Montes Taurus J. Pure Appl. Math. 4 (3), 41–60, 2022

Montes Taurus Journal of Pure and Applied Mathematics
https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

Decomposition Formulas for Second-Order Quadruple Gaussian Hypergeometric Series by Means of Operators $H(\alpha,\beta)$ and $\overline{H}(\alpha,\beta)$

Anvar Hasanov [a], Ainur Ryskan [b], Junesang Choi [c]

[a]*Institute of Mathematics, Uzbek Academy of Sciences, 81 Mirzo-Ulugbek street, Tashkent 700170, Uzbekistan*
[b]*National Pedagogical University, 86 Tole bi street, Almaty 0500012, Kazakhstan*
[c]*Department of Mathematics, Dongguk University, Gyeongju 38066, Republic of Korea*

Abstract

Numerous decomposition formulas for various hypergeometric functions of several variables have been offered. In this paper, we aim to establish symbolic operator identities and decomposition formulas for second-order quadruple Gaussian hypergeometric series associated with Appell functions and Saran hypergeometric functions by mainly using mutually inverse symbolic operators $H(\alpha,\beta)$ and $\overline{H}(\alpha,\beta)$, which were introduced in an earlier work. Mellin-Barnes type contour integrals are employed for proofs of the operator identities. Also we determine the regions of convergence of the 14 quadruple Gaussian hypergeometric series.

Keywords: Hypergeometric functions, Multiple hypergeometric functions, Inverse pairs of symbolic operators, Decomposition formulas, Mellin-Barnes contour integrals

2010 MSC: 33C20, 33C65, 44A45

1. Introduction and preliminaries

It has been acknowledged that a lot of problems of theoretical physics and contemporary mathematics lead to study of various hypergeometric functions of several complex variables. These comprise, for instance, problems of super string theory [5], analytic continuation of Mellin-Barnes type contour integrals [32, 33] and algebraic geometry [27, 44]. Hypergeometric functions of several variables emerge in quantum field theory as solutions of the Knizhnik-Zamolodchikov equations [44], which were discovered in conformal field theory and portray the behavior of the correlation functions in the Wess-Zumino-Witten model. Drinfeld [10, 11, 12, 13] has verified that one of the monodromies of the equation (Drinfeld's associator) gratifies the pentagonal equation and is a generating function for the values of the multi-argument hypergeometric functions at integer points. Such an approach enables us to relate special functions of the hypergeometric type to actual problems of the theory of representations of Lie algebras and quantum groups, as well as other applied problems [2, 19, 28, 30, 31, 39, 43].

Hypergeometric functions are also utilized in solving boundary value problems for degenerate differential equations [15, 20, 21, 22, 37, 41]. The Riemann functions and fundamental solutions of degenerate partial differential equations are expressed in terms of multidimensional hypergeometric functions. In fact, special functions have been

†*Article ID:* MTJPAM-D-21-00019

Email addresses: anvarhasanov@yahoo.com (Anvar Hasanov), ainur-ryskan@mail.ru (Ainur Ryskan), junesang@dongguk.ac.kr (Junesang Choi)

Received:2 February 2021, Accepted:15 July 2021, Published:30 July 2021

⋆*Corresponding Author:* Junesang Choi

introduced in many unrelated ways. Their occurrences have been determined, as a rule, with need to solve problems that lead to differential equations (or systems of such equations) that are not solvable in the class of elementary functions. For example, the Bessel functions, the Hermite functions, and the Gauss hypergeometric function have occurred. Also, the appearance of a large number of formulas relating to various special functions has, naturally, led to the desire to classify and systematize them.

We recall the intimate Gamma function $\Gamma(z)$ and the Pochhammer symbol $(\mu)_\nu$ together with some of their properties (see, e.g., [40, Chapter 1]). The Gamma function $\Gamma(\eta)$ is defined and extended by

$$\Gamma(\eta) = \begin{cases} \displaystyle\int_0^\infty t^{\eta-1}\, \mathrm{e}^{-t}\, \mathrm{d}t & (\Re(\eta) > 0) \\ \dfrac{\Gamma(\eta+n)}{\eta(\eta+1)\cdots(\eta+n-1)} & (n \in \mathbb{N},\ \eta \in \mathbb{C} \setminus \mathbb{Z}_{\le 0}). \end{cases} \tag{1.1}$$

Here and elsewhere, let $\mathbb{C}$, $\mathbb{R}_0$, $\mathbb{Z}$, and $\mathbb{N}$ indicate the sets of complex numbers, nonnegative real numbers, integers, and positive integers, respectively. Also put $\mathbb{Z}_{\le 0} := \mathbb{Z} \setminus \mathbb{N}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The Pochhammer symbol $(\mu)_\nu$ is defined (for $\mu,\ \nu \in \mathbb{C}$), in terms of the Gamma function Γ, by

$$(\mu)_\nu := \frac{\Gamma(\mu+\nu)}{\Gamma(\mu)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \mu(\mu+1)\cdots(\mu+n-1) & (\nu = n \in \mathbb{N};\ \mu \in \mathbb{C}), \end{cases} \tag{1.2}$$

it being accepted conventionally that $(0)_0 := 1$. The generalized binomial coefficient $\binom{\mu}{n}$ is defined (for $\mu \in \mathbb{C}$) by

$$\binom{\mu}{n} = \begin{cases} \dfrac{\mu(\mu-1)\cdots(\mu-n+1)}{n!} & (n \in \mathbb{N}) \\ 1 & (n = 0). \end{cases} \tag{1.3}$$

Making use of (1.2) and the ensuing known identity

$$\Gamma(\eta)\,\Gamma(1-\eta) = \frac{\pi}{\sin(\pi\eta)} \quad (\eta \in \mathbb{C} \setminus \mathbb{Z}), \tag{1.4}$$

we get

$$\binom{\mu}{n} = \frac{(-1)^n\,(-\mu)_n}{n!} = \frac{\Gamma(\mu+1)}{n!\,\Gamma(\mu-n+1)} \quad (\mu \in \mathbb{C},\ n \in \mathbb{N}_0). \tag{1.5}$$

Employing (1.2) and setting $\mu = \eta - 1$ in the second equality in (1.5), we get

$$(\eta)_{-n} = \frac{\Gamma(\eta-n)}{\Gamma(\eta)} = \frac{(-1)^n}{(1-\eta)_n} \quad (n \in \mathbb{N}_0,\ \eta \in \mathbb{C} \setminus \mathbb{Z}). \tag{1.6}$$

Equation (1.2) gives

$$(\mu)_{m+n} = (\mu)_m\,(\mu+m)_n \quad (\mu \in \mathbb{C},\ m,\ n \in \mathbb{N}_0), \tag{1.7}$$

which, upon using (1.6), yields

$$(\mu)_{n-\ell} = \frac{(-1)^\ell(\mu)_n}{(1-\mu-n)_\ell} \quad (0 \le \ell \le n). \tag{1.8}$$

Putting $\mu = 1$ in (1.8), we get

$$(-n)_\ell = \begin{cases} \dfrac{(-1)^\ell\, n!}{(n-\ell)!} & (0 \le \ell \le n;\ \ell,\ n \in \mathbb{N}_0), \\ 0 & (\ell > n). \end{cases} \tag{1.9}$$

Some of the quadruple (Gaussian) hypergeometric functions are recalled (see, e.g., [16, 17, 18, 38]):

$$F_{17}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w) = \sum_{m,n,p,q=0}^{\infty} \frac{(\mu_1)_{m+n+p}(\mu_2)_q(\nu_1)_{m+n}(\nu_2)_{p+q}}{(\xi_1)_{m+p}(\xi_2)_n(\xi_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \tag{1.10}$$

$$\left(|w|<1,\ \sqrt{|x|}+\sqrt{|y|}<1,\ \frac{|z|}{1-|w|}<\frac{1}{2}\left(1+|y|-|x|+\sqrt{(1+|y|-|x|)^2-4|y|}\right)\right);$$

$$F_{18}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\mu_2)_q(\nu_1)_{m+n}(\nu_2)_{p+q}}{(\xi_1)_{m+q}(\xi_2)_n(\xi_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \tag{1.11}$$

$$\left(|w|<1,\ (\sqrt{|x|}+\sqrt{|y|})^2+\frac{|z|}{1-|w|}<1\right);$$

$$F_{19}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\mu_2)_q(\nu_1)_{m+n}(\nu_2)_{p+q}}{(\xi_1)_m(\xi_2)_n(\xi_3)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \tag{1.12}$$

$$\left(|w|<1,\ \sqrt{\frac{|x|}{1-|z|}}+\sqrt{\frac{|y|}{1-|z|}}<1\right);$$

$$F_{20}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\mu_2)_q(\nu_1)_{m+n}(\nu_2)_p(\nu_3)_q}{(\xi_1)_{m+q}(\xi_2)_n(\xi_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \tag{1.13}$$

$$\left(|z|<1,\ |w|<1,\ \sqrt{\frac{|x|}{1-|z|}}+\sqrt{\frac{|y|}{1-|z|}}<1\right);$$

$$F_{21}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\mu_2)_q(\nu_1)_{m+n}(\nu_2)_p(\nu_3)_q}{(\xi_1)_m(\xi_2)_n(\xi_3)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \tag{1.14}$$

$$\left(|w|<1,\ (\sqrt{|x|}+\sqrt{|y|})^2+|z|<1\right);$$

$$F_{22}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\mu_2)_q(\nu_1)_{m+q}(\nu_2)_n(\nu_3)_p}{(\xi_1)_{m+n}(\xi_2)_p(\xi_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \tag{1.15}$$

$$\left(|z|<1,\ |w|<1,\ \frac{|x|}{(1-|z|)(1-|w|)}<1,\ \frac{|y|}{1-|z|}<1\right);$$

$$F_{23}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\mu_2)_q(\nu_1)_{m+q}(\nu_2)_n(\nu_3)_p}{(\xi_1)_m(\xi_2)_{n+p}(\xi_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \tag{1.16}$$

