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Mapping properties of generalized distribution series on univalent functions

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Abstract

In this work, the generalized distribution series which is constructed by probability mass function is considered to discuss certain properties of classes of univalent functions. Moreover, we discuss certain connections between different subclasses of univalent functions.

Keywords: Analytic functions, univalent functions, convex functions, starlike functions, generalized distribution series, probability mass function

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1. Introduction

Let $\mathcal{A}$ represent the class of normalized analytic functions in

$$\Delta := \{z \in \mathbb{C} \ : \ |z| < 1\}$$

which is given by

$$\mathcal{A} := \left\{ f \ : \ f(z) = z + \sum_{n \geq 2} a_n z^n, \quad f(0) = 0, \ f'(0) - 1 = 0, \quad z \in \Delta \right\}. \tag{1.1}$$

Also, we define the class

$$\mathcal{S} := \{f \in \mathcal{A} \ : \ f \text{ is univalent in } \Delta\}.$$

Further, we consider the following two subclasses with their analytic representations:

$$\gamma - \mathcal{UCV}^*(\varsigma) := \left\{ f \in \mathcal{A} \ : \ \Re\left(1 + \left(1 + \gamma e^{i\nu}\right) \frac{zf''(z)}{f'(z)}\right) < \varsigma, \quad \gamma \in [0, \ \infty), \ \nu \in \mathbb{R}, \ \varsigma \in \left(1, \frac{4+\gamma}{3}\right], \ z \in \Delta \right\}$$

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and

$$\gamma - \mathcal{S}_p^*(\varsigma) := \left\{ f \in \mathcal{A} \ : \ \Re\left(\left(1 + \gamma e^{i\nu}\right) \frac{zf'(z)}{f(z)}\right) < \varsigma, \quad \gamma \in [0, \infty), \ \nu \in \mathbb{R}, \ \varsigma \in (1, \frac{4+\gamma}{3}\right], \ z \in \Delta \right\}.$$

Let $\mathcal{V} \subseteq \mathcal{S}$, consisting of functions of the form

$$f(z) = z + \sum_{n \geq 2} |a_n| z^n. \tag{1.2}$$

Recently, Porwal and Dixit [18] considered the following two subclasses of $\mathcal{V}$

1. $\gamma - \mathcal{PUCV}^*(\varsigma) = \gamma - \mathcal{UCV}^*(\varsigma) \cap \mathcal{V}$
2. $\gamma - \mathcal{PS}_p^*(\varsigma) = \gamma - \mathcal{S}_p^*(\varsigma) \cap \mathcal{V}$

and studied its properties. The above said classes are especially extension of the classes studied in [5, 6, 8, 20] and [24].

In 1995, Dixit and Pal [7] defined a subclass of $\mathcal{A}$ as follows (see also [25]):

$$\mathcal{R}^{\varrho}(M, N) := \left\{ f \in \mathcal{A} \ : \ \left| \frac{f'(z) - 1}{(M - N)\varrho - N(f'(z) - 1)} \right| < 1, \quad 0 \neq \varrho \in \mathbb{C}, \quad -1 \leq N < M \leq 1 \right\}.$$

Also,

$$\mathcal{R}(\varsigma) := \{ f \in \mathcal{A} \ : \ \Re\left(f'(z)\right) < \varsigma, \quad \varsigma \in (1, 2] \}.$$

Probability mass (density) function (or namely pmf) of the generalized distribution namely *GD* is given by

$$p(n) = \frac{\psi_n}{\Psi}, \qquad n = 0, \ 1, \ 2, \ \cdots,$$

where $\psi_n \geq 0$, the series $\sum_{n \geq 0} \psi_n$ is convergent and

$$\Psi = \sum_{n \geq 0} \psi_n. \tag{1.3}$$

Let the series

$$\phi(x) = \sum_{n \geq 0} \psi_n x^n, \tag{1.4}$$

which is convergent in $|x| \leq 1$ (in view of (1.3)).

In 2018, first author introduced following *GD* series in [16] :

$$\mathcal{K}_\phi(z) = z + \sum_{n \geq 2} \frac{\psi_{n-1}}{\Psi} z^n. \tag{1.5}$$

Also,

$$\begin{aligned} \mathcal{K}_\phi(f, \ z) &= \mathcal{K}_\phi(z) * f(z) \\ &= z + \sum_{n \geq 2} \frac{1}{\Psi} a_n \psi_{n-1} z^n, \end{aligned}$$

where $*$ stands for Hadamard product.

Essentially intrigued by the works which are related tohypergeometric distribution series [1, 10], Poisson distribution series [2, 11, 15], generalized Bessel functions [3, 17], Wright functions [4, 21], Binomial distribution series [9, 12], hypergeometric functions [13, 18, 22, 23], confluent hyper-geometric distribution series [19] and generalized distribution series [16], the necessary and sufficient conditions for *GD* series $\mathcal{K}_\phi$ in $\gamma - \mathcal{PUCV}^*(\varsigma)$ and $\gamma - \mathcal{PS}_p^*(\varsigma)$ are discussed. Further, the inclusion relations in between the classes $\gamma - \mathcal{UCV}^*(\varsigma)$, $\gamma - \mathcal{S}_p^*(\varsigma)$, $\gamma - \mathcal{PUCV}^*(\varsigma)$, $\gamma - \mathcal{PS}_p^*(\varsigma)$, $\mathcal{R}^{\varrho}(M, N)$ and $\mathcal{R}(\varsigma)$ are studied.

