Montes Taurus J. Pure Appl. Math. 4 (3), 25–32, 2022

Montes Taurus Journal of Pure and Applied Mathematics

https://www.mtjpamjournal.com

MTJPAM
ISSN: 2687-4814

On Some Classes of Fredholm-Volterra Integral Equations in Two Variables

Adrian Petruşel [a], Ioan A. Rus[b]

[a]Department of Mathematics, Babeş-Bolyai University Cluj-Napoca and Academy of Romanian Scientists Bucharest, Romania
[b]Department of Mathematics, Babeş-Bolyai University Cluj-Napoca, Romania

Abstract

Let $a, b, c \in \mathbb{R}^2$, $a_i < c_i < b_i$, $i \in \{1,2\}$, $[a,b] := [a_1, b_1] \times [a_2, b_2]$, let $(\mathbb{B}, |\cdot|)$ be a (real or complex) Banach space, $K \in C([a,b] \times [a,c] \times \mathbb{B}, \mathbb{B})$, $H \in C([a,b] \times [a,b] \times \mathbb{B}, \mathbb{B})$ and $g \in C([a,b], \mathbb{B})$. In this paper we study the following integral equation

$$u(x) = \int\limits_{[a,c]} K(x,s,u(s))ds + \int\limits_{[a,x]} H(x,s,u(s))ds + g(x),\ \ x = (x_1, x_2) \in [a,b].$$

Using the Fibre Contraction Principle we give existence and uniqueness results, and we prove the convergence of the successive approximations. By the weakly Picard operator theory (in the framework of the ordered Banach space $\mathbb{B}$) we give Gronwall lemma type results and comparison theorems. Some other similar type of Fredholm-Volterra integral equations are also studied.

Keywords: Fredholm-Volterra integral equation, Existence and uniqueness, Successive approximations, Integral inequality, Gronwall lemma, Comparison lemma, Fibre contraction principle

2010 MSC: 45L05, 47H10, 47H09, 47J25, 45G10

1. Introduction

Let $a, b, c \in \mathbb{R}^2$, $a_i < c_i < b_i$, $i \in \{1,2\}$, $[a,b] := [a_1, b_1] \times [a_2, b_2]$, let $(\mathbb{B}, |\cdot|)$ be a (real or complex) Banach space, $K \in C([a,b] \times [a,c] \times \mathbb{B}, \mathbb{B})$, $H \in C([a,b] \times [a,b] \times \mathbb{B}, \mathbb{B})$ and $g \in C([a,b], \mathbb{B})$. In this paper we study the following integral equation of mixed type

$$u(x) = \int\limits_{[a,c]} K(x,s,u(s))ds + \int\limits_{[a,x]} H(x,s,u(s))ds + g(x),\ \ x = (x_1, x_2) \in [a,b].$$

Using the Fibre Contraction Principle we give existence and uniqueness results, and we prove the convergence of the successive approximations. By the weakly Picard operator theory (in the framework of the ordered Banach space $\mathbb{B}$) we give Gronwall lemma type results and comparison theorems. Some other similar type of Fredholm-Volterra integral equations are also studied.

†*Article ID:* MTJPAM-D-21-00021

Email addresses: petrusel@math.ubbcluj.ro (Adrian Petruşel), iarus@math.ubbcluj.ro (Ioan A. Rus)

Received:26 February 2021, Accepted:16 June 2021, Published:09 July 2021

⋆*Corresponding Author:* Adrian Petruşel

2. Preliminaries

2.1. *Weakly Picard operators*

Let $(X, \to)$ be an L-space, where X is a nonempty set and $\to$ is a convergence structure, in the sense of Fréchet, defined on X. Usual examples of L-spaces are: metric spaces (X, d), where $\to := \xrightarrow{d}$; topological spaces (X, τ), where $\to := \xrightarrow{\tau}$; normed spaces $(X, \|\cdot\|)$, where $\to := \xrightarrow{\|\cdot\|}$ or $\to := \rightharpoonup$; and many others.

Let us consider $A : X \to X$ a given operator. Then, we denote by F_A the fixed point set of A, i.e., $F_A := \{x \in X : x = A(x)\}$. In this context, $A : X \to X$ is said to be weakly Picard operator (briefly WPO) if, for all $x \in X$, the sequence $(A^n(x))_{n\in\mathbb{N}}$ of Picard iterates converges in $(X, \to)$ and the limit (which may depend on x) is a fixed point of A.