$$\left(|w|<1,\ \frac{|x|}{1-|w|}<1,\ \frac{|y|(1-|w|)}{1-|x|-|w|}<1,\ \frac{|z|(1-|w|)}{1-|x|-|w|}<1\right);$$

$$F_{24}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\mu_2)_q(\nu_1)_{m+n}(\nu_2)_p(\nu_3)_q}{(\xi_1)_m(\xi_2)_{n+q}(\xi_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \tag{1.17}$$

$$\left(|z|<1,\ |w|<1,\ \sqrt{\frac{|x|}{1-|z|}}+\sqrt{\frac{|y|}{1-|z|}}<1\right);$$

$$F_{25}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\mu_2)_q(\nu_1)_{m+q}(\nu_2)_n(\nu_3)_p}{(\xi_1)_{m+q}(\xi_2)_n(\xi_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \tag{1.18}$$

$$(|w|<1,\ |x|+|y|+|z|<1);$$

$$F_{26}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3,\nu_4;\xi_1,\xi_2,\xi_3;x,y,z,w)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\mu_2)_q(\nu_1)_m(\nu_2)_n(\nu_3)_p(\nu_4)_q}{(\xi_1)_{m+q}(\xi_2)_n(\xi_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \tag{1.19}$$

$$\left(|z|<1,\ \frac{|y|}{1-|z|}<1,\ \max\left\{|w|,\ \frac{|x|}{1-|y|-|z|}\right\}<1\right);$$

$$F_{27}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n}(\mu_2)_{p+q}(\nu_1)_{m+n}(\nu_2)_{p+q}}{(\xi_1)_{m+p}(\xi_2)_n(\xi_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!}; \tag{1.20}$$

$$\left(\sqrt{|x|}+\sqrt{|y|}<1,\ \sqrt{|z|}+\sqrt{|w|}<1\right);$$

$$F_{28}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n}(\mu_2)_{p+q}(\nu_1)_{m+p}(\nu_2)_{n+q}}{(\xi_1)_{m+n}(\xi_2)_p(\xi_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \tag{1.21}$$

$$\left(|x|<1,\ |y|<1,\ \frac{|z|}{1-|x|}+\frac{|w|}{1-|y|}<1\right);$$

$$F_{29}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n}(\mu_2)_{p+q}(\nu_1)_{m+p}(\nu_2)_{n+q}}{(\xi_1)_{m+q}(\xi_2)_n(\xi_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \tag{1.22}$$

$$\left(|y|<1,\ \frac{|x|}{1-|y|}<1,\ \frac{|w|}{1-|y|}+\frac{|z|(1-|y|)}{1-|x|-|y|}<1\right);$$

$$F_{30}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n}(\mu_2)_{p+q}(\nu_1)_{m+n}(\nu_2)_p(\nu_3)_q}{(\xi_1)_{m+p}(\xi_2)_n(\xi_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \tag{1.23}$$

$$\left(|w|<1,\ \sqrt{|x|}+\sqrt{|y|}<1,\ \frac{|z|}{1-|w|}<1\right).$$

We also recall the four Appell functions F_i ($i=1,\ 2,\ 3,\ 4$) (see, e.g., [1, 7, 14, 23, 24, 25, 26])

$$F_1(\mu;\nu_1,\nu_2;\xi;\ x,\ y)=\sum_{m,n=0}^{\infty}\frac{(\mu)_{m+n}\,(\nu_1)_m\,(\nu_2)_n}{(\xi)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.24}$$

$$(\max\{|x|,\ |y|\}<1);$$

$$F_2(\mu;\nu_1,\nu_2;\xi_1,\xi_2;\ x,\ y)=\sum_{m,n=0}^{\infty}\frac{(\mu)_{m+n}\,(\nu_1)_m\,(\nu_2)_n}{(\xi_1)_m(\xi_2)_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.25}$$

$$(|x|+|y|<1);$$

$$F_3(\mu_1,\mu_2;\nu_1,\nu_2;\xi;\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{(\mu_1)_m\,(\mu_2)_n\,(\nu_1)_m\,(\nu_2)_n}{(\xi)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!} \tag{1.26}$$

$$(\max\{|x|,\ |y|\} < 1)\,;$$

$$F_4(\mu;\nu;\xi_1,\xi_2;\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{(\mu)_{m+n}\,(\nu)_{m+n}}{(\xi_1)_m(\xi_2)_n} \frac{x^m}{m!}\frac{y^n}{n!} \tag{1.27}$$

$$\left(\sqrt{|x|} + \sqrt{|y|} < 1\right).$$

We further recall Saran hypergeometric functions F_F, F_G, F_R (see, e.g., [35, 36]):

$$F_F(\alpha;\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{1.28}$$

$$\left(\sqrt{|x|} + \sqrt{|z|} < 1,\ |y| < \frac{1}{2}(1 + |z| - |x| + \sqrt{(1 + |z| - |x|)^2 - 4|z|})\right);$$

$$F_G(\alpha;\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+n+p}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma_1)_m(\gamma_2)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{1.29}$$

$$(|x| + |y| < 1,\ |x| + |z| < 1)\,;$$

$$F_R(\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+p}(\alpha_2)_n(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{1.30}$$

$$\left(\sqrt{|x|} + \sqrt{|z|} < 1,\ |y| < 1\right).$$

Numerous decomposition formulas for various hypergeometric functions of one variable and several variables have been afforded (see, e.g., [3, 4, 6, 7, 8, 9, 23, 24, 25, 26]). In this paper, we aim to set up symbolic operator identities and decomposition formulas for second-order quadruple Gaussian hypergeometric series (1.10)-(1.23) associated with the Appell functions (1.24)-(1.27) and Saran hypergeometric functions (1.28)-(1.30) by mainly using mutually inverse symbolic operators $H(\alpha,\beta)$ and $\overline{H}(\alpha,\beta)$, which were introduced in [7], and Mellin-Barnes type contour integrals. We also try to determine regions of convergence of the quadruple hypergeometric series (1.10)-(1.23).

2. Determination of regions of convergence

For the regions of convergence of Appell functions (1.24)-(1.24) and Saran hypergeometric functions (1.28)-(1.30), among other sources, we refer to the monograph [42] which gives a systematic and comprehensive account of determination of regions of convergence for multiple Gaussian hypergeometric series. Here, we try to give regions of convergence of the quadruple hypergeometric series (1.10)-(1.23). To do this, throughout, we assume that all parameters and the four variables x, y, z, w are positive real numbers, and exceptional parameter values are excluded. Also we express

$$F_{17}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w)$$

as $F_{17}^{(4)}$. So do the other series in (1.10)-(1.23). Further, the sign '$\sim$' will denote equivalence in the sense of identical regions of convergence.

• Using Theorem 1 in [42, p. 108], which is called principle of cancellation, the region of convergence of convergence of the series (1.12) is equivalent to that of

$$\sum_{m,n,p,q=0}^{\infty} \frac{(\mu_1)_{m+n+p}(\mu_2)_q(\nu_1)_{m+n}}{(\xi_1)_m(\xi_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!}$$
$$= \sum_{q=0}^{\infty} \frac{(\mu_2)_q}{q!} w^q \sum_{m,n,p=0}^{\infty} \frac{(\mu_1)_{m+n+p}(\nu_1)_{m+n}}{(\xi_1)_m(\xi_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

whose region of convergence, since $(\mu_2)_q/q! = (\mu_2)_q/(1)_q$, by the principle of cancellation again, is equivalent to that of

$$(1-w)^{-1} \sum_{m,n,p=0}^{\infty} \frac{(\mu_1)_{m+n+p}(\nu_1)_{m+n}}{(\xi_1)_m(\xi_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}. \tag{2.1}$$

The region of convergence of the series in (2.1), by principle of cancellation, is equivalent to that of the series 20a in [42, p. 79], whose convergence region is given in [42, p. 112]. Hence proof of the region of convergence of convergence of the series (1.12) is complete.

• Using Theorem 1 in [42, p. 108], as in the proof of (1.12), the region of convergence of convergence of the series (1.18) is equivalent to that of the following series

$$(1-w)^{-1} \sum_{m,n,p=0}^{\infty} (\mu_1)_{m+n+p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}$$
$$= (1-w)^{-1} \sum_{j=0}^{\infty} (\mu_1)_j \frac{(x+y+z)^j}{j!} = (1-w)^{-1}\,[1-(x+y+z)]^{-\mu_1},$$

which implies the convergence condition in the series (1.18). Here we use the following series manipulation

$$\sum_{k_1,\ldots,k_n=0}^{\infty} f(m_1+\cdots+m_n) \frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!} = \sum_{k=0}^{\infty} f(k)\,\frac{(x_1+\cdots+x_n)^k}{k!} \quad (n \in \mathbb{N}). \tag{2.2}$$

By principle of cancellation, the series in (1.10)

$$F_{17}^{(4)} \sim \sum_{m,n,p,q=0}^{\infty} \frac{(\mu_1)_{m+n+p}(\nu_1)_{m+n}(1)_{p+q}}{(\xi_1)_{m+p}(\xi_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}$$
$$= \sum_{m,n,p=0}^{\infty} \frac{(\mu_1)_{m+n+p}\,(\nu_1)_{m+n}}{(\xi_1)_{m+p}\,(\xi_2)_m} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \sum_{q=0}^{\infty} (p+q)!\,\frac{t^q}{q!}.$$

Using the binomial theorem (see, e.g., [42, p. 111])

$$\sum_{q=0}^{\infty} (p+q)!\,\frac{w^q}{q!} = p!\,(1-w)^{-p-1} \quad (p \in \mathbb{N}_0,\ w<1), \tag{2.3}$$

we get

$$F_{17}^{(4)} \sim (1-w)^{-1} \sum_{m,n,p=0}^{\infty} \frac{(\mu_1)_{m+n+p}\,(\nu_1)_{m+n}\,(1)_p}{(\xi_1)_{m+p}\,(\xi_2)_m} \frac{x^m}{m!}\frac{y^n}{n!}\left(\frac{z}{1-w}\right)^p/p!. \tag{2.4}$$

The triple series in (2.4) ∼ the series [42, Entry 21a, p. 79], whose region of convergence is given in [42, Entry 21a, p. 94]. Thus we prove the region of convergence in (1.10).