2. Results Related with Probability Mass Function

Now, we recall certain results which are essentially required to discuss our results.

Lemma 2.1 ([7]). *If $f \in \mathcal{R}^{\varrho}(M, N)$ is of the form* (1.1) *then*

$$|a_n| \leq \frac{(M-N)|\varrho|}{n}, \qquad n \geq 2.$$

The sharpness of the result holds.

Lemma 2.2 ([18]). *If $f \in \mathcal{A}$ is of the form* (1.1) *and*

$$\sum_{n\geq 2} n(\gamma n + n - \gamma - \varsigma)|a_n| \leq \varsigma - 1, \tag{2.1}$$

then $f \in \gamma - \mathcal{UCV}^*(\varsigma)$.

Lemma 2.3 ([18]). *If $f \in \mathcal{A}$ is of the form* (1.1) *and*

$$\sum_{n\geq 2} (\gamma n + n - \gamma - \varsigma)|a_n| \leq \varsigma - 1, \tag{2.2}$$

then $f \in \gamma - \mathcal{S}_p^*(\varsigma)$.

Lemma 2.4 ([25]). *If $f \in \mathcal{R}(\varsigma)$ is of the form* (1.2) *then*

$$|a_n| \leq \frac{\varsigma - 1}{n}, \qquad n \geq 2.$$

Lemma 2.5 ([18]). *If $f \in \mathcal{S}$ is of the form* (1.2) *and $f \in \gamma - \mathcal{PUCV}^*(\varsigma)$*

$$|a_n| \leq \frac{\varsigma - 1}{n(\gamma n + n - \gamma - \varsigma)}, \qquad n \geq 2.$$

Lemma 2.6 ([18]). *If $f \in \mathcal{S}$ is of the form* (1.2) *and $f \in \gamma - \mathcal{PS}_p^*(\varsigma)$*

$$|a_n| \leq \frac{\varsigma - 1}{\gamma n + n - \gamma - \varsigma}, \qquad n \geq 2.$$

Remark 2.7. It should be worthy to note that for functions f of the form (1.2), the conditions (2.1) and (2.2) are also necessary.

Theorem 2.8. *The function $\mathcal{K}_\phi$ defined by* (1.5) *is in the class $\gamma - \mathcal{PUCV}^*(\varsigma)$ iff the condition*

$$(1+\gamma)\phi''(1) + (2\gamma + 3 - \varsigma)\phi'(1) \leq (\varsigma - 1)(2\phi(1) + \phi(0)) \tag{2.3}$$

is satisfied.

Proof. To prove that $\mathcal{K}_\phi \in \gamma - \mathcal{PUCV}^*(\varsigma)$ from Lemma 2.2, we have to prove that

$$\sum_{n\geq 2} n(\gamma n + n - \gamma - \varsigma)\frac{\psi_{n-1}}{\Psi} \leq \varsigma - 1.$$

Now,

$$\begin{aligned}
\sum_{n\geq 2} n(\gamma n + n - \gamma - \varsigma)\frac{\psi_{n-1}}{\Psi} &= \sum_{n\geq 2}\left[(1+\gamma)(n-1)(n-2) + (2\gamma+3-\varsigma)(n-1) + (1-\varsigma)\right]\frac{\psi_{n-1}}{\Psi} \\
&= \frac{1}{\Psi}\sum_{n\geq 1}\left[(1+\gamma)n(n-1) + (2\gamma+3-\varsigma)n + (1-\varsigma)\right]\psi_n \\
&= \frac{1}{\Psi}\left[(1+\gamma)\phi''(1) + (2\gamma+3-\varsigma)\phi'(1) + (1-\varsigma)(\phi(1)-\phi(0))\right].
\end{aligned}$$

From equation (2.3), the last expression is bounded above by $\varsigma - 1$ and the proof of above theorem is established. □

The proof of the next result is much similar to the proof of the aforementioned theorem, so we state without proof.

Theorem 2.9. *The function $\mathcal{K}_\phi$ defined by* (1.5) *is in the class $\gamma - \mathcal{PS}_p^*(\varsigma)$ iff the condition*

$$(1+\gamma)\phi'(1) \leq (\varsigma - 1)(2\phi(1) + \phi(0))$$

is satisfied.

Theorem 2.10. *If the function $f(z)$ given by* (1.1) *belongs to the class $\mathcal{R}^\varrho\,(M,\ N)$ and satisfies the inequality*

$$(M-N)\,|\varrho|\left[\frac{(1+\gamma)\phi'(1)}{\Psi} - (\varsigma-1)\left(1-\frac{\phi(0)}{\Psi}\right)\right] \leq \varsigma - 1, \tag{2.4}$$

then $\mathcal{K}_\phi(f,\ z) \in \gamma - \mathcal{UCV}^*(\varsigma)$.

