^{iy^*\gamma} - e^{iy^*\gamma^*}\right| < \frac{\varepsilon}{2\|h\|_{S_w^\alpha}} \tag{2.60}$$

when $\|y - y^*\| < \delta_4$ and also, by continuity at 0, there exists $\delta_5 > 0$ such that

$$\left\|T_{y-y^*} M_{\gamma-\gamma^*} h - h\right\|_{S_w^\alpha} < \frac{\varepsilon}{2} \tag{2.61}$$

when $\|y - y^*\| < \delta_5$. Let $\delta_6 = \min\{\delta_4, \delta_5\}$. Thus, combining (2.59), (2.60) and (2.61) we obtain

$$\begin{aligned} \left\|T_y M_\gamma g - T_{y^*} M_{\gamma^*} g\right\|_{S_w^\alpha} &\leq \left\|T_{y-y^*} M_{\gamma-\gamma^*} h - h\right\|_{S_w^\alpha} + \|h\|_{S_w^\alpha} \left|e^{iy^*\gamma} - e^{iy^*\gamma^*}\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon\|h\|_{S_w^\alpha}}{2\|h\|_{S_w^\alpha}} = \varepsilon \end{aligned}$$

when $\|y - y^*\| < \delta_6$. This completes the proof.

□

Proposition 2.18. *Let w be a weight function of regular growth on $\mathbb{R}^d$. If $S_w^\Theta(\mathbb{R}^d)$ is a solid space, $C_c\left(\mathbb{R}^d\right) \subset S_w^\Theta(\mathbb{R}^d)$ and $C_c\left(\mathbb{R}^d\right)$ is dense in $S_w^\Theta(\mathbb{R}^d)$, then $S_w^\alpha(\mathbb{R}^d)$ has an approximate identity with compactly supported fractional Fourier transforms.*

Proof. Let w be a weight function of regular growth on $\mathbb{R}^d$ and also S be a finite subset of $S_w^\alpha(\mathbb{R}^d)$ such that $S = \{g_1, ..., g_n\}$. Let $g \in S_w^\alpha(\mathbb{R}^d)$ and $\gamma = (-y_1 \cot\alpha_1, ..., -y_d \cot\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. Since $S_w^\Theta(\mathbb{R}^d)$ is a solid space, $C_c\left(\mathbb{R}^d\right) \subset S_w^\Theta(\mathbb{R}^d)$ and $C_c\left(\mathbb{R}^d\right)$ is dense in $S_w^\Theta(\mathbb{R}^d)$, then the mapping $y \to T_y M_\gamma g$ from $\mathbb{R}^d$ into $S_w^\alpha(\mathbb{R}^d)$ is continuous by Theorem 2.17. Let $\varepsilon > 0$ be given. There exist $\delta_i > 0$ such that

$$\left\|T_y M_\gamma g_i - g_i\right\|_{S_w^\alpha} < \frac{\varepsilon}{2}$$

when $\|y\| < \delta_i$ for all $i = 1, 2, ..., n$. Let $\delta = \min\{\delta_1, \delta_2, ..., \delta_n\}$. Therefore we write

$$\left\|T_y M_\gamma g_i - g_i\right\|_{S_w^\alpha} < \frac{\varepsilon}{2} \tag{2.62}$$

when $\|y\| < \delta$ for all $i = 1, 2, ..., n$. Let us take a positive function $h \in C_c\left(\mathbb{R}^d\right)$ such that $\operatorname{supp} h \subset B(0, \delta)$ and $\int_{\mathbb{R}^d} h(x)dx = 1$. Thus we have

$$(h\Theta g_i)(x) - g_i(x) = \int_{\mathbb{R}^d} h(y) T_y M_\gamma g_i(x) dy - g_i(x) = \int_{\mathbb{R}^d} h(y)\left(T_y M_\gamma g_i(x) - g_i(x)\right) dy$$

for all $x \in \mathbb{R}^d$ and $i = 1, ..., n$. Then by using (2.62), we get

$$\begin{aligned} \|(h\Theta g_i) - g_i\|_{S_w^\alpha} &= \left\|\int_{\mathbb{R}^d} h(y)\left(T_y M_\gamma g_i - g_i\right) dy\right\|_{S_w^\alpha} \leq \int_{\operatorname{supp} h} |h(y)| \left\|T_y M_\gamma g_i - g_i\right\|_{S_w^\alpha} dy \\ &< \frac{\varepsilon}{2} \int_{\operatorname{supp} h} h(y)dy = \frac{\varepsilon}{2} \end{aligned} \tag{2.63}$$