• The series

$$F_{22}^{(4)} \sim \sum_{m,n,p=0}^{\infty} \frac{(1)_{m+n+p}\,(\nu_2)_n}{(\xi_1)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \sum_{q=0}^{\infty} (m+q)!\,\frac{w^q}{q!}.$$

Using (2.3), we find

$$\begin{aligned} F_{22}^{(4)} &\sim (1-w)^{-1} \sum_{m,n,p=0}^{\infty} \frac{(1)_{m+n+p}\,(1)_m\,(\nu_2)_n}{(\xi_1)_{m+n}} \frac{\left(\frac{x}{1-w}\right)^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \\ &= (1-w)^{-1} \sum_{m,n=0}^{\infty} \frac{(1)_m\,(\nu_2)_n}{(\xi_1)_{m+n}} \frac{\left(\frac{x}{1-w}\right)^m}{m!}\frac{y^n}{n!} \sum_{p=0}^{\infty} (1)_{m+n+p}\frac{z^p}{p!}. \end{aligned}$$

Using (2.3) again, we get

$$\begin{aligned} F_{22}^{(4)} &\sim (1-w)^{-1} \sum_{m,n=0}^{\infty} \frac{(1)_m\,(1)_n}{(\xi_1)_{m+n}} \frac{\left(\frac{x}{1-w}\right)^m}{m!}\frac{y^n}{n!} \sum_{p=0}^{\infty} (1)_{m+n+p}\frac{z^p}{p!} \\ &= (1-z)^{-1}\,(1-w)^{-1} \sum_{m,n=0}^{\infty} \frac{(1)_{m+n}\,(1)_m\,(1)_n}{(\xi_1)_{m+n}} \frac{\left[\frac{x}{(1-z)(1-w)}\right]^m}{m!}\frac{\left(\frac{y}{1-z}\right)^n}{n!}. \end{aligned}$$

By principle of cancellation, we obtain

$$F_{22}^{(4)} \sim (1-z)^{-1}\,(1-w)^{-1}\left[1-\frac{x}{(1-z)(1-w)}\right]^{-1}\left(1-\frac{y}{1-z}\right)^{-1},$$

which proves the region of convergence of the series (1.15).

• We will prove the region of convergence of the series (1.14) by using estimations in both directions (see [42, Section 4.3]). We have

$$\begin{aligned} F_{21}^{(4)} &\sim \sum_{m,n,p,q=0}^{\infty} \frac{(\mu_1)_{m+n+p}(1)_q(\nu_1)_{m+n}(\nu_2)_p(1)_q}{(\xi_1)_m(\xi_2)_n(1)_{p+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \\ &\sim S := \sum_{m,n,p=0}^{\infty} \frac{(\mu_1)_{m+n+p}\,(\nu_1)_{m+n}}{(\xi_1)_m\,(\xi_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \sum_{q=0}^{\infty} \frac{(1)_q}{(1+p)_q}\,w^q. \end{aligned} \tag{2.5}$$

Let D denote the region of convergence of the triple series in S. The series S can be estimated in three different ways as specified below:

$$\begin{aligned} &S > \sum_{q=0}^{\infty} w^q, \quad S > \sum_{m,n,p=0}^{\infty} \frac{(\mu_1)_{m+n+p}\,(\nu_1)_{m+n}}{(\xi_1)_m\,(\xi_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \\ &S < \sum_{m,n,p=0}^{\infty} \frac{(\mu_1)_{m+n+p}\,(\nu_1)_{m+n}}{(\xi_1)_m\,(\xi_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \sum_{q=0}^{\infty} w^q. \end{aligned}$$

These inequalities imply that the quadruple series S is (i) divergent for $w \geq 1$, (ii) divergent for $(x,y,z) \in \mathrm{int}(\mathbb{R}_0^3 \setminus D)$, and (iii) convergent if $(x,y,z) \in D$ and $w < 1$. Note that D is given, for example, in [42, Entry 20a, p. 79 and p. 93]. The region of convergence of (1.14) is now proved.

• The series $F_{29}^{(4)}$ in (1.22)

$$\sim S_1 = \sum_{m,n,p,q=0}^{\infty} \frac{(m+n)!\,(n+q)!\,(p+q)!\,(m+p)!}{(m+q)!\,n!\,p!} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!}.$$

Considering the inequality (see, e.g., [42, p. 116])

$$(m+n)!\,(n+q)! \leq n!\,(m+n+q)! \quad (m,\,n,\,q \in \mathbb{N}_0), \tag{2.6}$$

we have

$$\begin{aligned} S_1 &< \sum_{m,p,q=0}^{\infty} \frac{(p+q)!\,(m+p)!}{p!} \frac{x^m}{m!} \frac{z^p}{p!} \frac{w^q}{q!} \sum_{n=0}^{\infty} (1+m+q)_n \frac{y^n}{n!} \\ &= (1-y)^{-1} \sum_{p,q=0}^{\infty} (p+q)! \frac{z^p}{p!} \frac{\left(\frac{w}{1-y}\right)^q}{q!} \sum_{m=0}^{\infty} (1+p)_m \frac{\left(\frac{x}{1-y}\right)^m}{m!} \\ &= (1-y)^{-1} \left(1 - \frac{x}{1-y}\right)^{-1} \sum_{p,q=0}^{\infty} (p+q)! \frac{\left[\frac{z(1-y)}{1-x-y}\right]^p}{p!} \frac{\left(\frac{w}{1-y}\right)^q}{q!} \\ &= (1-y)^{-1} \left(1 - \frac{x}{1-y}\right)^{-1} \sum_{j=0}^{\infty} \left[\frac{w}{1-y} + \frac{z(1-y)}{1-x-y}\right]^j. \end{aligned}$$

Also,

$$S_1 > \sum_{n=0}^{\infty} y^n, \quad S_1 > \sum_{m,n=0}^{\infty} (m+n)! \frac{x^m}{m!} \frac{y^n}{n!} = \sum_{j=0}^{\infty} (x+y)^j.$$

From these inequality we find that (i) S_1 is convergent under the condition in (1.22), (ii) divergent for $y \geq 1$, (iii) and divergent for $x + y < 1$. Hence the condition in (1.22) is *sufficient* for its convergence.

- The series $F_{23}^{(4)}$ in (1.20)

$$\sim S_2 = \sum_{m,n,p,q=0}^{\infty} \frac{\{(m+n)!\}^2 \, \{(p+q)!\}^2}{(m+p)!\, n!\, q!} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{w^q}{q!}. \tag{2.7}$$

Using the inequality

$$m!\, p! \leq (m+p)! \quad (m,\ p \in \mathbb{N}_0), \tag{2.8}$$

we have

$$S_2 < \sum_{m,n=0}^{\infty} \frac{\{(m+n)!\}^2}{m!\, n!} \frac{x^m}{m!} \frac{y^n}{n!} \sum_{p,q=0}^{\infty} \frac{\{(p+q)!\}^2}{p!\, q!} \frac{z^p}{p!} \frac{w^q}{q!}.$$

We find that the two double series are equivalent to the Appell series F_4 (see, e.g., [42, Eq. (5), p. 23]), each of which is convergent under the given condition.

Indeed,

$$S_3 = \sum_{p,q=0}^{\infty} \frac{\{(p+q)!\}^2}{p!\, q!} \frac{z^p}{p!} \frac{w^q}{q!} = \sum_{p,q=0}^{\infty} \{(p+q)!\}^2 \frac{(\sqrt{z}^p)^2}{(p!)^2} \frac{(\sqrt{w}^q)^2}{(q!)^2}.$$

Using [42, Theorem 8, p. 117], we find

$$\begin{aligned} S_3 &\sim \left(\sum_{p,q=0}^{\infty} (p+q)! \frac{\sqrt{z}^p}{p!} \frac{\sqrt{w}^q}{q!}\right)^2 = \left(\sum_{k=0}^{\infty} (\sqrt{z} + \sqrt{w})^k\right)^2 \\ &\sim \sum_{k=0}^{\infty} (\sqrt{z} + \sqrt{w})^{2k}. \end{aligned}$$

The last series is a geometric series and convergent for

$$(\sqrt{z} + \sqrt{w})^2 < 1 \quad \Leftrightarrow \quad \sqrt{z} + \sqrt{w} < 1.$$

- The series $F_{18}^{(4)}$ in (1.11)

$$\sim S_4 = \sum_{m,n,p,q=0}^{\infty} \frac{(m+n+p)!\, q!\, (m+n)!\, (p+q)!}{(m+q)!\, n!\, p!} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{w^q}{q!}.$$

Using (2.8), we have

$$S_4 < \sum_{m,n,p=0}^{\infty} \frac{(m+n+p)!\,(m+n)!}{m!\,n!} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \sum_{q=0}^{\infty} (1+p)_q \frac{w^q}{q!}$$
$$= (1-w)^{-1} \sum_{m,n,p=0}^{\infty} \frac{(m+n+p)!\,(m+n)!}{m!\,n!} \frac{x^m}{m!} \frac{y^n}{n!} \frac{\left(\frac{z}{1-w}\right)^p}{p!},$$

in which the last triple series ∼ [42, Entry 20a, pp. 79 and 93]. Thus the region of convergence is sufficient for series $F_{18}^{(4)}$ in (1.11).

