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On estimates for the first Hankel-Clifford transform

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Abstract

In this work, we obtain new inequalities for the first Hankel-Clifford transform in the space $\mathrm{L}^2((0,+\infty), x^\mu)$, $\mu \geq 0$, using a generalized translation operator for proving these estimates in certain classes of functions characterized by a generalized continuity modulus.

Keywords: First Hankel-Clifford transform, generalized translation operator

2010 MSC: 47G30

1. Definitions and preliminaries

In [1, 2], Abilov et al. proved two estimates for the Fourier transform and for the Bessel transform on certain classes of functions characterized by the generalized continuity modulus.

In this paper, we prove the generalization of Abilov's results in the first Hankel-Clifford transform on the interval $(0,+\infty)$. For this work, we use a generalized translation operator. We point out that similar results have been established in the Jacobi transform and the Dunkl transform (see [5, 6]).

Now, we collect some basic facts on the first Hankel-Clifford transform, and more details about this transform can be found in [4, 10, 12, 13].

Assume that $\mathrm{L}^p_\mu = \mathrm{L}^p_\mu((0,+\infty))$, $p \in [1,+\infty)$ and $\mu \geq 0$, as the space of all those real-valued measurable functions f on $(0,+\infty)$, such that

$$\|f\|_{\mathrm{L}^p_\mu} = \left(\int_0^\infty |f(x)|^p x^\mu dx\right)^{1/p} < \infty .$$

The first Hankel-Clifford transform of a function $f \in \mathrm{L}^1_\mu$ is defined for $\mu \geq 0$ as (see [4, 10, 12])

$$h_{1,\mu}(f)(\lambda) = \lambda^\mu \int_0^{+\infty} c_\mu(\lambda x) f(x) dx ,$$

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where c_μ is the Bessel-Clifford function of the first kind order μ (see [7]) and defined by

$$c_\mu(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k x^k}{k!\Gamma(\mu+k+1)}$$

and $x \to c_\mu(x)$ is the solution on $(0,+\infty)$ of equation differential equation

$$x\frac{d^2}{dx^2}y + (\mu+1)\frac{d}{dx}y + y = 0\ .$$

The inverse first Hankel-Clifford transform $h_{1,\mu}^{-1}$ of $f \in \mathrm{L}_\mu^1$ is given by

$$f(x) = x^\mu \int_0^{+\infty} c_\mu(\lambda x) h_{1,\mu}(f)(\lambda) d\lambda\ .$$

For $\mu \geq 0$, let $F(\lambda) = h_{1,\mu}(f)(\lambda)$ and $G(\lambda) = h_{1,\mu}(g)(\lambda)$ denote the first Hankel-Clifford transform of order μ of $f(x)$ and $g(x)$, respectively. From [10], we have the following Parseval's relation

$$\int_0^{+\infty} F(\lambda)G(\lambda)\lambda^{-\mu} d\lambda = \int_0^{+\infty} f(x)g(x)x^{-\mu} dx\ .$$

Then the first Hankel-Clifford transform $h_{1,\mu} : f \to h_{1,\mu}(f)$ is a linear isomorphism of the space L_μ^2 into itself, and for any function $f \in \mathrm{L}_\mu^2$, we have the Parseval's identity

$$\|\lambda^{-\mu} h_{1,\mu}(f)(\lambda)\|_{\mathrm{L}_\mu^2} = \|x^{-\mu} f(x)\|_{\mathrm{L}_\mu^2}\ .$$

The normalized spherical Bessel function of index μ is defined by

$$j_\mu(x) = \frac{2^\mu \Gamma(\mu+1) J_\mu(x)}{x^\mu}, \tag{1.1}$$

where J_μ is the Bessel function of the first kind and Γ is the gamma-function (see [9]).

From [7] and formula (1.1), we have

$$c_\mu(x) = \frac{1}{\Gamma(\mu+1)} j_\mu(2\sqrt{x})\ . \tag{1.2}$$

From [3], we have the following lemma:

Lemma 1.1. *Let* $\mu \geq -\frac{1}{2}$. *Then*

1. $|j_\mu(t)| \leq 1$.
2. $1 - j_\mu(t) = O(t),\ t \geq 1$.
3. $1 - j_\mu(t) = O(t^2),\ 0 \leq t \leq 1$.
4. $\sqrt{t} J_\mu(t) = O(1)$.