If A is WPO and $F_A = \{x^*\}$, then by definition A is called a Picard operator (PO).

If $A : X \to X$ is a WPO, then we define the operator $A^\infty : X \to X$, by $A^\infty(x) := \lim\limits_{n\to\infty} A^n(x)$. It is obvious that $A^\infty(X) = F_A$, i.e., A^∞ is a set retraction of X on F_A.

In our next considerations, let us consider the case of an ordered L-space, i.e., an L-space endowed with a partial ordering ”$\leq$” which is closed with respect to $\to$.

Abstract Gronwall Lemma. Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ be an operator. We suppose that:

(1) A is increasing with respect to $\leq$;

(2) A is WPO with respect to $\to$.

Then:

(i) $x \leq A(x) \Rightarrow x \leq A^\infty(x)$;

(ii) $x \geq A(x) \Rightarrow x \geq A^\infty(x)$.

Abstract Comparison Lemma. Let $(X, \to, \leq)$ be an ordered L-space and $A, B, C : X \to X$ three operators having the following properties:

(1) $A \leq B \leq C$;

(2) the operators A, B, C are WPO with respect to $\to$;

(3) the operator B is increasing with respect to $\leq$.

Then:

$$x \leq y \leq z \Rightarrow A^\infty(x) \leq B^\infty(y) \leq C^\infty(z).$$

For the abstract Gronawall Lemma and the abstract comparison principle see [9], [10], [12], [13], [14].

2.2. *Generalized fibre contraction theorem*

The following result is know as the Fiber Contraction Principle.

Fiber Contraction Principle. Let $(X, \to)$ be an L-space, (Y, ρ) be a metric space, $B : X \to X$, $C : X \times Y \to Y$ and $A : X \times Y \to X \times Y$, $A(x, y) := (B(x), C(x, y))$. We suppose that:

(1) (Y, ρ) is complete;

(2) B is WPO with respect to $\to$;

(3) $C(x, \cdot) : Y \to Y$ is an α-contraction, for all $x \in X$;

(4) $C : X \times Y \to Y$ is continuous.

Then, A is WPO. Moreover, if B is a PO, then A is a PO.

In order to obtain our main results, we need the following extension of the above theorem.
Generalized Fiber Contraction Principle. Let $(X_0, \to)$ be an L-space, $m \geq 1$ and (X_k, d_k), $k \in \{1, \cdots, m\}$ be metric spaces. For $k \in \{0, 1, \cdots, m\}$ let us consider $A_k : \prod_{i=0}^{k} X_i \to X_k$. We suppose that:

(1) for each $k \in \{1, 2, \cdots, m\}$ the metric spaces (X_k, d_k) are complete;

(2) A_0 is a WPO;

(3) for each $k \in \{1, 2, \cdots, m\}$ the operators $A_k(x_0, \ldots, x_{k-1}, \cdot) : X_k \to X_k$ are α_i-contractions;

(4) for each $k \in \{1, 2, \cdots, m\}$ the operators A_k are continuous.

Then, the operator $A : \prod_{k=0}^{m} X_k \to \prod_{k=0}^{m} X_k$, defined by,

$$A(x_0, \ldots, x_m) := (A_0(x_0), A_1(x_0, x_1), \ldots, A_m(x_0, \ldots, x_m))$$

is a WPO. Moreover, if A_0 is a PO, then A is a PO too.
For the fibre contraction principle, its generalization and applications see also [12], [13], [14], [16], [17].

3. Main results

Let $a, b, c \in \mathbb{R}^2$, $a_i < c_i < b_i$, $i \in \{1, 2\}$, let $(\mathbb{B}, |\cdot|)$ be a (real or complex) Banach space. We shall use the following notations:
$I := [a_1, b_1] \times [a_2, b_2]$,
$I_1 := [a_1, c_1] \times [a_2, c_2]$, $I_2 := [c_1, b_1] \times [a_2, c_2]$,
$I_3 := [a_1, c_1] \times [c_2, b_2]$, $I_4 := [c_1, b_1] \times [c_2, b_2]$,
$\Gamma_{12} := I_1 \cap I_2$, $\Gamma_{13} := I_1 \cap I_3$, $\Gamma_{24} := I_2 \cap I_4$.
We also consider the following L-spaces:

$$X := \left(C(I, \mathbb{B}), \xrightarrow{unif}\right),\ X_i = \left(C(I_i, \mathbb{B}), \xrightarrow{unif}\right),\ i \in \{1, 2, 3, 4\}.$$