Remark 2.1. The convergence conditions given in (1.11), (1.20) and (1.22) are sufficient but not necessary. The regions of the quadruple hypergeometric series whose proofs are not given, including those of (1.11), (1.20) and (1.22), are left to the interested readers. □

3. Symbolic operator identities

Burchnall and Chaundy [3, 4] offered a number of series expansions of double hypergeometric functions in terms of elementary hypergeometric functions by introducing to employ certain intriguing symbolic operators (see also [6]). We recall the ensuing symbolic operators (see [3, 4, 6]):

$$\nabla_{x,y}(h) := \frac{\Gamma(h)\,\Gamma\left(\delta_x+\delta_y+h\right)}{\Gamma(h+\delta_x)\,\Gamma\left(h+\delta_y\right)} = \sum_{i=0}^{\infty} \frac{(-\delta_x)_i\left(-\delta_y\right)_i}{(h)_i\, i!}; \tag{3.1}$$

$$\Delta_{x,y}(h) := \frac{\Gamma(h+\delta_x)\,\Gamma\left(h+\delta_y\right)}{\Gamma(h)\,\Gamma\left(h+\delta_x+\delta_y\right)} = \sum_{i=0}^{\infty} \frac{(-\delta_x)_i\left(-\delta_y\right)_i}{\left(1-h-\delta_x-\delta_y\right)_i i!} \tag{3.2}$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i\,(h)_{2i}(-\delta_x)_i\left(-\delta_y\right)_i}{(h+i-1)_i(\delta_x+h)_i\left(\delta_y+h\right)_i i!}; \tag{3.3}$$

$$\nabla_{x,y}(h)\,\Delta_{x,y}(g) = \frac{\Gamma(h)\,\Gamma\left(\delta_x+\delta_y+h\right)}{\Gamma(h+\delta_x)\,\Gamma\left(h+\delta_y\right)} \frac{\Gamma(g+\delta_x)\,\Gamma\left(g+\delta_y\right)}{\Gamma(g)\,\Gamma\left(g+\delta_x+\delta_y\right)} \tag{3.4}$$

$$= \sum_{i=0}^{\infty} \frac{(g-h)_i(g)_{2i}(-\delta_x)_i\left(-\delta_y\right)_i}{(h)_i(g+i-1)_i(\delta_x+g)_i\left(\delta_y+g\right)_i i!} \tag{3.5}$$

$$= \sum_{i=0}^{\infty} \frac{(h-g)_i(-\delta_x)_i\left(-\delta_y\right)_i}{(h)_i\left(1-g-\delta_x-\delta_y\right)_i i!}, \tag{3.6}$$

where

$$\delta_x := x\frac{\partial}{\partial x} \quad \text{and} \quad \delta_y := y\frac{\partial}{\partial y}. \tag{3.7}$$

Many expansion formulas for hypergeometric functions of two variables were obtained by using the operators (3.1) - (3.6) (see [14]). Operators (3.1) - (3.6) are applicable only in hypergeometric functions of two variables. So, in order to be applied to hypergeometric functions of more than two variables, multidimensional mutually inverse symbolic operators have been introduced and employed (see, e.g., [7, 23, 24, 25, 26]):

$$\begin{aligned}\tilde{\nabla}_{x_1;x_2,\ldots,x_r}(h) &= \frac{\Gamma(h)\Gamma(h+\delta_1+\delta_2+\cdots+\delta_r)}{\Gamma(h+\delta_1)\Gamma(h+\delta_2+\cdots+\delta_r)} \\ &= \sum_{k_2,k_3,\ldots,k_r=0}^{\infty} \frac{(-\delta_1)_{k_2+\cdots+k_r}(-\delta_2)_{k_2}\cdots(-\delta_r)_{k_r}}{(h)_{k_2+\cdots+k_r} k_2!k_3!\cdots k_r!};\end{aligned} \tag{3.8}$$

$$\begin{aligned}\tilde{\Delta}_{x_1;x_2,\ldots,x_r}(h) &= \frac{\Gamma(h+\delta_1)\Gamma(h+\delta_2+\cdots+\delta_r)}{\Gamma(h)\Gamma(h+\delta_1+\delta_2+\cdots+\delta_r)} \\ &= \sum_{k_2,k_3,\ldots,k_r=0}^{\infty} \frac{(-\delta_1)_{k_2+\cdots+k_r}(-\delta_2)_{k_2}\cdots(-\delta_r)_{k_r}}{(1-h-\delta_2-\cdots-\delta_r)_{k_2+\cdots+k_r} k_2!k_3!\cdots k_r!};\end{aligned} \tag{3.9}$$

$$\begin{aligned}H_{x_1,\ldots x_r}(\alpha,\beta) &= \frac{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)}{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)} \\ &= \sum_{k_1,\cdots,k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(\beta)_{k_1+\cdots+k_r} k_1!\cdots k_r!};\end{aligned} \tag{3.10}$$

$$\begin{aligned}\overline{H}_{x_1,\ldots x_r}(\alpha,\beta) &= \frac{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)}{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)} \\ &= \sum_{k_1,\ldots k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(1-\alpha-\delta_1-\cdots-\delta_r)_{i+j} i!j!},\end{aligned} \tag{3.11}$$

where

$$\delta_{x_j} = x_j \frac{\partial}{\partial x_j} \quad (j = 1, \ldots, r;\ r \in \mathbb{N}). \tag{3.12}$$

We offer certain identities involving symbolic operators in Theorem 3.1.

Theorem 3.1. *The following formulas hold.*

$$F_{17}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w) = H_w(\mu_2,\xi_3)(1-w)^{-\nu_2} F_F\left(\mu_1;\nu_1,\nu_2;\xi_2,\xi_1;y,\frac{z}{1-w},x\right); \tag{3.13}$$

$$(1-w)^{-\nu_2} F_F\left(\mu_1;\nu_1,\nu_2;\xi_2,\xi_1;y,\frac{z}{1-w},x\right) = \overline{H}_w(\mu_2,\xi_3) F_{17}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w); \tag{3.14}$$

$$F_{19}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w) = H_{z.w}(\nu_2,\xi_3)(1-w)^{-\mu_2}(1-z)^{-\mu_1} F_4\left(\mu_1,\nu_1;\xi_1,\xi_2;\frac{x}{1-z},\frac{y}{1-z}\right); \tag{3.15}$$

$$(1-w)^{-\mu_2}(1-z)^{-\mu_1} F_4\left(\mu_1,\nu_1;\xi_1,\xi_2;\frac{x}{1-z},\frac{y}{1-z}\right) = \overline{H}_{z,w}(\nu_2,\xi_3) F_{19}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w); \tag{3.16}$$

$$F_{20}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w) = H_z(\nu_2,\xi_3)(1-z)^{-\mu_1} F_R\left(\mu_1,\mu_2,\nu_1,\nu_3;\xi_2,\xi_1;\frac{y}{1-z},w,\frac{x}{1-z}\right); \tag{3.17}$$

$$(1-z)^{-\mu_1} F_R\left(\mu_1,\mu_2,\nu_1,\nu_3;\xi_2,\xi_1;\frac{y}{1-z},w,\frac{x}{1-z}\right) = \overline{H}_z(\nu_2,\xi_3) F_{20}^{(4)}(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w); \tag{3.18}$$

$$\begin{aligned} &F^{(4)}_{22}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w\right) = H_z\left(\nu_3,\xi_2\right)H_w\left(\mu_2,\xi_3\right)\\ &\times(1-z)^{-\mu_1}(1-w)^{-\nu_1}F_1\left(\mu_1;\nu_1,\nu_2;\xi_1;\frac{x}{(1-z)(1-w)},\frac{y}{1-z}\right); \end{aligned} \tag{3.19}$$

$$\begin{aligned} &(1-z)^{-\mu_1}(1-w)^{-\nu_1}F_1\left(\mu_1;\nu_1,\nu_2;\xi_1;\frac{x}{(1-z)(1-w)},\frac{y}{1-z}\right)\\ &=\overline{H}_z\left(\nu_3,\xi_2\right)\overline{H}_w\left(\mu_2,\xi_3\right)F^{(4)}_{22}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w\right); \end{aligned} \tag{3.20}$$

$$F^{(4)}_{23}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w\right) = H_w\left(\mu_2,\xi_3\right)(1-w)^{-\nu_1}F_G\left(\mu_1;\nu_1,\nu_2,\nu_3;\xi_1,\xi_2;\frac{x}{1-w},y,z\right); \tag{3.21}$$

$$(1-w)^{-\nu_1}F_G\left(\mu_1;\nu_1,\nu_2,\nu_3;\xi_1,\xi_2;\frac{x}{1-w},y,z\right) = \overline{H}_w\left(\mu_2,\xi_3\right)F^{(4)}_{23}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w\right); \tag{3.22}$$

$$F^{(4)}_{24}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w\right) = H_z\left(\nu_2,\xi_3\right)(1-z)^{-\mu_1}F_R\left(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2;\frac{x}{1-z},w,\frac{y}{1-z}\right); \tag{3.23}$$

$$(1-z)^{-\mu_1}F_R\left(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2;\frac{x}{1-z},w,\frac{y}{1-z}\right) = \overline{H}_z\left(\nu_2,\xi_3\right)F^{(4)}_{24}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w\right); \tag{3.24}$$

$$\begin{aligned} &F^{(4)}_{25}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w\right) = H_y\left(\nu_2,\xi_2\right)H_z\left(\nu_3,\xi_3\right)\\ &\times(1-y-z)^{-\mu_1}F_1\left(\nu_1;\mu_1,\mu_2;\xi_1;\frac{x}{1-y-z},w\right); \end{aligned} \tag{3.25}$$

$$\begin{aligned} &(1-y-z)^{-\mu_1}F_1\left(\nu_1;\mu_1,\mu_2;\xi_1;\frac{x}{1-y-z},w\right)\\ &=\overline{H}_y\left(\nu_2,\xi_2\right)\overline{H}_z\left(\nu_3,\xi_3\right)F^{(4)}_{25}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w\right); \end{aligned} \tag{3.26}$$

$$\begin{aligned} &F^{(4)}_{26}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3,\nu_4;\xi_1,\xi_2,\xi_3;x,y,z,w\right) = H_y\left(\nu_2,\xi_2\right)H_z\left(\nu_3,\xi_3\right)\\ &\times(1-y-z)^{-\mu_1}F_3\left(\mu_1,\mu_2,\nu_1,\nu_4;\xi_1;\frac{x}{1-y-z},w\right); \end{aligned} \tag{3.27}$$

$$\begin{aligned} &(1-y-z)^{-\mu_1}F_3\left(\mu_1,\mu_2,\nu_1,\nu_4;\xi_1;\frac{x}{1-y-z},w\right)\\ &=\overline{H}_y\left(\nu_2,\xi_2\right)\overline{H}_z\left(\nu_3,\xi_3\right)F^{(4)}_{26}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3,\nu_4;\xi_1,\xi_2,\xi_3;x,y,z,w\right); \end{aligned} \tag{3.28}$$