For $\mu \geq 0$, let K be the map defined by

$$Af(x) = x^\mu f(x)\ .$$

Let $S = S(x,y,z)$ be the area of triangle with sides $a,\ b,\ c$ ([8, 14]). Set

$$\mathrm{D}_\mu(a,b,c) = \frac{S^{2\mu-1}}{2^{2\mu}(abc)^\mu \Gamma(\mu+\frac{1}{2})\sqrt{\pi}}.$$

If S exists and zero otherwise. We note that $\mathrm{D}_\mu(a,b,c) \geq 0$ and it is symmetric in $a,\ b,\ c$.

The generalized translation operator is defined by

$$\mathrm{T}_h(f)(x) = \int_0^{+\infty} f(z)D_\mu(h,x,z)z^\mu dz,\ 0 < x, h < \infty\ .$$

From [12], we have

$$h_{1,\mu}(A\mathrm{T}_h f(.))(\lambda) = c_\mu(h\lambda)h_{1,\mu}(Af(.))(\lambda);\ \lambda \in (0,+\infty), \tag{1.3}$$

where $f \in \mathrm{L}^2_\mu$.

We have in [10] the following differential operator

$$\mathrm{B}_\mu = x\frac{d^2}{dx^2} + (1-\mu)\frac{d}{dx}$$

and the following relation holds

$$h_{1,\mu}(\mathrm{B}_\mu f)(\lambda) = -\lambda h_{1,\mu}(f)(\lambda). \tag{1.4}$$

The finite differences of the first and higher orders are defined by

$$\Delta_h Af(x) = (\Gamma(\mu+1)A\mathrm{T}_h f(x) - Af(x))$$

and

$$\begin{aligned} \Delta_h^k Af(x) &= \Delta_h(\Delta_h^{k-1}Af(x)) \\ &= (\Gamma(\mu+1)A\mathrm{T}_h - A)^k f(x) \\ &= \sum_{i=0}^{k}(-1)^{k-i}\binom{i}{k}(\Gamma(\mu+1)A\mathrm{T}_h)^i f(x)A^{k-i}f(x), \end{aligned}$$

where $i \in \{1,2,...,k\}$ and $k \in \{1,2,...\}$.

The k^{th} order generalized modulus of continuity of functions $f \in \mathrm{L}^2_\mu$ is defined as

$$\Omega_k(Af,\delta) = \sup_{0<h\leq\delta} \|x^{-\mu}\Delta_h^k Af(x)\|_{\mathrm{L}^2_\mu}.$$

Denote by $\mathrm{W}^{r,k}_{2,\psi}(\mathrm{B}_\mu)$ the space of functions $f \in \mathrm{L}^2_\mu$, $\mathrm{B}^j_\mu f \in \mathrm{L}^2_\mu$, $j \in \{1,2,...,r\}$, (in the sense of Levi, see [11]) and

$$\Omega_k(\mathrm{B}^r_\mu Af,\delta) = O\left(\psi(\delta^k)\right)$$

and $\psi(t)$ is an arbitrary function defined on $[0,\infty)$

2. Main Results

In this section, we establish the estimates for the integral

$$\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda,$$

where $N > 0$.

Lemma 2.1. *For $f \in \mathrm{W}^{r,k}_{2,\psi}(\mathrm{B}_\mu)$, then we have*

$$\begin{aligned} \|x^{-\mu}\Delta_h^k \mathrm{B}^r_\mu Af(x)\|_{\mathrm{L}^2_\mu} &= \|\lambda^{-\mu}\lambda^r(1-j_\mu(2\sqrt{\lambda h}))h_{1,\mu}(Af)(\lambda)\|_{\mathrm{L}^2_\mu} \\ &= \left(\int_0^{+\infty}\lambda^{2r}|1-j_\mu(2\sqrt{\lambda h})|^{2k}|h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{1/2}. \end{aligned}$$

Proof. From formula (1.4), we obtain

$$h_{1,\mu}(\mathrm{B}_\mu^r Af)(\lambda) = (-1)^r \lambda^r h_{1,\mu}(Af)(\lambda),\ \forall f \in \mathrm{L}_\mu^2, \tag{2.1}$$

where $r \in \{0, 1, 2, ...\}$.