Proof. To prove that $\mathcal{K}_\phi(f,\ z) \in \gamma - \mathcal{UCV}^*(\varsigma)$, by virtue of Lemma 2.1 it is enough to show that

$$\sum_{n\geq 2} n(\gamma n + n - \gamma - \varsigma)\frac{\psi_{n-1}}{\Psi}\,|a_n| \leq \varsigma - 1.$$

Now,

$$\begin{aligned}
\sum_{n\geq 2} n(\gamma n + n - \gamma - \varsigma)\frac{\psi_{n-1}}{\Psi}\,|a_n| &= \frac{(M-N)\,|\varrho|}{\Psi}\sum_{n\geq 2}(\gamma n + n - \gamma - \varsigma)\psi_{n-1}\\
&= \frac{(M-N)\,|\varrho|}{\Psi}\sum_{n\geq 2}\left[(1+\gamma)(n-1) - (\varsigma-1)\right]\psi_{n-1}\\
&= \frac{(M-N)\,|\varrho|}{\Psi}\sum_{n\geq 1}\left[(1+\gamma)n - (\varsigma-1)\right]\psi_{n}\\
&= \frac{(M-N)\,|\varrho|}{\Psi}\left[(1+\gamma)\phi'(1) - (\varsigma-1)\left(\phi(1)-\phi(0)\right)\right].
\end{aligned}$$

From equation (2.4), the last expression is bounded above by $\varsigma - 1$ and the proof is established. □

On putting $\psi_n = \dfrac{m^n}{n!}$, $m > 0$, then we get the following result of Porwal [15].

Corollary 2.11. *Let $m > 0$ and $f \in \mathcal{R}^\varrho(M,\ N)$, if for some $\gamma (0 \leq \gamma < \infty)$, the inequality*

$$(M-N)\,|\varrho|\left[(1+\gamma)m - (\varsigma-1)(1-e^{-m})\right] \leq \varsigma - 1,$$

is satisfied then $\mathcal{K}_\phi(f,\ z) \in \gamma - \mathcal{UCV}^*(\varsigma)$.

The proof of the next result is much similar to the proof of the aforementioned theorem, so we state without proof.

Theorem 2.12. *If the function $f(z)$ given by* (1.2) *belongs to the class $\mathcal{R}(\varsigma)$ and satisfies the inequality*

$$\frac{(1+\gamma)\phi'(1)}{\Psi} - (\varsigma-1)\left(1-\frac{\phi(0)}{\Psi}\right) \leq 1,$$

then $\mathcal{K}_\phi(f,\ z) \in \gamma - \mathcal{PUCV}^*(\varsigma)$.

Theorem 2.13. *If the function $f(z)$ given by* (1.1) *belongs to the class $\mathcal{R}^\varrho\,(M,\ N)$ and satisfies the condition*

$$\frac{(M-N)\,|\varrho|}{\Psi}\left[(1+\gamma)(\Psi - \phi(0)) - (\gamma+\varsigma)\int_0^1 \phi(t)dt\right] \leq \varsigma - 1,$$

then $\mathcal{K}_\phi(f,\ z) \in \gamma - \mathcal{S}_p^*(\varsigma)$.

Proof. To prove that $\mathcal{K}_\phi \in \gamma - \mathcal{S}_p^*(\varsigma)$ by virtue of Lemma 2.1 and Lemma 2.3, it is enough to prove that

$$\sum_{n\geq 2}(\gamma n + n - \gamma - \varsigma)\frac{\psi_{n-1}}{\Psi}\,|a_n| \leq \varsigma - 1.$$

Now,

$$\begin{aligned}\sum_{n\geq 2}(\gamma n + n - \gamma - \varsigma)\frac{\psi_{n-1}}{\Psi}\,|a_n| &= \frac{(M-N)\,|\varrho|}{\Psi}\sum_{n\geq 2}\left[1+\gamma-\frac{\gamma+\varsigma}{n}\right]\psi_{n-1}\\ &= \frac{(M-N)\,|\varrho|}{\Psi}\sum_{n\geq 1}\left[1+\gamma-\frac{\gamma+\varsigma}{n+1}\right]\psi_{n}\\ &= \frac{(M-N)\,|\varrho|}{\Psi}\left[(1+\gamma)(\phi(1)-\phi(0))-(\gamma+\varsigma)\int_0^1\phi(t)dt\right]\\ &\leq \varsigma - 1.\end{aligned}$$

Thus, the proof of Theorem 2.13 is established. □

The evidence for the following outcome is very similar to the evidence for the above-mentioned theory, so we say without evidence.

Theorem 2.14. *Let f be of the form* (1.2) *in the class $\mathcal{R}(\varsigma)$ and the condition*

$$(1+\gamma)\left(1-\frac{\phi(0)}{\Psi}\right)-\frac{(\gamma+\varsigma)}{\Psi}\int_0^1\phi(t)dt \leq 1$$

is satisfied then $\mathcal{K}_\phi(f,\ z) \in \gamma - \mathcal{PS}_p^*(\varsigma)$.

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