$$F^{(4)}_{28}\left(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w\right) = H_{x.y}\left(\mu_1,\xi_1\right)(1-x)^{-\nu_1}(1-y)^{-\nu_2}F_2\left(\mu_2;\nu_1,\nu_2;\xi_2,\xi_3;\frac{z}{1-x},\frac{w}{1-y}\right); \tag{3.29}$$

$$(1-x)^{-\nu_1}(1-y)^{-\nu_2}F_2\left(\mu_2;\nu_1,\nu_2;\xi_2,\xi_3;\frac{z}{1-x},\frac{w}{1-y}\right) = \overline{H}_{x.y}\left(\mu_1,\xi_1\right)F^{(4)}_{28}\left(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w\right); \tag{3.30}$$

$$F_{30}^{(4)}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w\right)=H_w\left(\nu_3,\xi_3\right)(1-w)^{-\mu_2}F_R\left(\mu_1,\mu_2,\nu_1,\nu_2;\xi_2,\xi_1;y,\frac{z}{1-w},x\right); \tag{3.31}$$

$$(1-t)^{-\mu_2}F_R\left(\mu_1,\mu_2,\nu_1,\nu_2;\xi_2,\xi_1;y,\frac{z}{1-w},x\right)=\overline{H}_w\left(\nu_3,\xi_3\right)F_{30}^{(4)}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w\right). \tag{3.32}$$

Proof. The operator identities (3.13) - (3.32) can be verified by utilizing the Mellin-Barnes transformation (e.g., [29]):

$$f(x)\leftrightarrow f^*(s)=\int_0^\infty f(x)\,x^{s-1}\,\mathrm{d}s, \tag{3.33}$$

where $f^*(s)$ is the Mellin-Barnes transformation of the original function $f(x)$.

We will prove only the identity (3.13). We observe the following Mellin-Barnes integral representation for the hypergeometric function $F_{17}^{(4)}$:

$$\begin{aligned}&\frac{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma(\xi_1)\Gamma(\xi_2)\Gamma(\xi_3)}F_{17}^{(4)}\left(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w\right)\\&=\frac{1}{(2\pi i)^4}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\Gamma(\mu_1+s_1+s_2+s_3)\Gamma(\mu_2+s_4)}{\Gamma(\xi_1+s_1+s_3)\Gamma(\xi_2+s_2)\Gamma(\xi_3+s_4)}\\&\quad\times\Gamma(\nu_1+s_1+s_2)\Gamma(\nu_2+s_3+s_4)\Gamma(-s_1)\Gamma(-s_2)\Gamma(-s_3)\Gamma(-s_4)\\&\quad\times(-x)^{s_1}(-y)^{s_2}(-z)^{s_3}(-w)^{s_4}\,\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4.\end{aligned} \tag{3.34}$$

The right member of (3.13), denoted by $\mathcal{R}$, can be written in the following form:

$$\mathcal{R}=\sum_{i=0}^{\infty}\frac{(\xi_3-\mu_2)_i(-\delta_t)_i}{(\xi_3)_i i!}\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\nu_1)_{m+n}(\nu_2)_{p+q}}{(\xi_1)_{m+p}(\xi_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!}. \tag{3.35}$$

Given the known transformation (see [29])

$$(-\delta)_k g(x)\leftrightarrow\frac{\Gamma(k-s)}{\Gamma(-s)}g^*(s)\quad\left(k\in\mathbb{N}_0\right), \tag{3.36}$$

and

$$\begin{aligned}&\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\nu_1)_{m+n}(\nu_2)_{p+q}}{(\xi_1)_{m+p}(\xi_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!}\\&\leftrightarrow\frac{1}{(2\pi i)^4}\frac{\Gamma(\xi_1)\Gamma(\xi_2)}{\Gamma(\mu_1)\Gamma(\nu_1)\Gamma(\nu_2)}\\&\times\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\Gamma(\mu_1+s_1+s_2+s_3)\Gamma(\nu_1+s_1+s_2)}{\Gamma(\xi_1+s_1+s_3)\Gamma(\xi_2+s_2)}\\&\times\Gamma(\nu_2+s_3+s_4)\Gamma(-s_1)\Gamma(-s_2)\Gamma(-s_3)\Gamma(-s_4)\\&\times(-x)^{s_1}(-y)^{s_2}(-z)^{s_3}(-w)^{s_4}\,\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4,\end{aligned} \tag{3.37}$$

we get

$$
\begin{aligned}
&(-\delta_t)_i \sum_{m,n,p,q=0}^{\infty} \frac{(\mu_1)_{m+n+p}(\nu_1)_{m+n}(\nu_2)_{p+q}}{(\xi_1)_{m+p}(\xi_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!} \\
&\leftrightarrow \frac{1}{(2\pi i)^4} \frac{\Gamma(\xi_1)\Gamma(\xi_2)}{\Gamma(\mu_1)\Gamma(\nu_1)\Gamma(\nu_2)} \frac{\Gamma(i-s_4)}{\Gamma(-s_4)} \\
&\quad \times \int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} \frac{\Gamma(\mu_1+s_1+s_2+s_3)\Gamma(\nu_1+s_1+s_2)}{\Gamma(\xi_1+s_1+s_3)\Gamma(\xi_2+s_2)} \\
&\quad \times \Gamma(\nu_2+s_3+s_4)\Gamma(-s_1)\Gamma(-s_2)\Gamma(-s_3)\Gamma(-s_4) \\
&\quad \times (-x)^{s_1}(-y)^{s_2}(-z)^{s_3}(-w)^{s_4}\,\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4.
\end{aligned}
\tag{3.38}
$$

Then, using (3.38) in (3.35), we obtain

$$
\begin{aligned}
\mathcal{R} &= \frac{1}{(2\pi i)^4} \frac{\Gamma(\xi_1)\Gamma(\xi_2)}{\Gamma(\mu_1)\Gamma(\nu_1)\Gamma(\nu_2)} \\
&\times \int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} \frac{\Gamma(\mu_1+s_1+s_2+s_3)\Gamma(\nu_1+s_1+s_2)\Gamma(\nu_2+s_3+s_4)}{\Gamma(\xi_1+s_1+s_3)\Gamma(\xi_2+s_2)} \\
&\times \Gamma(-s_1)\Gamma(-s_2)\Gamma(-s_3)\Gamma(-s_4)(-x)^{s_1}(-y)^{s_2}(-z)^{s_3}(-w)^{s_4} \\
&\times \sum_{i=0}^{\infty} \frac{(-s_4)_i(\xi_3-\mu_2)_i}{(\xi_3)_i i!}\,\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4.
\end{aligned}
\tag{3.39}
$$

Using Gauss summation theorem (see, e.g., [40, p. 64])

$$
{}_2F_1(\lambda,\mu;\nu;1) = F(\lambda,\mu;\nu;1) = \frac{\Gamma(\nu)\Gamma(\nu-\lambda-\mu)}{\Gamma(\nu-\lambda)\Gamma(\nu-\mu)} \tag{3.40}
$$

$$
\left(\Re(\nu-\lambda-\mu)>0,\ \nu\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}\right),
$$

$$
\sum_{i=0}^{\infty} \frac{(-s_4)_i(\xi_3-\mu_2)_i}{(\xi_3)_i i!} = F(-s_4,\xi_3-\mu_2;\xi_3;1) = \frac{\Gamma(\xi_3)\Gamma(\mu_2+s_4)}{\Gamma(\xi_3+s_4)\Gamma(\mu_2)},
$$

we finally obtain

$$
\begin{aligned}
\mathcal{R} &= \frac{1}{(2\pi i)^4} \frac{\Gamma(\xi_1)\Gamma(\xi_2)\Gamma(\xi_3)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\nu_1)\Gamma(\nu_2)} \\
&\times \int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} \frac{\Gamma(\mu_1+s_1+s_2+s_3)\Gamma(\mu_2+s_4)\Gamma(\nu_1+s_1+s_2)}{\Gamma(\xi_1+s_1+s_3)\Gamma(\xi_2+s_2)\Gamma(\xi_3+s_4)} \\
&\times \Gamma(\nu_2+s_3+s_4)\Gamma(-s_1)\Gamma(-s_2)\Gamma(-s_3)\Gamma(-s_4) \\
&\times (-x)^{s_1}(-y)^{s_2}(-z)^{s_3}(-w)^{s_4}\,\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4.
\end{aligned}
\tag{3.41}
$$

From relations (3.34) and (3.41), we see that the left and right members of the identity (3.13) have the same Mellin-Barnes integral representation. Hence, on the basis of the Slater's theorem [29], the operator identity (3.13) is proved.

□

4. Decomposition formulas

Using operator identities (3.13) - (3.32), we can derive the corresponding decomposition formulas for the hypergeometric functions of four variables which are given in following theorem.