Let $R : X \to \prod_{i=1}^{4} X_i$, $u \mapsto (u|_{I_1}, u|_{I_2}, u|_{I_3}, u|_{I_4})$ and

$$U := \left\{ u \in \prod_{i=1}^{4} X_i \mid u_1|_{\Gamma_{12}} = u_2|_{\Gamma_{12}}, u_1|_{\Gamma_{13}} = u_3|_{\Gamma_{13}}, u_1|_{\Gamma_{14}} = u_4|_{\Gamma_{14}}, u_3|_{\Gamma_{34}} = u_4|_{\Gamma_{34}} \right\}.$$

It is clear that $u \in X \Leftrightarrow R(u) \in U$ and $R : X \to U$ is a bijection.
Moreover, $R : \left(X, \xrightarrow{unif}\right) \to \left(U, \xrightarrow{unif}\right)$ is homeomorphism.
For $K \in C(I \times I_1 \times \mathbb{B}, \mathbb{B})$, $H \in C(I \times I \times \mathbb{B}, \mathbb{B})$ and $g \in C(I, \mathbb{B})$ we consider the following Fredholm-Volterra integral equation

$$\begin{aligned} u(x_1, x_2) &= \int_{a_1}^{c_1} \int_{a_2}^{c_2} K(x_1, x_2, s_1, s_2, u(s_1, s_2)) ds_1 ds_2 \\ &+ \int_{a_1}^{x_1} \int_{a_2}^{x_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2)) ds_1 ds_2 \\ &+ g(x_1, x_2),\ (x_1, x_2) \in I. \end{aligned} \tag{3.1}$$

Let $A : X \to X$ be defined by

$$\begin{aligned} A(u)(x_1, x_2) := &\int_{a_1}^{c_1}\int_{a_2}^{c_2} K(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ \int_{a_1}^{x_1}\int_{a_2}^{x_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ g(x_1, x_2),\ (x_1, x_2) \in I. \end{aligned} \tag{3.2}$$

From the definition (3.2) of the operator A we observe that:

$$\begin{aligned} A(u)(x_1, x_2) = &\int_{a_1}^{c_1}\int_{a_2}^{c_2} K(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ \int_{a_1}^{x_1}\int_{a_2}^{x_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ g(x_1, x_2),\ (x_1, x_2) \in I_1 \end{aligned} \tag{3.3}$$

$$\begin{aligned} A(u)(x_1, x_2) = &\int_{a_1}^{c_1}\int_{a_2}^{c_2} K(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ \int_{a_1}^{c_1}\int_{a_2}^{x_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ \int_{c_1}^{x_1}\int_{a_2}^{x_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ g(x_1, x_2),\ (x_1, x_2) \in I_2 \end{aligned} \tag{3.4}$$

$$\begin{aligned} A(u)(x_1, x_2) = &\int_{a_1}^{c_1}\int_{a_2}^{c_2} K(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ \int_{a_1}^{x_1}\int_{a_2}^{c_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ \int_{a_1}^{x_1}\int_{c_2}^{x_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ g(x_1, x_2),\ (x_1, x_2) \in I_3 \end{aligned} \tag{3.5}$$

$$\begin{aligned} A(u)(x_1, x_2) = &\int_{a_1}^{c_1}\int_{a_2}^{c_2} K(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ \int_{a_1}^{c_1}\int_{a_2}^{c_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ \int_{c_1}^{x_1}\int_{a_2}^{c_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ \int_{a_1}^{c_1}\int_{c_2}^{x_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ \int_{c_1}^{x_1}\int_{c_2}^{x_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1ds_2 \\ &+ g(x_1, x_2),\ (x_1, x_2) \in I_4. \end{aligned} \tag{3.6}$$

From (3.3) – (3.6), the operator A induces the following operators:

$$\begin{aligned}
&T_1 : X_1 \to X_1,\ T_1(u_1)(x_1,x_2) := \text{second part of (3.3)},\\
&T_2 : X_1 \times X_2 \to X_2,\ T_2(u_1,u_2)(x_1,x_2) := \text{second part of (3.4)},\\
&T_3 : X_1 \times X_2 \times X_3 \to X_3,\ T_3(u_1,u_2,u_3)(x_1,x_2) := \text{second part of (3.5)},\\
&T_4 : X_1 \times X_2 \times X_3 \to X_4,\ T_4(u_1,u_2,u_3,u_4)(x_1,x_2) := \text{second part of (3.6)},
\end{aligned}$$

and

$$T : \prod_{i=1}^{4} X_i \to \prod_{i=1}^{4} X_i,\ T := (T_1, T_2, T_3, T_4),$$

i.e.