for all $i = 1, ..., n$. Let $C = \max\left\{\|g_1\|_{S^\alpha_w}, ..., \|g_n\|_{S^\alpha_w}\right\}$. There exists a function $f \in F^\alpha_{0,w}\left(\mathbb{R}^d\right)$ such that

$$\|h - f\|_{1,w} < \frac{\varepsilon}{2C} \tag{2.64}$$

by Corollary 1. Since $C_c\left(\mathbb{R}^d\right) \subset S^\Theta_w(\mathbb{R}^d)$, then $f \in S^\alpha_w(\mathbb{R}^d)$. Therefore combining (2.63) and (2.64), we obtain

$$\begin{aligned} \|(f\Theta g_i) - g_i\|_{S^\alpha_w} &\leq \|(f\Theta g_i) - (h\Theta g_i)\|_{S^\alpha_w} + \|(h\Theta g_i) - g_i\|_{S^\alpha_w} \\ &= \|h - f\|_{1,w}\|g_i\|_{S^\alpha_w} + \|(h\Theta g_i) - g_i\|_{S^\alpha_w} \\ &\leq \|h - f\|_{1,w} C + \|(h\Theta g_i) - g_i\|_{S^\alpha_w} \\ &< \frac{\varepsilon}{2C} C + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $i = 1, ..., n$. Hence $S^\alpha_w(\mathbb{R}^d)$ has an approximate identity with compactly supported fractional Fourier transforms by 1.3. Proposition in [8]. □

Since the algebra $S^\alpha_w(\mathbb{R}^d)$ which satisfies the conditions in Proposition 2.18 has an approximate identity, then it is an algebra without order. Now, we will give definition of multipliers for the Banach algebra $S^\alpha_w(\mathbb{R}^d)$.

Definition 2.19. Let w be a weight function of regular growth on $\mathbb{R}^d$. Let $S^\Theta_w(\mathbb{R}^d)$ be a solid space and let $C_c\left(\mathbb{R}^d\right)$ be dense in $S^\Theta_w(\mathbb{R}^d)$ such that $C_c\left(\mathbb{R}^d\right) \subset S^\Theta_w(\mathbb{R}^d)$. A multiplier T is a continuous linear operator from $S^\alpha_w(\mathbb{R}^d)$ into $S^\alpha_w(\mathbb{R}^d)$ which commutes with operator $T_y M_\gamma$, where $\gamma = (-y_1 \cot\alpha_1, ..., -y_d \cot\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. The set of multipliers on $S^\alpha_w(\mathbb{R}^d)$ is denoted by $M\left(S^\alpha_w(\mathbb{R}^d)\right)$.

Theorem 2.20. *Let w be a weight function of regular growth on $\mathbb{R}^d$. Let $S^\Theta_w(\mathbb{R}^d)$ be a solid space and let $C_c\left(\mathbb{R}^d\right)$ be dense in $S^\Theta_w(\mathbb{R}^d)$, where $C_c\left(\mathbb{R}^d\right) \subset S^\Theta_w(\mathbb{R}^d)$. Suppose T is an operator from $S^\alpha_w(\mathbb{R}^d)$ into $S^\alpha_w(\mathbb{R}^d)$. Then $T \in M\left(S^\alpha_w(\mathbb{R}^d)\right)$ if and only if*

$$T(f\Theta g) = Tf\Theta g = f\Theta Tg \tag{2.65}$$

for all $f, g \in S^\alpha_w(\mathbb{R}^d)$.

Proof. Let w be a weight function of regular growth on $\mathbb{R}^d$. Let $S^\Theta_w(\mathbb{R}^d)$ be a solid space and let $C_c\left(\mathbb{R}^d\right)$ be dense in $S^\Theta_w(\mathbb{R}^d)$, where $C_c\left(\mathbb{R}^d\right) \subset S^\Theta_w(\mathbb{R}^d)$. Suppose T is an operator from $S^\alpha_w(\mathbb{R}^d)$ into $S^\alpha_w(\mathbb{R}^d)$. First of all, we assume that equality (2.65) holds for all $f, g \in S^\alpha_w(\mathbb{R}^d)$. Since $S^\alpha_w(\mathbb{R}^d)$ is an algebra without order, then we get $T \in M\left(S^\alpha_w(\mathbb{R}^d)\right)$, similar to the first part of the proof of Theorem 2.5.