Theorem 4.1. *The following decomposition formulas hold.*

$$F^{(4)}_{17}(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w) = (1-w)^{-\nu_2}\sum_{i=0}^{\infty}\frac{(-1)^i(\nu_2)_i(\xi_3-\mu_2)_i}{(\xi_3)_i i!}\Big(\frac{w}{1-w}\Big)^i \times F_F\left(\mu_1;\nu_1,\nu_2+i;\xi_2,\xi_1;y,\frac{z}{1-w},x\right); \tag{4.1}$$

$$(1-w)^{-\nu_2}F_F\left(\mu_1;\nu_1,\nu_2;\xi_2,\xi_1;y,\frac{z}{1-w},x\right) = \sum_{i=0}^{\infty}\frac{(\nu_2)_i(\xi_3-\mu_2)_i}{(\xi_3)_i i!}w^i F^{(4)}_{17}(\mu_1,\mu_2,\nu_1,\nu_2+i;\xi_1,\xi_2,\xi_3+i;x,y,z,w); \tag{4.2}$$

$$F^{(4)}_{19}(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w) = (1-z)^{-\mu_1}(1-w)^{-\mu_2} \times\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(\mu_1)_i(\mu_2)_j(\xi_3-\nu_2)_{i+j}}{(\xi_3)_{i+j}i!j!}\Big(\frac{z}{1-z}\Big)^i\Big(\frac{w}{1-w}\Big)^j \times F_4\left(\mu_1+i,\nu_1;\xi_1,\xi_2;\frac{x}{1-z},\frac{y}{1-z}\right); \tag{4.3}$$

$$(1-w)^{-\mu_2}(1-z)^{-\mu_1}F_4\left(\mu_1,\nu_1;\xi_1,\xi_2;\frac{x}{1-z},\frac{y}{1-z}\right) = \sum_{i,j=0}^{\infty}\frac{(\mu_1)_i(\mu_2)_j(\xi_3-\nu_2)_{i+j}}{(\xi_3)_{i+j}i!j!}z^i w^j \times F^{(4)}_{19}(\mu_1+i,\mu_2+j,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3+i+j;x,y,z,w); \tag{4.4}$$

$$F^{(4)}_{20}(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w) = (1-z)^{-\mu_1}\sum_{i=0}^{\infty}\frac{(-1)^i(\mu_1)_i(\xi_3-\nu_2)_i}{(\xi_3)_i i!}\Big(\frac{z}{1-z}\Big)^i \times F_R\left(\mu_1+i,\mu_2,\nu_1,\nu_3;\xi_2,\xi_1;\frac{y}{1-z},w,\frac{x}{1-z}\right); \tag{4.5}$$

$$(1-z)^{-\mu_1}F_R\left(\mu_1,\mu_2,\nu_1,\nu_3;\xi_2,\xi_1;\frac{y}{1-z},w,\frac{x}{1-z}\right) = \sum_{i=0}^{\infty}\frac{(\mu_1)_i(\xi_3-\nu_2)_i}{(\xi_3)_i i!}z^i \times F^{(4)}_{20}(\mu_1+i,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3+i;x,y,z,w); \tag{4.6}$$

$$F^{(4)}_{22}(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w) = (1-z)^{-\mu_1}(1-w)^{-\nu_1} \times\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(\mu_1)_i(\nu_1)_j(\xi_2-\nu_3)_i(\xi_3-\mu_2)_j}{(\xi_2)_i(\xi_3)_j i!j!}\Big(\frac{z}{1-z}\Big)^i\Big(\frac{w}{1-w}\Big)^j \times F_1\left(\mu_1+i;\nu_1+j,\nu_2;\xi_1;\frac{x}{(1-z)(1-w)},\frac{y}{1-z}\right); \tag{4.7}$$

$$(1-z)^{-\mu_1}(1-w)^{-\nu_1}F_1\left(\mu_1;\nu_1,\nu_2;\xi_1;\frac{x}{(1-z)(1-w)},\frac{y}{1-z}\right) = \sum_{i,j=0}^{\infty}\frac{(\mu_1)_i(\nu_1)_j(\xi_2-\nu_3)_i(\xi_3-\mu_2)_j}{(\xi_2)_i(\xi_3)_j i!j!}z^i w^j \times F_{22}{}^{(4)}(\mu_1+i,\mu_2,\nu_1+j,\nu_2,\nu_3;\xi_1,\xi_2+i,\xi_3+j;x,y,z,w); \tag{4.8}$$

$$F^{(4)}_{23}(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w) = (1-w)^{-\nu_1}\sum_{i=0}^{\infty}\frac{(-1)^i(\nu_1)_i(\xi_3-\mu_2)_i}{(\xi_3)_i i!}\Big(\frac{w}{1-w}\Big)^i \times F_G\left(\mu_1;\nu_1+i,\nu_2,\nu_3;\xi_1,\xi_2;\frac{x}{1-w},y,z\right); \tag{4.9}$$

$$\begin{aligned}(1-w)^{-\nu_1} F_G\left(\mu_1;\nu_1,\nu_2,\nu_3;\xi_1,\xi_2;\frac{x}{1-w},y,z\right) &= \sum_{i=0}^{\infty}\frac{(\nu_1)_i(\xi_3-\mu_2)_i}{(\xi_3)_i i!}w^i \\ &\times F_{23}^{(4)}\left(\mu_1,\mu_2,\nu_1+i,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3+i;x,y,z,w\right);\end{aligned} \tag{4.10}$$

$$\begin{aligned}F_{24}^{(4)}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w\right) &= (1-z)^{-\mu_1}\sum_{i=0}^{\infty}\frac{(-1)^i(\mu_1)_i(\xi_3-\nu_2)_i}{(\xi_3)_i i!}\left(\frac{z}{1-z}\right)^i \\ &\times F_R\left(\mu_1+i,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2;\frac{x}{1-z},w,\frac{y}{1-z}\right);\end{aligned} \tag{4.11}$$

$$\begin{aligned}(1-z)^{-\mu_1} F_R\left(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2;\frac{x}{1-z},w,\frac{y}{1-z}\right) &= \sum_{i=0}^{\infty}\frac{(\mu_1)_i(\xi_3-\nu_2)_i}{(\xi_3)_i i!}z^i \\ &\times F_{24}^{(4)}\left(\mu_1+i,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3+i;x,y,z,w\right);\end{aligned} \tag{4.12}$$

$$\begin{aligned}F_{25}^{(4)}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w\right) &= (1-y-z)^{-\mu_1}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(\mu_1)_{i+j}(\xi_2-\nu_2)_i(\xi_3-\nu_3)_j}{(\xi_2)_i(\xi_3)_j i!j!}\left(\frac{y}{1-y-z}\right)^i \\ &\times\left(\frac{z}{1-y-z}\right)^j F_1\left(\nu_1;\mu_1+i+j,\mu_2;\xi_1;\frac{x}{1-y-z},w\right);\end{aligned} \tag{4.13}$$

$$\begin{aligned}(1-y-z)^{-\mu_1} F_1\left(\nu_1;\mu_1,\mu_2;\xi_1;\frac{x}{1-y-z},w\right) &= \sum_{i,j=0}^{\infty}\frac{(\mu_1)_{i+j}(\xi_2-\nu_2)_i(\xi-\nu_3)_j}{(\xi_2)_i(\xi_3)_j i!j!}y^iz^j \\ &\times F_{25}^{(4)}\left(\mu_1+i+j,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2+i,\xi_3+j;x,y,z,w\right);\end{aligned} \tag{4.14}$$

$$\begin{aligned}F_{26}^{(4)}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3,\nu_4;\xi_1,\xi_2,\xi_3;x,y,z,w\right) &= (1-y-z)^{-\mu_1} \\ &\times\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(\mu_1)_{i+j}(\xi_2-\nu_2)_i(\xi_3-\nu_3)_j}{(\xi_2)_i(\xi_3)_j i!j!}\left(\frac{y}{1-y-z}\right)^i \\ &\times\left(\frac{z}{1-y-z}\right)^j F_3\left(\mu_1+i+j,\mu_2,\nu_1,\nu_4;\xi_1;\frac{x}{1-y-z},w\right);\end{aligned} \tag{4.15}$$

$$\begin{aligned}(1-y-z)^{-\mu_1} F_3\left(\mu_1,\mu_2,\nu_1,\nu_4;\xi_1;\frac{x}{1-y-z},w\right) &= \sum_{i,j=0}^{\infty}\frac{(\mu_1)_{i+j}(\xi_2-\nu_2)_i(\xi_3-\nu_3)_j}{(\xi_2)_i(\xi_3)_j i!j!}y^iz^j \\ &\times F_{26}^{(4)}\left(\mu_1+i+j,\mu_2,\nu_1,\nu_2,\nu_3,\nu_4;\xi_1,\xi_2+i,\xi_3+j;x,y,z,w\right);\end{aligned} \tag{4.16}$$

$$\begin{aligned}F_{28}^{(4)}\left(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w\right) &= (1-x)^{-\nu_1}(1-y)^{-\nu_2} \\ &\times\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(\nu_1)_i(\nu_2)_j(\xi_1-\mu_1)_{i+j}}{(\xi_1)_{i+j}i!j!}\left(\frac{x}{1-x}\right)^i\left(\frac{y}{1-y}\right)^j \\ &\times F_2\left(\mu_2;\nu_1+i,\nu_2+j;\xi_2,\xi_3;\frac{z}{1-x},\frac{w}{1-y}\right);\end{aligned} \tag{4.17}$$

$$(1-x)^{-\nu_1}(1-y)^{-\nu_2}F_2\left(\mu_2;\nu_1,\nu_2;\xi_2,\xi_3;\frac{z}{1-x},\frac{w}{1-y}\right)=\sum_{i,j=0}^{\infty}\frac{(\nu_1)_i(\nu_2)_j(\xi_1-\mu_1)_{i+j}}{(c_1)_{i+j}i!\,j!}x^iy^j \times F_{28}^{(4)}\left(\mu_1,\mu_2,\nu_1+i,\nu_2+j;\xi_1+i+j,\xi_2,\xi_3;x,y,z,w\right); \tag{4.18}$$

$$F_{30}^{(4)}\left(\mu_1,\mu_2,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3;x,y,z,w\right)=(1-t)^{-\mu_2}\sum_{i=0}^{\infty}\frac{(-1)^i(\mu_2)_i(\xi_3-\nu_3)_i}{(\xi_3)_i i!}\left(\frac{w}{1-w}\right)^i \times F_R\left(\mu_1,\mu_2+i,\nu_1,\nu_2;\xi_2,\xi_1;y,\frac{z}{1-w},x\right); \tag{4.19}$$

$$(1-w)^{-\mu_2}F_R\left(\mu_1,\mu_2,\nu_1,\nu_2;\xi_2,\xi_1;y,\frac{z}{1-w},x\right)=\sum_{i=0}^{\infty}\frac{(\mu_2)_i(\xi_3-\nu_3)_i}{(\xi_3)_i i!}w^i \times F_{30}^{(4)}\left(\mu_1,\mu_2+i,\nu_1,\nu_2,\nu_3;\xi_1,\xi_2,\xi_3+i;x,y,z,w\right). \tag{4.20}$$

Proof. Proofs of the operational identities (3.13) - (3.32) are indeed carried out in parallel with those that were presented in earlier works, which have been quoted in previous sections. In addition to the operators (3.8) - (3.9), we also use the following differential operators [34, p. 93]:

$$(\delta+\alpha)_n f(\xi)=\xi^{1-\alpha}\frac{d^n}{d\xi^n}\left\{\xi^{\alpha+n-1}f(\xi)\right\}, \tag{4.21}$$

$$(-\delta)_n f(\xi)=(-\xi)^n\frac{d^n}{d\xi^n}f(\xi) \tag{4.22}$$

$$\left(\delta=\xi\frac{d}{d\xi},\ \alpha\in\mathbb{C},\ n\in\mathbb{N}_0\right),$$

where $f(\xi)$ is an analytic function. We will prove only (4.1) in two ways.