$$T(u_1,u_2,u_3,u_4) := (T_1(u_1), T_2(u_1,u_2), T_3(u_1,u_2,u_3), T_4(u_1,u_2,u_3,u_4)).$$

Remark 3.1. From the definition of T and R we have that $A = R^{-1}TR$ and $A^n = R^{-1}T^nR$. So we have the following equivalence:

$$A : \left(X, \xrightarrow{unif}\right) \to \left(X, \xrightarrow{unif}\right) \text{ is a PO}$$

if and only if

$$T : \left(\prod_{i=1}^{4} X_i, \xrightarrow{unif}\right) \to \left(\prod_{i=1}^{4} X_i, \xrightarrow{unif}\right) \text{ is a PO.}$$

Now we present our first existence, uniqueness and approximation result for the equation (3.1).

Theorem 3.2. *We suppose that:*

(1) there exists $L_1 > 0$ such that:

$$|K(x_1,x_2,s_1,s_2,\eta_1) - K(x_1,x_2,s_1,s_2,\eta_2)| \le L_1|\eta_1 - \eta_2|,$$

for all $(x_1,x_2) \in I$, $(s_1,s_2) \in I_1$ and $\eta_1, \eta_2 \in \mathbb{B}$;

(2) there exists $L_2 > 0$ such that:

$$|H(x_1,x_2,s_1,s_2,\eta_1) - H(x_1,x_2,s_1,s_2,\eta_2)| \le L_2|\eta_1 - \eta_2|,$$

for all $(x_1,x_2), (s_1,s_2) \in I$ and $\eta_1, \eta_2 \in \mathbb{B}$;

(3) $(L_1 + L_2)(c_1 - a_1)(c_2 - a_2) < 1$.

Then:

(i) the equation (3.1) has a unique solution $u^* \in C(I,\mathbb{B})$;

(ii) for each $u^0 \in C(I,\mathbb{B})$ the sequence $\{u^n\}_{n\in\mathbb{N}}$ of successive approximations, defined by

$$\begin{aligned}
u^{n+1}(x_1,x_2) = &\int_{a_1}^{c_1}\int_{a_2}^{c_2} K(x_1,x_2,s_1,s_2,u^n(s_1,s_2))ds_1ds_2\\
&+ \int_{a_1}^{x_1}\int_{a_2}^{x_2} H(x_1,x_2,s_1,s_2,u^n(s_1,s_2))ds_1ds_2\\
&+ g(x_1,x_2),\ (x_1,x_2) \in I,
\end{aligned}$$

converges with respect to the uniform convergence on I to u^*.

Proof. To prove the theorem is equivalent to prove that A is Picard operator. For to prove that A is a PO, by Remark 3.1, we shall prove that the operator T is Picard. Since the operator T is triangular, the conclusion will follow by applying the Generalized Fibre Contraction Principle with $m = 4$.

Let us consider:

a) on X_1 the norm $\max_{I_1} |u_1(x_1,x_2)|$;

b) on X_2 the norm $\max\limits_{I_2}\left(|u_2(x_1,x_2)|e^{-\tau_2(x_2-a_2)}\right)$;

c) on X_3 the norm $\max\limits_{I_3}\left(|u_3(x_1,x_2)|e^{-\tau_1(x_1-a_1)}\right)$;

d) on X_4 the norm $\max\limits_{I_4}\left(|u_4(x_1,x_2)|e^{-\tau(x_1+x_2-a_1-a_2)}\right)$.

We remark that in the condition (3) and for a suitable chosen of the constants $\tau_1,\tau_2,\tau > 0$ the operators T_1, $T_2(u_1,\cdot)$, $T_3(u_1,u_2,\cdot)$, $T_4(u_1,u_2,u_3,\cdot)$ are contractions. From the Generalized Fibre Contraction Principle we have that T is a PO. From Remark 3.1 the proof is complete. □

Remark 3.3. In the case that $\mathbb{B} := \mathbb{R}^p$ or $\mathbb{B} := \mathbb{C}^p$ and $|\cdot|$ a norm on $\mathbb{R}^p$ or on $\mathbb{C}^p$, Theorem 3.2 is a result for a system of p integral equations of Fredholm-Volterra type.