We use the formulas (1.2), (1.3) and (2.1), we conclude

$$h_{1,\mu}(\mathrm{T}_h^i \mathrm{B}_\mu^r Af)(\lambda) = (-1)^r j_\mu^i(2\sqrt{\lambda h})\lambda^r \lambda h_{1,\mu}(Af)(\lambda)\ . \tag{2.2}$$

From the definition of finite differences and formula (2.2) the image $\Delta_h^k \mathrm{B}_\mu^r Af(x)$ under the first Hankel-Clifford transform has the form $(-1)^r\lambda^r(j_\mu(2\sqrt{\lambda h}) - 1)^k h_{1,\mu}(Af)(\lambda)$. By Parseval's identity, we have the result. □

Theorem 2.2. *For function $f \in \mathrm{L}_\mu^2$ in the class $\mathrm{W}_{2,\psi}^{r,k}(\mathrm{B}_\mu)$,*

$$\sup_{\mathrm{W}_{2,\psi}^{r,k}(\mathrm{B}_\mu)} \int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda = O\Big(N^{-2r}\psi^2((\frac{c}{N})^k)\Big),$$

where $r \in \{0, 1, 2, ...\}$; $k \in \{1, 2, ...\}$; $c > 0$ is a fixed constant, and $\psi(t)$ is any nonnegative function defined on the interval $[0, +\infty)$

Proof. Let $f \in \mathrm{W}_{2,\psi}^{r,k}(\mathrm{B}_\mu)$. By Hölder inequality

$$\begin{aligned}
& \int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu} d\lambda - \int_N^{+\infty} j_\mu(2\sqrt{\lambda h})|h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda \\
= \ & \int_N^{+\infty} (1 - j_\mu(2\sqrt{\lambda h}))|h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda \\
= \ & \int_N^{+\infty} (1 - j_\mu(2\sqrt{\lambda h}))\left(|h_{1,\mu}(Af)(\lambda)|\lambda^{\frac{-\mu}{2}}\right)^2 d\lambda \\
= \ & \int_N^{+\infty} (1 - j_\mu(2\sqrt{\lambda h}))\left(|h_{1,\mu}(Af)(\lambda)|\lambda^{\frac{-\mu}{2}}\right)^{2-\frac{1}{k}} \left(|h_{1,\mu}(Af)(\lambda)|\lambda^{\frac{-\mu}{2}}\right)^{\frac{1}{k}} d\lambda \\
\leq \ & \left(\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{\frac{2k-1}{2k}} \left(\int_N^{+\infty} |1 - j_\mu(2\sqrt{\lambda h})|^{2k}|h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{\frac{1}{2k}} \\
\leq \ & \left(\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{\frac{2k-1}{2k}} \left(\int_N^{+\infty} \lambda^{-2r}|1 - j_\mu(2\sqrt{\lambda h})|^{2k}\lambda^{2r}|h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{\frac{1}{2k}} \\
\leq \ & N^{-\frac{r}{k}} \left(\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{\frac{2k-1}{2k}} \left(\int_N^{+\infty} \lambda^{2r}|1 - j_\mu(2\sqrt{\lambda h})|^{2k}|h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{\frac{1}{2k}} .
\end{aligned}$$

Lemma 2.1 gives

$$\|x^{-\mu}\Delta_h^k \mathrm{B}_\mu^r Af(x)\|_{\mathrm{L}_\mu^2}^2 = \int_0^{+\infty} \lambda^{2r}|1 - j_\mu(2\sqrt{\lambda h})|^{2k}|h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda$$

$$\begin{aligned}
\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda \ \ \leq \ & \int_N^{+\infty} j_\mu(2\sqrt{\lambda h})|h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda \\
+ \ & N^{-\frac{r}{k}} \left(\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{\frac{2k-1}{2k}} \|x^{-\mu}\Delta_h^k \mathrm{B}_\mu^r Af(x)\|_{\mathrm{L}_\mu^2}^{\frac{1}{k}} .
\end{aligned}$$