Method 1: Using the series definition in (1.28) and the following expansion

$$(1-u)^{-\alpha}=\sum_{\ell=0}^{\infty}\frac{(\alpha)_\ell}{\ell!}u^\ell \quad (\alpha\in\mathbb{C},\ |u|<1), \tag{4.23}$$

we obtain

$$(1-w)^{-\nu_2}F_F\left(\mu_1;\nu_1,\nu_2;\xi_2,\xi_1;y,\frac{z}{1-w},x\right)=\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\nu_1)_{m+n}(\nu_2)_{p+q}}{(\xi_1)_{m+p}(\xi_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!}. \tag{4.24}$$

Multiplying both sides of (4.24) by the operator $H_w(\mu_2,\xi_3)$ in (3.10), we find from (3.13) that

$$F_{17}^{(4)}\left(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w\right)=\sum_{j=0}^{\infty}\frac{(\xi_3-\mu_2)_j(-\delta_w)_j}{(\xi_3)_j\, j!}\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\nu_1)_{m+n}(\nu_2)_{p+q}}{(\xi_1)_{m+p}(\xi_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!}. \tag{4.25}$$

Using (4.22) to get $(-\delta_w)_j\, w^q=(-q)_j\, w^q$, from (4.25), we have

$$F_{17}^{(4)}\left(\mu_1,\mu_2,\nu_1,\nu_2;\xi_1,\xi_2,\xi_3;x,y,z,w\right)=\sum_{j=0}^{\infty}\frac{(\xi_3-\mu_2)_j(-q)_j}{(\xi_3)_j\, j!}\sum_{m,n,p,q=0}^{\infty}\frac{(\mu_1)_{m+n+p}(\nu_1)_{m+n}(\nu_2)_{p+q}}{(\xi_1)_{m+p}(\xi_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{w^q}{q!}. \tag{4.26}$$

Let $\mathcal{R}_1$ be the right member of (4.26). Then, in view of (4.26), it is enough to show that $\mathcal{R}_1$ is equal to the right member of (4.1). Now we have

$$\mathcal{R}_1 = \sum_{m,n,p=0}^{\infty} \frac{(\mu_1)_{m+n+p}\,(\nu_1)_{m+n}}{(\xi_1)_{m+p}\,(\xi_2)_n} \frac{x^m\,y^n\,z^p}{m!\,n!\,p!} \times \sum_{q=0}^{\infty}\sum_{j=0}^{q} \frac{(\nu_2)_{p+q}}{q!} \frac{(\xi_3-\mu_2)_j(-q)_j}{(\xi_3)_j\,j!}\,w^q.$$

Using $(-q)_j/q! = (-1)^j/(q-j)!$, we have

$$\mathcal{R}_1 = \sum_{m,n,p=0}^{\infty} \frac{(\mu_1)_{m+n+p}\,(\nu_1)_{m+n}}{(\xi_1)_{m+p}\,(\xi_2)_n} \frac{x^m\,y^n\,z^p}{m!\,n!\,p!} \times \sum_{q=0}^{\infty}\sum_{j=0}^{q} \frac{(\nu_2)_{p+q}}{(q-j)!} \frac{(\xi_3-\mu_2)_j(-1)^j}{(\xi_3)_j\,j!}\,w^q.$$

Using the ensuing double series manipulation

$$\sum_{q=0}^{\infty}\sum_{j=0}^{q} \mathscr{A}_{q,j} = \sum_{q=0}^{\infty}\sum_{j=0}^{\infty} \mathscr{A}_{q+j,j}, \tag{4.27}$$

where $\mathscr{A}_{q,j}$ is a function from $\mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ such that the right member double series converges absolutely,

$$\mathcal{R}_1 = \sum_{m,n,p=0}^{\infty} \frac{(\mu_1)_{m+n+p}\,(\nu_1)_{m+n}}{(\xi_1)_{m+p}\,(\xi_2)_n} \frac{x^m\,y^n\,z^p}{m!\,n!\,p!} \times \sum_{q=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\nu_2)_{p+q+j}}{q!} \frac{(\xi_3-\mu_2)_j(-1)^j}{(\xi_3)_j\,j!}\,w^{q+j}.$$

Using $(\nu_2)_{p+q+j} = (\nu_2)_j\,(\nu_2+j)_{p+q} = (\nu_2)_j\,(\nu_2+j)_p\,(\nu_2+j+p)_q$, we have

$$\mathcal{R}_1 = \sum_{j=0}^{\infty} \frac{(-1)^j(\nu_2)_j(\xi_3-\mu_2)_j}{(\xi_3)_j\,j!} w^j \times \sum_{m,n,p=0}^{\infty} \frac{(\mu_1)_{m+n+p}(\nu_1)_{m+n}(\nu_2+j)_p}{(\xi_1)_{m+p}(\xi_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \times \sum_{q=0}^{\infty} \frac{(\nu_2+j+p)_q}{q!}\,w^q. \tag{4.28}$$

Applying (4.23) to the last summation in (4.28), we obtain

$$\mathcal{R}_1 = (1-w)^{-\nu_2} \sum_{j=0}^{\infty} \frac{(-1)^j(\nu_2)_j(\xi_3-\mu_2)_j}{(\xi_3)_j\,j!} \left(\frac{w}{1-w}\right)^j \sum_{m,n,p=0}^{\infty} \frac{(\mu_1)_{m+n+p}(\nu_1)_{m+n}(\nu_2+j)_p}{(\xi_1)_{m+p}(\xi_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{\left(\frac{z}{1-w}\right)^p}{p!}, \tag{4.29}$$

which, in view of (1.28), leads to the right member of (4.1).

Method 2: Let $\mathcal{R}_2$ denote the right member of (4.1). Using the series definition of (1.28), we get

$$\mathcal{R}_2 = \sum_{i=0}^{\infty} \frac{(-1)^i(\nu_2)_i(\xi_3-\mu_2)_i}{(\xi_3)_i\,i!} w^i \sum_{m,n,p=0}^{\infty} \frac{(\mu_1)_{n+p+m}(\nu_1)_{n+m}(\nu_2+i)_p}{(\xi_2)_n(\xi_1)_{p+m}\,m!\,n!\,p!} x^m y^n z^p\,(1-w)^{-\nu_2-i-p}. \tag{4.30}$$

Applying (4.23) to expand $(1-w)^{-\nu_2-i-p}$ in (4.30), we obtain

$$\begin{aligned}\mathcal{R}_2 =& \sum_{i=0}^{\infty} \frac{(-1)^i(\nu_2)_i(\xi_3-\mu_2)_i}{(\xi_3)_i i!} w^i \\ &\times \sum_{m,n,p,q=0}^{\infty} \frac{(\mu_1)_{n+p+m}(\nu_1)_{n+m}(\nu_2+i)_p(\nu_2+i+p)_q}{(\xi_2)_n(\xi_1)_{p+m}\, m!\, n!\, p!\, q!} x^m y^n z^p w^q.\end{aligned} \tag{4.31}$$

Noting $(\nu_2+i)_p(\nu_2+i+p)_q = (\nu_2+i)_{p+q}$, we have

$$\begin{aligned}\mathcal{R}_2 =& \sum_{i=0}^{\infty} \frac{(-1)^i(\nu_2)_i(\xi_3-\mu_2)_i}{(\xi_3)_i i!} \\ &\times \sum_{m,n,p,q=0}^{\infty} \frac{(\mu_1)_{n+p+m}(\nu_1)_{n+m}(\nu_2+i)_{p+q}}{(\xi_2)_n(\xi_1)_{p+m}\, m!\, n!\, p!\, q!} x^m y^n z^p w^{q+i}.\end{aligned} \tag{4.32}$$

Employing the following double series manipulation

$$\sum_{q=0}^{\infty}\sum_{i=0}^{\infty} \mathscr{A}_{q,i} = \sum_{q=0}^{\infty}\sum_{i=0}^{q} \mathscr{A}_{q-i,i},$$

which is equivalent to the relation (4.27), in the double series of indices i and q in (4.32), we derive

$$\begin{aligned}\mathcal{R}_2 =& \sum_{m,n,p,q=0}^{\infty} \frac{(\mu_1)_{n+p+m}(\nu_1)_{n+m}}{(\xi_2)_n(\xi_1)_{p+m}\, m!\, n!\, p!} x^m y^n z^p w^q \\ &\times \sum_{i=0}^{q} \frac{(-1)^i(\nu_2)_i(\xi_3-\mu_2)_i}{(\xi_3)_i i!} \frac{(\nu_2+i)_{p+q-i}}{(q-i)!}\end{aligned} \tag{4.33}$$

Using (1.2) and (1.9), we get

$$(\nu_2+i)_{p+q-i} = \frac{\Gamma(\nu_2+p+q)}{\Gamma(\nu_2+i)} = \frac{(\nu_2)_{p+q}}{(\nu_2)_i}$$

and

$$\frac{1}{(q-i)!} = \frac{(-1)^i\,(-q)_i}{q!}.$$

Then, by using Gauss summation theorem (3.40),

$$\begin{aligned}&\sum_{i=0}^{q} \frac{(-1)^i(\nu_2)_i(\xi_3-\mu_2)_i}{(\xi_3)_i i!} \frac{(\nu_2+i)_{p+q-i}}{(q-i)!} \\ &= \frac{(\nu_2)_{p+q}}{q!} \sum_{i=0}^{q} \frac{(-q)_i\,(\xi_3-\mu_2)_i}{(\xi_3)_i\, i!} = \frac{(\nu_2)_{p+q}}{q!}\, F\left(-q,\, \xi_3-\mu_2;\, \xi_3\,;\, 1\right) \\ &= \frac{(\nu_2)_{p+q}}{q!} \frac{\Gamma(\xi_3)\,\Gamma(\mu_2+q)}{\Gamma(\xi_3+q)\,\Gamma(\mu_2)} = \frac{(\nu_2)_{p+q}}{q!} \frac{(\mu_2)_q}{(\xi_3)_q}.\end{aligned} \tag{4.34}$$

Using the last equality of (4.34) in (4.33), we have

$$\mathcal{R}_2 = \sum_{m,n,p,q=0}^{\infty} \frac{(\mu_1)_{n+p+m}(\mu_2)_q(\nu_1)_{n+m}(\nu_2)_{p+q}}{(\xi_2)_n(\xi_1)_{p+m}\,(\xi_3)_q\, m!\, n!\, p!\, q!} x^m y^n z^p w^q, \tag{4.35}$$

which, in view of (1.10), is equal to the left member of (4.1). □

Acknowledgments

This paper is dedicated to Professor Themistocles M. Rassias on the occasion of his 70th birthday.