Remark 3.4. $\mathbb{B} \subset s(\mathbb{R})$ or $\mathbb{B} \subset s(\mathbb{C})$ is a Banach space

$$(l^2(\mathbb{R}), l^2(\mathbb{C}), m(\mathbb{R}), m(\mathbb{C}), C_0(\mathbb{R}), C_0(\mathbb{C}), \ldots)$$

then Theorem 3.2 is a result for an infinite system of integral equations of Fredholm-Volterra type.

Remark 3.5. Let us consider the equation (3.1) in the conditions (1) and (2) of Theorem 3.2. We consider on $C(I,\mathbb{B})$ the max-norm. If

$$L_1(c_1-a_1)(c_2-a_2) + L_2(b_1-a_1)(b_2-a_2) < 1,$$

then A is a PO. Notice that the condition (3) is less restrictive than the above condition.

Remark 3.6. In a similar way we can study the following integral equation

$$\begin{aligned} u(x_1,x_2) &= \int_{b_1}^{c_1}\int_{b_2}^{c_2} K(x_1,x_2,s_1,s_2,u(s_1,s_2))ds_1ds_2 \\ &+ \int_{b_1}^{x_1}\int_{b_2}^{x_2} H(x_1,x_2,s_1,s_2,u(s_1,s_2))ds_1ds_2 \\ &+ g(x_1,x_2),\ \ x_1\in[a_1,b_1],\ \ x_2\in[a_2,b_2]. \end{aligned}$$

Remark 3.7. For the fixed point techniques in the integral equation theory see, for example, the following works: [1]–[11], [13], [14].

4. Gronwall lemma type result

Let $(\mathbb{B}, |\cdot|, \leq)$ be an ordered Banach space, $I := [a_1,b_1]\times[a_2,b_2]$, $I_1 := [a_1,c_1]\times[a_2,c_2]$ and the following inequation

$$\begin{aligned} u(x_1,x_2) &\leq \int_{a_1}^{c_1}\int_{a_2}^{c_2} K(x_1,x_2,s_1,s_2,u(s_1,s_2))ds_1ds_2 \\ &+ \int_{a_1}^{x_1}\int_{a_2}^{x_2} H(x_1,x_2,s_1,s_2,u(s_1,s_2))ds_1ds_2 \\ &+ g(x_1,x_2),\ \ x_1,x_2\in I. \end{aligned} \tag{4.1}$$

Then we have the following Gronwall lemma type result for our integral equation of mixed type.

Theorem 4.1. *Let us consider the integral equation (3.1). We suppose that:*
(1) K, H and g are as in Theorem 3.2;
(2) the functions $K(x_1,x_2,s_1,s_2,\cdot):\mathbb{B}\to\mathbb{B}$, $H(x_1,x_2,s_1,s_2,\cdot):\mathbb{B}\to\mathbb{B}$ *are increasing.*
Let u^* *be the unique solution of equation (3.1) and* $u \in C(I,\mathbb{B})$ *be a solution of inequation (4.1).*
Then we have that $u \leq u^*$.

Proof. The conclusion follows by applying the Abstract Gronawall Lemma for the Picard operator $A : X \to X$ be defined by

$$\begin{aligned} A(u)(x_1, x_2) := &\int_{a_1}^{c_1}\int_{a_2}^{c_2} K(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1 ds_2 \\ &+ \int_{a_1}^{x_1}\int_{a_2}^{x_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1 ds_2 \\ &+ g(x_1, x_2),\ (x_1, x_2) \in I, \end{aligned}$$

where $X := (C(I, \mathbb{B}), \stackrel{unif}{\longrightarrow})$. □

By a similar approach one can prove the following result.

Remark 4.2. If u is a solution of

$$\begin{aligned} u(x_1, x_2) \geq &\int_{a_1}^{c_1}\int_{a_2}^{c_2} K(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1 ds_2 \\ &+ \int_{a_1}^{x_1}\int_{a_2}^{x_2} H(x_1, x_2, s_1, s_2, u(s_1, s_2))ds_1 ds_2 \\ &+ g(x_1, x_2), \end{aligned} \tag{4.2}$$

and K, h, g are as in Theorem 4.1, then $u \geq u^*$.