From formula (4) of Lemma 1.1, we have

$$j_\mu(x) = O(x^{-\mu-\frac{1}{2}}).$$

Then

$$j_\mu(2\sqrt{\lambda h}) = O\left((\lambda h)^{-\frac{1}{2}\mu-\frac{1}{4}}\right) .$$

Thus we have

$$\begin{aligned}
\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda &= O\left(\int_N^{+\infty} (\lambda h)^{-\frac{1}{2}\mu-\frac{1}{4}}|h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)\\
&+ N^{-\frac{r}{k}}\left(\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{\frac{2k-1}{2k}} \|x^{-\mu}\Delta_h^k \mathrm{B}_\mu^r Af(x)\|_{\mathrm{L}^2_\mu}^{\frac{1}{k}}\\
&= O\left((Nh)^{-\frac{1}{2}\mu-\frac{1}{4}}\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)\\
&+ N^{-\frac{r}{k}}\left(\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{\frac{2k-1}{2k}} \|x^{-\mu}\Delta_h^k \mathrm{B}_\mu^r Af(x)\|_{\mathrm{L}^2_\mu}^{\frac{1}{k}} ,
\end{aligned}$$

or

$$\begin{aligned}
(1-O((Nh)^{-\frac{1}{2}\mu-\frac{1}{4}}))\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda &= O(N^{-\frac{r}{k}})\left(\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{\frac{2k-1}{2k}}\\
&\times \|x^{-\mu}\Delta_h^k \mathrm{B}_\mu^r Af(x)\|_{\mathrm{L}^2_\mu}^{\frac{1}{k}} .
\end{aligned}$$

Setting $h \to \frac{c}{h}$ in the last inequality and choosing $c > 0$ such that $1 - O((c)^{-\frac{1}{2}\mu-\frac{1}{4}}) \geq \frac{1}{2}$, we obtain

$$\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda = O(N^{-\frac{r}{k}})\left(\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{\frac{2k-1}{2k}} \psi^{\frac{1}{k}}((\frac{c}{N})^k) .$$

We conclude that

$$\left(\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{\frac{1}{2k}} = O(N^{-\frac{r}{k}})\psi^{\frac{1}{k}}((\frac{c}{N})^k) ,$$

then

$$\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda = O\left(N^{-2r}\psi^2((\frac{c}{N})^k)\right) .$$

This completes the proof of theorem. □

Theorem 2.3. *If* $\psi(t) = t^m$, $0 < m < 2$. *Then*

$$\sqrt{\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda} = O(N^{-r-km}) \Longleftrightarrow f \in \mathrm{W}^{r,k}_{2,\psi}(\mathrm{B}_\mu).$$

where $r \in \{0, 1, ...\}$; $k \in \{1, 2, ...\}$.

Proof. Let $f \in \mathrm{W}^{r,k}_{2,\psi}(\mathrm{B}_\mu)$, we prove sufficiency by using Theorem 2.2, then

$$\left(\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right)^{1/2} = O(N^{-r-km}).$$

To prove necessity, let

$$\sqrt{\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda} = O(N^{-r-km})$$

i.e.,

$$\int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda = O(N^{-2r-2km}) .$$

It is easy to show there exists a function $f \in \mathrm{L}^2_\mu$ such that $\mathrm{B}^r_\mu Af \in \mathrm{L}^2_\mu$ and

$$\mathrm{B}^r_\mu Af(x) = (-1)^r x^\mu \int_0^{+\infty} c_\mu(\lambda x)\lambda^r h_{1,\mu}(Af)(\lambda)\lambda . \tag{2.3}$$

From formulas (1.2), (2.3) and Parseval's identity, we have

$$\|x^{-\mu}\Delta^k_h \mathrm{B}^r_\mu Af(x)\|^2_{\mathrm{L}^2_\mu} = \int_0^{+\infty} |1 - j_\mu(2\sqrt{\lambda h})|^{2k} \lambda^{2r} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda .$$