Conversely, let $T \in M\left(S^\alpha_w(\mathbb{R}^d)\right)$. Let $\left(S^\alpha_w(\mathbb{R}^d)\right)'$ be dual space of the space $S^\alpha_w(\mathbb{R}^d)$. For $\varphi \in \left(S^\alpha_w(\mathbb{R}^d)\right)'$, let us consider the functional l is given by $l(g) = \langle Tg, \varphi\rangle$ for all $g \in S^\alpha_w(\mathbb{R}^d)$. Then, by using linearity of the operators T and φ, we may write

$$l(ag + bh) = \langle T(ag + bh), \varphi\rangle = a\langle Tg, \varphi\rangle + b\langle Th, \varphi\rangle = al(g) + bl(h)$$

for all $f, g \in S^\alpha_w(\mathbb{R}^d)$ and $a, b \in \mathbb{C}$. Also, let $\|T\|$ and $\|\varphi\|$ be operator norms of T and φ, respectively. Thus, we have

$$|\langle Tg, \varphi\rangle| \leq \|\varphi\| \|Tg\|_{S^\alpha_w} \leq \|\varphi\| \|T\| \|g\|_{S^\alpha_w}$$

for all $g \in S^\alpha_w(\mathbb{R}^d)$. Hence l is a bounded linear functional on $S^\alpha_w(\mathbb{R}^d)$. Since $\left(S^\alpha_w(\mathbb{R}^d)\right)'$ is dual space of the space $S^\alpha_w(\mathbb{R}^d)$, then there exists a function $\psi \in \left(S^\alpha_w(\mathbb{R}^d)\right)'$ such that

$$\langle g, \psi\rangle = \langle Tg, \varphi\rangle \tag{2.66}$$

for all $g \in S^\alpha_w(\mathbb{R}^d)$. Now, let $\gamma = (-y_1 \cot\alpha_1, ..., -y_d \cot\alpha_d)$ for all $y = (y_1, ..., y_d) \in \mathbb{R}^d$. Then we define a function k by

$$k(y) = f(y) T_y M_\gamma g \tag{2.67}$$

for $f, g \in S_w^\alpha(\mathbb{R}^d)$. Also, it is known that the mapping $y \to T_y M_\gamma g$ from $\mathbb{R}^d$ into $S_w^\alpha(\mathbb{R}^d)$ is continuous by Theorem 2.17. Thus the function k from $\mathbb{R}^d$ into $S_w^\alpha(\mathbb{R}^d)$ is a measurable. Besides, by using (2.51), we have

$$\int_{\mathbb{R}^d} \|k(y)\|_{S_w^\alpha} dy = \int_{\mathbb{R}^d} \left\|f(y)T_y M_\gamma g\right\|_{S_w^\alpha} dy \leq \int_{\mathbb{R}^d} |f(y)|\, w(y)\, \|g\|_{S_w^\alpha} dy = \|g\|_{S_w^\alpha} \|f\|_{1,w} < \infty.$$

Therefore we may write

$$\left\langle \int_{\mathbb{R}^d} k(y)dy, \varphi \right\rangle = \int_{\mathbb{R}^d} \langle k(y), \varphi \rangle\, dy$$

by the definition of vector-valued integrals, [22]. Hence, by using (2.67) and definition of the Θ convolution, we get

$$\langle f\Theta g, \varphi \rangle = \int_{\mathbb{R}^d} \left\langle f(y)T_y M_\gamma g, \varphi \right\rangle dy = \int_{\mathbb{R}^d} f(y) \left\langle T_y M_\gamma g, \varphi \right\rangle dy. \tag{2.68}$$

Thus, by using (2.66), (2.68), we obtain

$$\langle f\Theta Tg, \varphi \rangle = \int_{\mathbb{R}^d} f(y) \left\langle TT_y M_\gamma g, \varphi \right\rangle dy = \int_{\mathbb{R}^d} f(y) \left\langle T_y M_\gamma g, \psi \right\rangle dy = \langle f\Theta g, \psi \rangle = \langle T(f\Theta g), \varphi \rangle$$

for all $f, g \in S_w^\alpha(\mathbb{R}^d)$. By using Hahn-Banach theorem, we obtain

$$T(f\Theta g) = f\Theta Tg.$$

Consequently, by using commutativity of Θ convolution, the equality (2.65) holds. □

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