The authors are very thankful for the anonymous reviewers' valuable and encouraging comments to improve this paper as it stands.

References

[1] P. App ell, J. Kampé de Fériet, *Fonctions hypergétriques et hypersphriques, polynômes d'Hermite*, Gauthier-villars, Paris, 1926.

[2] L. Bers, *Mathematical aspects of subsonic and transonic gas dynamics*, John Wiley and Sons, **176**, New York, 1958.

[3] J. L. Burchnall, T. W. Chaundy, *Expansions of Appell's double hypergeometric functions*, Quart. J. Math (Oxford) **11 (1)**, 249–270, 1940. https://doi.org/10.1093/qmath/os-11.1.249

[4] J. L. Burchnall, T. W. Chaundy, *Expansions of Appell's double hypergeometric functions (II)*, Quart. J. Math (Oxford) **12 (1)**, 112–128, 1941. https://doi.org/10.1093/qmath/os-12.1.112

[5] P. Candelas, X. C. De La Ossa, P. S. Green, L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nucl. Phys. B **359 (1)**, 21–74, 1991. https://doi.org/10.1016/0550-3213(91)90292-6

[6] T. W. Chaundy, *Expansions of hypergeometric functions*, Quart. J. Math (Oxford) **13 (1)**, 159–171, 1942. https://doi.org/10.1093/qmath/os-13.1.159

[7] J. Choi, A. Hasanov, *Applications of the operator $H(\alpha,\beta)$ to the Humbert double hypergeometric functions*, Comput. Math. Appl. **61**, 663—671, 2011. doi:10.1016/j.camwa.2010.12.012

[8] J. Choi, A. Hasanov, *Certain decomposition formulas of generalized hypergeometric functions ${}_pF_q$ and some formulas of an analytic continuation of the Clausen function ${}_3F_2$*, Commun. Korean Math. Soc. **27 (1)**, 107–116, 2012. http://dx.doi.org/10.4134/CKMS.2012.27.1.107

[9] J. Choi, Y. S. Kim, A. Hasanov, *Relations between the hypergeometric function of Appell F_3 and Kampé de Fériet functions*, Miskolc Math. Notes **12 (2)**, 131–148, 2011.

[10] V. G. Drinfeld, *Hamiltonian structures on Lie groups, Lie bialgebras, and the geometric meaning of the classical Yang-Baxter equations*, Reports of the USSR Academy of Sciences **268 (2)**, 285–287, 1983.

[11] V. G. Drinfeld, *On constant quasiclassical solutions of the quantum Yang-Baxter equation*, Reports of the USSR Academy of Sciences **273 (3)**, 531–535, 1983.

[12] V. G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Reports of the USSR Academy of Sciences **283 (5)**, , 1060–1064, 1985.

[13] V. G. Drinfeld, *Quantum groups*, Zap. scientific Semin. Leningrad Dep. Mat., Institute of the USSR Academy of Sciences 155, 19–49, 1986.

[14] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. **I**, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[15] T. G. Ergashev, A. Hasanov, *Fundamental solutions of the bi-axially symmetric Helmholtz equation*, Uzbek Math. J. **1**, 55–64, 2018.

[16] H. Exton, *Certain hypergeometric functions of four variables*, Bull. Soc. Math. Grece (N.S.) **13**, 104–113, 1972.

[17] H. Exton, *Some integral representations and transformations of hypergeometric functions of four variables*, Bull. Soc. Math. Grece (N.S.) **14**, 132–140, 1973.

[18] H. Exton, *Multiple hypergeometric functions and applications*, Ellis Horwood Ltd., John Wiley and Sons, London, New York, Sydney, Toronto, 1976.

[19] F. I. Frankl, *Selected works in gas dynamics*, Nauka. Moscow (in Russian), 1973.

[20] A. Hasanov, *Fundamental solutions of generalized bi-axially symmetric Helmholtz equation*, Complex Var. Elliptic Equ. **52 (8)**, 673—683, 2007.

[21] A. Hasanov, *The solution of the Cauchy problem for generalized Euler–Poisson–Darboux equation*, Inter. J. Appl. Math. Statist. **8 (7)**, 30—44, 2007.

[22] A. Hasanov, *On a mixed problem for the equation $sign y|y|^m u_{xx} + x^n u_{yy} = 0$*, Izv. Akad. Nauk UzSSR, ser. Fiz.-mat. Nauk **2**, 28—32, 1982 (in Russian).

[23] A. Hasanov, H. M. Srivastava, *Decomposition formulas associated with the Lauricella multivariable hypergeometric functions*, Comput. Math. Appl. **53 (7)**, 1119–1128, (2007). https://doi.org/10.1016/j.camwa.2006.07.007

[24] A. Hasanov, H. M. Srivastava, *Some decomposition formulas associated with the Lauricella function $F_A^{(r)}$ and other multiple hypergeometric functions*, Appl. Math. Lett. **19 (2)**, 113–121, 2006. https://doi.org/10.1016/j.aml.2005.03.009

[25] A. Hasanov, H. M. Srivastava, M. Turaev, *Decomposition formulas for some triple hypergeometric functions*, J. Math. Anal. Appl. **324 (2)**, 955–969, 2006. https://doi.org/10.1016/j.jmaa.2006.01.006

[26] A. Hasanov, M. Turaev, J. Choi, *Decomposition formulas for the generalized hypergeometric ${}_4F_3$ function*, Honam Math. J. **32 (1)**, 1–16, 2010.

[27] R. P. Horja, *Hypergeometric functions and mirror symmetry in toric varieties*, Preprint. math., AG/9912109, 1-103, 1999.

[28] G. Lohofer, *Theory of an electromagnetically levitated metal sphere 1: Absorbed power*, SIAM J. Appl. Math. **49 (2)**, 567—581, 1989.

[29] O. I. Marichev, *Handbook of integral transforms of higher transcendental functions: Theory and Algorithmic Tables*, Halsted Press (Ellis Horwood Limited, Chichester), Wiley, New York, Brisbane, Chichester and Toronto, 1982.

[30] A. M. Mathai, R. K. Saxena, *Generalized hypergeometric functions with applications in statistics and physical sciences*, Springer-Verlag, Berlin, Heidelberg, New York, Vol. **348**, 1973.

[31] A. W. Niukkanen, *Generalised hypergeometric series ${}^N F(x_1, \dots, x_N)$ arising in physical and quantum chemical applications*, J. Phys. A: Math. Gen. **16**, 1813—1825, 1983. DOI:10.1088/0305-4470/16/9/007

[32] M. Passare, A. Tsikh, A. A. Cheshel, *Multiple Mellin-Barnes integrals as periods of Calabi-Yau manifolds with several moduli*, Theor. Math. Phys. **109 (3)**, 1544–1555, 1996.

[33] M. Passare, A. Tsikh, O. Zhdanov, *A multidimensional Jordan residue lemma with an application to Mellin-Barnes integrals*, Aspects Math. E. **26**, 233–241, 1994.

[34] E. G. Poole, *Introduction to the theory of linear differential equations*, Clarendon (Oxford University) Press, Oxford, 1936.

[35] S. Saran, *Relations between functions contiguous to certain hypergeometric functions of three variables*, Ganita **5**, 69–76, 1954.

[36] S. Saran, *Transformations of certain hypergeometric functions of three variables*, Acta Math. **93 (3-4)**, 292–312, 1955.

[37] R. B. Seilkhanova, A. Hasanov, *Particular solutions of generalized Euler-Poisson-Darboux equation*, Electron. J. Diff. Equ. **9**, 1–10, 2015.

[38] C. Sharma, C. L. Parihar, *Hypergeometric functions of four variables (I).*, J. Indian Acad. Math. **11 (2)**, 99–115, 1989.

[39] I. N. Sneddon, *Special Functions of Mathematical Physics and Chemistry*, 3rd ed., **182**, Longman, London, New York, 1980.

[40] H. M. Srivastava, J. Choi, *Zeta and q-Zeta functions and associated series and integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[41] H. M. Srivastava, A. Hasanov, J. Choi, *Double-layer potentials for a generalized bi-axially symmetric Helmholtz equation*, Sohag J. Math. **2 (1)**, 1–10, 2015.

[42] H. M. Srivastava, P. W. Karlsson, *Multiple Gaussian hypergeometric series*, Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1985.

[43] H. M. Srivastava, B. R. K. Kashyap, *Special functions in queuing theory and related stochastic processes*, Academic Prees, **308**, New York, London, San Francisco, 1982.

[44] A. Varchenko, *Multidimensional hypergeometric functions and representation theory of Lie algebras and quantum groups*, Advanced Series in Mathematical Physics **21**, 371, World Scientific, Singapore, 1995.