For abstract Gronwall lemma see also [12], [13], [15].

5. Comparison theorems

Let $(\mathbb{B}, |\cdot|, \leq)$ be an ordered Banach space, $I := [a_1, b_1]\times[a_2, b_2]$ and $I_1 := [a_1, c_1]\times[a_2, c_2]$. Using the main results in Section 3 and the Abstract Comparison Lemma we can obtain a comparison theorem for our integral equation of mixed type. More exactly, we have the following theorem.

Theorem 5.1. *Let us consider the integral equation (3.1). We will consider, for $i \in \{1, 2, 3\}$, the operators K_i, H_i, g_i as in Theorem 3.2 and the corresponding equation (E_i), for $i \in \{1, 2, 3\}$.*

In addition, we suppose that:

(i) $K_1 \leq K_2 \leq K_3$, $H_1 \leq H_2 \leq H_3$, $g_1 \leq g_2 \leq g_3$;

(ii) $K_2(x_1, x_2, s_1, s_2, \cdot)$, $H_2(x_1, x_2, s_1, s_2, \cdot)$ are increasing.

Then, if u_i^* is the unique solution of (E_i) for $i \in \{1, 2, 3\}$, we have that $u_1^* \leq u_2^* \leq u_3^*$.

For abstract comparison lemma see [12], [13], [15].

Acknowledgments

This paper is dedicated to Professor Themistocles M. Rassias on the occasion of his 70th birthday.

References

[1] D. Bainov and P. Simenonov, *Integral Inequalities and Applications*, Kluwer Acad. Publ., Dordrecht, 1992.
[2] H. Brunner and E. Messina, *Time-stepping methods for Volterra-Fredholm integral equations*, Rend. Mat. (Roma) **23**, 329–342, 2003.
[3] C. Corduneanu, *Bielecki's method in the theory of integral equations*, Ann. Univ. Mariae Curie-Sklodowska, Sec. A **38**, 23–40, 1984.
[4] C. Corduneanu, *Integral Equation and Applications*, Cambridge Univ. Press, Cambridge, 1991.
[5] M. A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, Oxford, 1964.
[6] M. Kwapisz, *Weighted norms and existence and uniqueness of L^p solutions for integral equations in several variables*, J. Diff. Eq. **97**, 246–262, 1992.
[7] N. Lungu and I. A. Rus, *On a functional Volterra-Fredholm integral equations, via Picard operator*, J. Math. Inequalities **3**, 519–527, 2009.
[8] B. G. Pachpatte, *Multidimensional Integral Equations and Inequalities*, Atlantis Press, Amsterdam-Paris, 2011.

[9] A. Petruşel and I. A. Rus, *A class of functional-integral equations with application to a bilocal problem*, In: T.M. Rassias, L. Toth (eds.), Topics in Mathematical Analysis and Applications, 609–631, Springer, Berlin, 2014.

[10] A. Petruşel, I. A. Rus and M. A. Şerban, *Fixed point structures, invariant operators, invariant partitions and applications to Carathéodory integral equations*, 497–515. In: P.M. Pardalos, T.M. Rassias (eds.), Contributions in Mathematics and Engineering, Springer, 2016.

[11] R. Precup, *Methods in Nonlinear Integral Equations*, Kluwer Acad. Publ., Dordrecht, 2002.

[12] I. A. Rus, *Picard operators and applications*, Sci. Math. Japon. **58 (1)**, 191–219, 2003.

[13] I. A. Rus, *Some nonlinear functional differential and integral equations, via weakly Picard operator theory: a survey*, Carpathian J. Math. **26 (2)**, 230–258, 2010.

[14] I. A. Rus, *Some variants of contraction principle in the case of operators with Volterra property: step by step contraction principle*, Adv. Theory of Nonlinear Anal. Appl. **3 (3)**, 111–120, 2019.

[15] I. A. Rus, A. Petruşel and G. Petruşel, *Fixed Point Theory*, Cluj Univ. Press Cluj-Napoca, 2003.

[16] I. A. Rus and M. A. Şerban, *Operators on infinite dimensional cartesian product*, Analele Univ. Vest Timişoara, Mat. Infor. **48 (1-2)**, 253–263, 2010.

[17] M. A. Şerban, *Teoria punctului fix pentru operatori definiţi pe produs cartesian*, Presa Univ. Clujeană, Cluj-Napoca, 2002.