We put

$$\int_0^{+\infty} = \int_0^N + \int_N^{+\infty} = \mathrm{I}_1 + \mathrm{I}_2 ,$$

where $N = [(4h)^{-1}]$. We estimate I_2 then seperately from (2) of Lemma 1.1, we obtain

$$\begin{aligned}
\mathrm{I}_2 &= \int_N^{+\infty} |1 - j_\mu(2\sqrt{\lambda h})|^{2k} \lambda^{2r} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda \\
&= O\left(\int_N^{+\infty} \lambda^{2r} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda\right) \\
&= O\left(\sum_{l=0}^{+\infty} \int_{N+l}^{N+l+1} \lambda^{2r} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda\right) \\
&= O\left(\sum_{l=0}^{+\infty} (N+l+1)^{2r} \int_{N+l}^{N+l+1} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda\right) \\
&= O\left(\sum_{l=0}^{+\infty} (N+l+1)^{2r} \left[\int_{N+l}^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda - \int_{N+l+1}^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda\right]\right) \\
&= \left(\sum_{l=0}^{+\infty} (N+l+1)^{2r} \int_{N+l}^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda - \sum_{l=0}^{+\infty} (N+l+1)^{2r} \int_{N+l+1}^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda\right) \\
&= O(N^{2r} \int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda \\
&+ \sum_{l=0}^{+\infty} ((N+l+1)^{2r} - (N+l)^{2r}) \int_{N+l}^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda) \\
&= O(N^{2r} \int_N^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda \\
&+ \sum_{l=0}^{+\infty} (N+l)^{2r-1} \int_{N+l}^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2 \lambda^{-\mu} d\lambda)
\end{aligned}$$

$$
\begin{aligned}
&= O(N^{2r}N^{-2r-2km}) + O(\sum_{l=1}^{+\infty}(N+l)^{2r-1}(N+l)^{-2r-2km}) \\
&= O(N^{-2km}) + O(N^{-2km}) \\
&= O((4h)^{2km}) \\
&= O(h^{2km})\ .
\end{aligned}
$$

Then

$$\mathrm{I}_2 = O(h^{2km}).$$

We estimate I_1, by formula (3) of Lemma 1.1

$$
\begin{aligned}
\mathrm{I}_1 &= \int_0^N |1 - j_\mu(2\sqrt{\lambda h})|^{2k}\lambda^{2r}|h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda \\
&= O(h^{2k})\int_0^N \lambda^{2k}\lambda^{2r}|h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda \\
&= O(h^{2k})\int_0^N \lambda^{2r+2k}|h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda \\
&= O(h^{2k})\sum_{l=0}^{N-1}\int_l^{l+1} \lambda^{2r+2k}|h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda \\
&= O(h^{2k})\sum_{l=0}^{N-1}(l+1)^{2r+2k}\int_l^{l+1} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda \\
&= O(h^{2k})\sum_{l=0}^{N-1}(l+1)^{2r+2k}\left[\int_l^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda - \int_{l+1}^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right] \\
&= O(h^{2k})\left[1 + \sum_{l=0}^{N-1}((l+1)^{2r+2k} - l^{2r+2k})\int_l^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right] \\
&= O(h^{2k})\left[1 + \sum_{l=1}^{N-1} l^{2r+2k-1}\int_l^{+\infty} |h_{1,\mu}(Af)(\lambda)|^2\lambda^{-\mu}d\lambda\right] \\
&= O(h^{2k})\left[1 + \sum_{l=1}^{N-1} l^{2r+2k-1}l^{-2r-2km}\right] \\
&= O(h^{2k})\left[1 + \sum_{l=1}^{N-1} l^{2k-2km-1}\right] \\
&= O(h^{2k}).O(N^{2k-2km}) = O(h^{2km})
\end{aligned}
$$

i.e.,

$$\mathrm{I}_1 = O(h^{2km})\ .$$

Combining the estimates for I_1 and I_2 gives

$$\|x^{-\mu}\Delta_h^k \mathrm{B}_\mu^r Af(x)\|_{\mathrm{L}^2_\mu} = O(h^{km})\ ,$$

which means that $f \in \mathrm{W}^{r,k}_{2,\psi}(\mathrm{B}_\mu)$. Then the necessity is proved. □

3. Conclusion

In this work, by using a generalized translation operator, we have succeded to generalise two thoerems in [1, 2] for the first Hankel-Clifford transform on certain classes of functions characterized by the generalized continuity modulus.

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